Schur's lemma

Theorem (16): 1) Let (ϱ_1, V_1) and (ϱ_2, V_2) be irreducible representations of G. Let S be an intertwining operator between these representations satisfying $S: V_1 \to V_2$, $S\varrho_1(g)\vec{v}_1 = \varrho_2(g) S \vec{v}_1$ for all $g \in G$.

Then either $S = 0$ (zero mapping, $S\vec{v} = \vec{0}$ for all $\vec{v} \in V_1$)

or S is an isomorphism and thus ϱ_1 and ϱ_2 are equivalent.

Proof: - Ker S and Im S are invariant subspaces under the action of G: 1) if $\vec{v} \in \operatorname{Ker} S$ $(S\vec{v} = \vec{0})$ then $S\varrho_1(g)\vec{v} = \varrho_2(g) S\vec{v} = \vec{0}$

and thus $\varrho_1(g)\vec{v} \in \operatorname{Ker} S$ for all $g \in G$

2) if $\vec{w} \in \operatorname{Im} S$, then there exists $\vec{v} \in V_1$: $S\vec{v} = \vec{w}$

and thus $\varrho_2(g)\vec{w} = \varrho_2(g) S\vec{v} = S\varrho_1(g)\vec{v} = S\vec{v}\,'$ for some $\vec{v}\,' \in V_1$

and $\varrho_2(g)\vec{w} \in \operatorname{Im} S$ for all $g \in G$

- now, as V_1 and V_2 are irreducible, we have

$\operatorname{Ker} S = 0$ or $\operatorname{Ker} S = V_1$ and $\operatorname{Im} S = 0$ or $\operatorname{Im} S = V_2$

if $\operatorname{Im} S = \vec{0}$ then $S = \vec{0}$

if $\operatorname{Ker} S = V_1$ then $S = 0$

if $\operatorname{Ker} S = \vec{0}$ and $\operatorname{Im} S = V_2$, then S is an isomorphism and (ϱ_1, V_1) is equivalent to (ϱ_2, V_2)

2) Let (ϱ, V) be an irreducible complex finite-dimensional representation of G and let S be an intertwining operator on V, i.e. $S: V \to V$ that commutes with all $\varrho(g)$

Then $S = \lambda \mathbb{1}$, where $\lambda \in \mathbb{C}$, i.e. S is a multiple of identity

Proof: Let $A = S - \lambda\mathbb{1}$ where λ solves $\det(S - \lambda\mathbb{1}) = 0$ (that's why we need complex IR with $\dim < +\infty$)

then A is also intertwining but it has no inverse. Thus, according 1), $A = 0$ and $S = \lambda\mathbb{1}$.

- this theorem is valid also for IR of Lie algebras

Corollaries of Schur's lemma

1) Irreducibility criterion for finite (and compact Lie) groups

Theorem 17: – each reducible finite-dimensional repr. of these groups is necessarily completely reducible, and corresponding matrices can be converted to the form $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$ using a similarity transformation, but such matrices commute with $\left(\begin{array}{c|c} \lambda_1 \mathbb{1} & 0 \\ \hline 0 & \lambda_2 \mathbb{1} \end{array}\right)$ where $\lambda_1 \neq \lambda_2$. Thus, if the only matrix that commutes with all matrices of the given repr. is a multiple of the identity matrix, then this repr. must be irreducible.

(we will use this criterion later to show that constructed repr. for SU(2) group are irreducible)

2) Complex finite-dimensional IR of an abelian group G are necessarily one-dimensional

Theorem 18:

– let $T(g)$ be a representation of an abelian group G on V, then $T(g_1)T(g_2) = T(g_2)T(g_1)$ for all g_1 and $g_2 \in G$

that is, $T(g_1)$ commutes with all $T(g_2)$ and if it is IR then (Schur's lemma 2) $T(g_1) = \lambda(g_1)\mathbb{1}$ for all $g_1 \in G$

but then $T(g)$ is reducible, unless $\dim V = 1$

(notice that complexity is important as we have seen for SO(2))

3) Orthogonality relations for complex irreducible matrix repr.

Theorem 19: – let $D^\mu(g)$ and $D^\nu(g)$ are two complex IR for a finite (or compact Lie) group, then

a) if these IR are not equivalent then

$$\sum_{g \in G} D^\mu_{ij}(g^{-1}) D^\nu_{kl}(g) = 0 \quad \left(\text{or} \int_G D^\mu_{ij}(g^{-1}) D^\nu_{kl}(g)\, dg = 0 \right)$$

this is a left-invariant measure on G

later, we will not write relations for compact Lie groups, just $\sum \leftrightarrow \int dg$ and $\#G \leftrightarrow V_G$

b) if these IR are equivalent, i.e. there exists S such that $D^\mu(g) = S^{-1} D^\nu(g) S$, then

