Irreducible representations of finite groups

- summary of main results Theorem:

1) number of IR (non-equivalent) = number of conjugacy classes

$$N_{IR} = N_c \qquad (\text{not proved yet, only } N_{IR} \leq N_c)$$

$\uparrow$ from ortogonality relations $\downarrow$

2) dimensions of IR are restricted by

$$\sum_{\mu=1}^{N_{IR}} d_\mu^2 = \#G \qquad (\text{not proved yet, only } \sum d_\mu^2 \leq \#G)$$

3) orthogonality relations for characters

a) $$\sum_{k=1}^{N_c} n_k \chi^\mu(C_k)^* \chi^\nu(C_k) = \#G\, \delta_{\mu\nu} \qquad (\text{proved})$$

b) $$\sum_{\alpha=1}^{N_{IR}} \chi^\alpha(C_i)^* \chi^\alpha(C_j) = \frac{\#G}{n_i} \delta_{ij} \qquad (\text{not proved yet})$$

$\uparrow$ sum over irred. (non-equiv.) representations

Proof of 1): It follows from 3b) that $N_c \leq N_{IR}$ (3b later :-)) and together with $N_{IR} \leq N_c$ we get $N_{IR} = N_c$

Proof of 2): Let us define ~~the~~ the regular representation of a finite group G with $n = \#G$ elements and $g_1, \dots, g_n \in G$

as a matrix repr. constructed by

$$D^{reg}(g_s)_{k\ell} = \begin{cases} 1 & \text{for } g_s g_\ell = g_k \\ 0 & \text{for } g_s g_\ell \neq g_k \end{cases} \Rightarrow \text{only one 1 and zeroes in every row and column of } n\times n \text{ matrices}$$

Example:

	e	a	b
$g_1 = e$	e	a	b
$g_2 = a$	a	b	e
$g_3 = b$	b	e	a

gives $D^{reg}(e) = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$, $D^{reg}(a) = \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$, $D^{reg}(b) = \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}$

- it is really a representation: a) $D^{reg}(e) = \mathbb{1}_{n\times n}$ as $e\, g_\ell = g_k$ only for $k = \ell$

b) $D^{reg}(g_r) D^{reg}(g_s) = D^{reg}(g_r g_s)$

$$\| $$

$$\sum_{\ell=1}^{n} \underbrace{D^{reg}(g_r)_{k\ell}}_{1 \text{ for } g_r g_\ell = g_k} \underbrace{D^{reg}(g_s)_{\ell m}}_{1 \text{ for } g_s g_m = g_\ell} = \begin{cases} 1 & \text{for } g_r \overbrace{(g_s g_m)}^{g_\ell} = g_k \\ 0 & \text{otherwise} \end{cases}$$

- character of the regular representation

$$\chi^{reg}(g_s) = \begin{cases} \#G & \text{for } g_s = e \quad \text{(all ones on the diagonal)} \\ 0 & \text{for } g_s \neq e \quad \text{(all zeroes on the diagonal)} \end{cases}$$

- let us determine how many times are IRs contained in this regular repr.

$$\alpha_\mu^{reg} = \frac{1}{\#G} \sum_{g \in G} \chi^{reg}(g)^* \chi^\mu(g) = \frac{1}{\#G} \chi^{reg}(e)^* \chi^\mu(e) = d_\mu$$

and thus each IR of G is contained in $D^{reg}(g)$ d_μ times.

or $\quad \dim D^{reg} = \#G = \sum_{\mu=1}^{N_{IR}} d_\mu^2$

<u>Proof of 3b)</u>: here we use multiplication of conjugacy classes

$$C_i C_j = \sum_{k=1}^{N_c} c_{ij}^k C_k$$

and show that in general $\quad n_i n_j \chi^\alpha(C_i) \chi^\alpha(C_j) = d_\alpha \sum_{k=1}^{N_c} c_{ij}^k n_k \chi^\alpha(C_k) \quad (*)$ for all IRs of G

- to prove $(*)$:

a) as $\quad g C_k = C_k g \quad$ for all $g \in G$ (conjugacy classes),

by setting $\quad A_k = \sum_{h \in C_k} D^\alpha(h) \quad$ [sum of matrices which represent all $h \in C_k$ in a certain IR α of G]

we get

$$D^\alpha(g) A_k = A_k D^\alpha(g) \quad \text{for all } g \in G$$

(notice that, in general, $D^\alpha(g) D^\alpha(h) \neq D^\alpha(h) D^\alpha(g)$ for $h \in C_k$)

and using Schur's lemma 2) (as D^α is irred. repr.)

we have $\quad A_k = \lambda_k \mathbb{1}$

where λ_k can be obtained by taking the trace

$$\operatorname{Tr} A_k \underset{\substack{\uparrow \\ \text{definition of } A_k}}{=} n_k \chi^\alpha(C_k) = \lambda_k d_\alpha \Rightarrow \lambda_k = \frac{n_k \chi^\alpha(C_k)}{d_\alpha}$$

b) from $C_i C_j = \sum_k c_{ij}^k C_k$, it follows

$$A_i A_j = \sum_{k=1}^{N_c} c_{ij}^k A_k$$

and by substitution for $A_i = \lambda_i \mathbb{1}$ we get $(*)$

- to finish the proof of 3b), we sum (*) over all irred. repr. of G, obtaining

$$n_i n_j \sum_{\alpha=1}^{N_{IR}} \chi^\alpha(C_i)\chi^\alpha(C_j) = \sum_{k=1}^{N_C} c_{ij}^k \, n_k \boxed{\sum_{\alpha=1}^{N_{IR}} d_\alpha \chi^\alpha(C_k)}$$

from decomposition of the regular repr. into irred. repr.

$$\chi^{res}(C_k) = \delta_{k1} \#G$$

if $C_1 = (E)$

thus

$$n_i n_j \sum_{\alpha=1}^{N_{IR}} \chi^\alpha(C_i)\chi^\alpha(C_j) = c_{ij}^1 \cdot \#G$$

- to determine c_{ij}^1 we have to figure out ~~for~~ which classes contains inverse elements to some class C_j
- as $a = gbg^{-1}$ implies $a^{-1} = (gbg^{-1})^{-1} = gb^{-1}g^{-1}$ we see that the class $C_{j'}$ of inverse elements of C_j has the same number of elements as C_j but in general $C_j \neq C_{j'}$ (for example in the cyclic groups)
- thus, c_{ij}^1 is non-zero $i = j'$ and then

$$c_{ij}^1 = n_i \delta_{ij'}$$

- moreover, $n_j = n_{j'}$ and $\chi^\alpha(C_{j'}) = \chi^\alpha(C_j)^*$ thanks to equivalence with some unitary representation for which $\chi^\alpha(g^{-1}) = \chi^\alpha(g)^*$
- finally, we have

$$n_i n_j \sum_{\alpha=1}^{N_{IR}} \chi^\alpha(C_i)\chi^\alpha(C_j) = n_i n_{j'} \sum_{\alpha=1}^{N_{IR}} \chi^\alpha(C_i)\chi^\alpha(C_{j'})^* = n_i \delta_{ij'}$$

and by cancelling n_i and relabelling j' back to j' we get 3b)