$$\sum_{g\in G} D^\mu_{ij}(g^{-1}) D^\nu_{ke}(g) = \frac{\# G}{d_\mu} S_{kj} (S^{-1})_{ie}$$

and for $\mu = \nu$ and $S = \mathbb{1}$ we have

$$\sum_{g\in G} D^\mu_{ij}(g^{-1}) D^\nu_{ke}(g) = \frac{\# G}{d_\mu} \delta_{kj} \delta_{ie} \delta_{\mu\nu}$$

c) if these IR are unitary, then $D^\mu_{ij}(g^{-1}) = D^\mu_{ji}(g)^*$

and thus
$$\boxed{\sum_{g\in G} D^\mu_{ji}(g)^* D^\nu_{ke}(g) = \frac{\# G}{d_\mu} \delta_{\mu\nu} \delta_{jk} \delta_{ie}}$$

Proof: let B be an arbitrary $d_\mu \times d_\nu$ matrix, where d_μ and d_ν are dimensions of IRs $D^\mu(g)$ and $D^\nu(g)$,

then a matrix $A = \sum_{g\in G} D^\mu(g^{-1}) B D^\nu(g)$ satisfies $D^\mu(h) A = A D^\nu(h)$ for all $h \in G$

since
$$D^\mu(h) A = \sum_{g\in G} D^\mu(h) D^\mu(g^{-1}) B D^\nu(g) = \sum_{g'\in G} D^\mu(g'^{-1}) B D^\nu(g') D^\nu(h) = A D^\nu(h)$$

here, we used substitution $g'^{-1} = h g^{-1}$ or $g = g' h$ and the rearrangement theorem

- now, if $D^\mu(g)$ is not equivalent to $D^\nu(g)$ then (Schur) $A = 0$ and we take B as a matrix with only one non-zero element $b_{rs} = 1$, obtaining

$$A = 0 = \sum_{j,k} \sum_{g\in G} D^\mu_{ij}(g^{-1}) \underset{\delta_{jr}\delta_{ks}}{B_{jk}} D^\nu_{ke}(g) = \sum_{g\in G} D^\mu_{ir}(g^{-1}) D^\nu_{se}(g) = 0$$

(orthog. relation a))

but if $D^\mu(g) = S^{-1} D^\nu(g) S$, i.e. $D^\mu(g)$ and $D^\nu(g)$ are equivalent

then $D^\mu(h) A = A S D^\mu(h) S^{-1}$ or $D^\mu(h) AS = AS D^\mu(h)$ for all $h \in G$

and (Schur 2) $AS = \lambda \mathbb{1}$ or $A = \lambda S^{-1}$

- λ is determined using the trace of AS:

$$\lambda d_\mu = \mathrm{Tr}\, \lambda \mathbb{1} = \mathrm{Tr}\, AS = \sum_{g\in G} \mathrm{Tr} [\underbrace{D^\mu(g^{-1}) B S \overbrace{D^\mu(g) S^{-1}}^{D^\nu(g)}}_{A} \cancel{S^{-1}} \cdot \cancel{S}] =$$

$$= \sum_{g\in G} \mathrm{Tr}\, BS = \# G \sum_{i,j} B_{ij} S_{ji}$$

and choosing $B_{ij} = \delta_{ir}\delta_{js}$ we finally get $d_\mu \lambda = \#G\, S_{sr}$

and
$$\lambda S^{-1}_{i\ell} = A_{i\ell} = \sum_{j,k}\sum_{g\in G} D^\mu_{ij}(g^{-1})\underbrace{B_{jk}}_{\text{again } \delta_{jr}\delta_{ks}} D^\nu_{k\ell}(g) = \underbrace{\sum_{g\in G} D^\mu_{ir}(g^{-1}) D_{s\ell}(g)}_{\text{(ort. relation b))}} = \frac{\#G}{d_\mu} S_{sr}(S^{-1})_{i\ell}$$

- other relations are just simple consequences of this one

- what is orthogonal in the orthogonality relations?

if we would have all non-equivalent IRs of some group

and put together vectors such as
$$\left(D^\mu_{ij}(g_1), D^\mu_{ij}(g_2), \dots, D^\mu_{ij}(g_{\#G})\right) \text{ for all } \begin{matrix}\mu = 1,\dots,N_{IR}\\ i,j = 1,\dots,d_\mu\end{matrix}$$

then these vectors would be all orthogonal to each other

- there are d_μ^2 different vectors for each IR $D^\mu(g)$

and thus we have a condition (restriction)
$$\sum_\mu d_\mu^2 \leq \#G$$

as there cannot be more orthogonal vectors than components of the vectors

(later we will prove $\sum_\mu d_\mu^2 = \#G$)

Example: Irreducible representations of cyclic groups

- because a group generated by a single element $g \in G$ satisfying $g^n = e$ is commutative (abelian), all its complex IRs are one-dimensional, that is, $D^\mu(g)$ are just numbers also satisfying
$$D^\mu(g)^n = 1 = D^\mu(e) \quad \text{for all } \mu$$

and other elements of G are represented by multiples of this number $D^\mu(g)$

- thus we get n different complex IRs, for each root of $z^n = 1$ one IR.

- let $\varepsilon = e^{i\frac{2\pi}{n}}$ denote one of these roots then we have

„the character table" (see below for the definition of the character of IR)

cyclic point group → C_n

C_n	$E=e$	$C_n=g$	$C_n^2=g^2$	....	$C_n^{n-1}=g^{n-1}$
$\Gamma^1 = A$	1	1	1	...	1
Γ^2	1	ε	ε^2	....	$\varepsilon^{n-1} = \varepsilon^*$
Γ^3	1	ε^2	ε^4	....	$\varepsilon^{2(n-1)} = (\varepsilon^*)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Γ^{n-1}	1	ε^{n-2}	$\varepsilon^{2(n-2)}$	...	$\varepsilon^{(n-1)(n-2)}$
Γ^n	1	$\varepsilon^{n-1} = \varepsilon^*$	$\varepsilon^{2(n-1)} = (\varepsilon^*)^2$	...	$\varepsilon^{(n-1)(n-1)} = (\varepsilon^*)^{n-1} = \varepsilon$

(*) $\{\Gamma^2, \dots, \Gamma^n\}$; E_1: Γ^2, Γ^n; E_2: Γ^3, Γ^{n-1}

← completely symmetric IR

orthogonal rows (but also columns) (later)

(*) for odd n, there are $\frac{n-1}{2}$ pairs of IRs that form two-dimen. real IRs

for even n, there are $\frac{n-2}{2}$ pairs and one more IR that is antisymmetric generated by $D(g) = -1$ for which $D(g^{2k}) = 1$ and $D(g^{2k+1}) = -1$

usually denoted as B

- particularly for C_4 we have

C_4	E	C_4	C_2	C_4^3
A	1	1	1	1
B	1	−1	1	−1
E	1	i	−1	$-i$
	1	$-i$	−1	i

<u>note</u>: for another commutative group with 4 elements (Klein 4-group)

we have condition $D^\mu(g)^2 = 1$ for all its elements

thus $D^\mu(g) = \pm 1$

as for C_{2v} we have

E	C_2	σ_v	σ_v'
C_2	E	σ_v'	σ_v
σ_v	σ_v'	E	C_2
σ_v'	σ_v	C_2	E

again only for IRs

C_{2v}	E	C_2	σ_v	σ_v'
A_1	1	1	1	1
A_2	1	1	−1	−1
B_1	1	−1	1	−1
B_2	1	−1	−1	1

here we have to satisfy $D(\sigma_v') = D(C_2) D(\sigma_v)$

again orthogonal rows (and columns)

Character of a representation

- all equivalent representations form a class of equivalence and a common property for all of them is it character

Def: The character of a representation is defined
28 as the traces of all matrices that represent all $g \in G$ on some vector space V where we chose a certain basis,

or $$\boxed{\chi(g) = \mathrm{Tr}\, D(g)} = \sum_{i=1}^{\dim V} D_{ii}(g)$$

- for a single element g, $\chi(g)$ is its character
- for all $g \in G$, $\chi(g)$ as a function of g is the character of the representation

- using cyclic property of the trace, we can see that the definition is independent of the chosen basis since for $D'(g) = A^{-1} D(g) A$ we have

$$\chi'(g) = \mathrm{Tr}\, D'(g) = \mathrm{Tr}\,(A^{-1} D(g) A) = \mathrm{Tr}\, D(g) = \chi(g)$$

and thus all equiv. repr. have the same character

- but the inverted statement is not true in general, only (again) for representations of finite (and compact Lie) groups

Example: for $G = (\mathbb{R}, +)$ we had two repr. $D(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ which have the same character but they are not equivalent

- Character properties
  1) for $g = e$ we have $D(e) = \mathbb{1}_{d \times d} \Rightarrow \boxed{\chi(e) = d = \dim V}$
  2) characters of conjugate elements are the same
     - again using the cyclic property of the trace

- for $g_1 \sim g_2$ we have $g_2 = h^{-1} g_1 h$ for some $h \in G$ and
$$\chi(g_2) = \mathrm{Tr}\, D(g_2) = \mathrm{Tr}\left[D(h^{-1}) D(g_1) D(h)\right] = \mathrm{Tr}\, D(g_1) = \chi(g_1)$$

therefore, <u>only characters of conjugacy classes are tabulated</u>

3) if $D(g)$ is completely reducible and decomposed as
$$D(g) = D_1(g) \oplus D_2(g) \oplus \ldots \oplus D_r(g)$$
then
$$\chi(g) = \sum_{i=1}^{r} \chi_i(g)$$

<u>Orthogonality relations for characters of finite groups</u>

- let $\chi^\mu(g)$ and $\chi^\nu(g)$ are characters of two IR on complex finite-dim. spaces for a group G, if they are not equivalent for $\mu \neq \nu$ then from the orthog. relations for D's
$$\sum_{g\in G} D^\mu_{ij}(g^{-1})\, D^\nu_{k\ell}(g) = \frac{\#G}{d_\mu} \delta_{\mu\nu} \delta_{jk} \delta_{i\ell}$$
we get by setting $i=j$, $k=\ell$ and summing over i and k
$$\sum_{i,k} \sum_{g\in G} D^\mu_{ii}(g^{-1}) D^\nu_{kk}(g) = \boxed{\sum_{g\in G} \chi^\mu(g^{-1}) \chi^\nu(g) =}$$
$$= \sum_{i,k} \frac{\#G}{d_\mu} \delta_{\mu\nu} \delta_{ki} \delta_{ik} = \frac{\#G}{d_\mu} \sum_i \delta_{ii} \delta_{\mu\nu} \boxed{= \#G\, \delta_{\mu\nu}}$$
and because for finite groups each repr. is equivalent to some unitary repr., we have
$$\chi(g^{-1}) = \mathrm{Tr}\, D(g^{-1}) = \mathrm{Tr}\left[A D^U(g^{-1}) A^{-1}\right] = \mathrm{Tr}\, D^U(g)^\dagger = \chi^U(g)^* = \chi(g)^*$$
and thus
$$\boxed{\sum_{g\in G} \chi^\mu(g)^* \chi^\nu(g) = \#G\, \delta_{\mu\nu}}$$

- as all elements in a conjugacy classes C_k, $k = 1, \ldots, N_c$ have the same character $\chi(C_k) = \chi(g)$ for any $g \in C_k$ we can also write $\boxed{\sum_{k=1}^{N_c} n_k \chi^\mu(C_k)^* \chi^\nu(C_k) = \#G\, \delta_{\mu\nu}}$
where $n_k = \# C_k$
- we see that characters of non-equivalent IRs form an orthogonal system of vectors in the N_c-dim. space $\Rightarrow N_{IR} \leq N_c$
(later $=$)

Decomposition of reducible representations to IRs

- for a finite (or compact Lie) group, every finite-dim. repr. is completely reducible and in a suitable basis we get

$$D(g) = \begin{pmatrix} \left.\begin{matrix} D^1(g) & 0 \\ 0 & D^1(g) \end{matrix}\right\}\alpha_1 & & & 0 \\ & D^2(g)\}\alpha_2 & & \\ & & \ddots & \\ 0 & & & D^{N_{IR}}(g)\}\alpha_{N_{IR}} \end{pmatrix}$$

where the smallest subrepresentations are equivalent to IRs of the given group

and α_μ denotes how many times is the IR $D^\mu(g)$ present in $D(g)$.

- we set

$$\chi(g) = \sum_{\mu=1}^{N_{IR}} \alpha_\mu \chi^\mu(g)$$

N_{IR} ← number of all non-equivalent IRs of G

- from the orthogonality relations we have

$$\sum_{g\in G} \chi^\nu(g)^* \chi(g) = \sum_{\mu=1}^{N_r} \alpha_\mu \sum_{g\in G} \chi^\nu(g)^* \chi^\mu(g) = \alpha_\nu \#G$$

and thus

$$\boxed{\alpha_\mu = \frac{1}{\#G} \sum_{g\in G} \chi^\mu(g)^* \chi(g) = \frac{1}{\#G} \sum_{k=1}^{N_C} n_k \chi^\mu(C_k)^* \chi(C_k)}$$

- it is sufficient to know only characters of conjugacy classes
- α_μ can be zero (if $D(g)$ does not contain $D^\mu(g)$)
- decomposition is unique up to reordering

Frobenius' criterion of irreducibility (Theorem 20)

- from

$$\sum_{g\in G} \chi(g)^* \chi(g) = \sum_{g\in G} \sum_{\mu,\nu} \alpha_\mu \alpha_\nu \chi^\mu(g)^* \chi^\nu(g) = \#G \sum_\mu \alpha_\mu^2$$

- if $D(g)$ is irreducible then $\chi(g) = 1\cdot\chi^\mu(g)$ for certain μ

and $\sum_\mu \alpha_\mu^2 = 1$

- but for a reducible repr., at least two α_μ's are non-zero

and $\sum_\mu \alpha_\mu^2 > 1$

- thus we have

$$\boxed{\sum_{g\in G} \chi(g)^* \chi(g) = \#G \iff D(g) \text{ is irreducible}}$$