Direct product of two representations

- if a group G acts on vector spaces $V^{(1)}$ and $V^{(2)}$ with bases

$$\{e_j^{(1)}\}_{j=1}^{n_1=\dim V^{(1)}} \quad \text{and} \quad \{e_\ell^{(2)}\}_{\ell=1}^{n_2=\dim V^{(2)}}$$

as $$T^{(1)}(g)\, e_j^{(1)} = \sum_{i=1}^{n_1} e_i^{(1)} D_{ij}^{(1)}(g) \quad \text{and} \quad T^{(2)}(g)\, e_\ell^{(2)} = \sum_{k=1}^{n_2} e_k^{(2)} D_{k\ell}^{(2)}(g),$$

that is we have two (in general reducible) representations $\Gamma^{(1)}$ and $\Gamma^{(2)}$,

then the group G acts also on the tensor product $V^{(1)} \otimes V^{(2)}$

with the basis $\{e_j^{(1)} \otimes e_\ell^{(2)}\}_{j,\ell=1}^{n_1,n_2}$

as $$\left[T^{(1)}(g) \otimes T^{(2)}(g)\right] e_j^{(1)} \otimes e_\ell^{(2)} = \sum_{i,k} e_i^{(1)} \otimes e_k^{(2)} \underbrace{D_{ij}^{(1)}(g)\, D_{k\ell}^{(2)}(g)}_{D_{ik,j\ell}^{(1)\otimes(2)}(g)}$$

where the matrix $D^{(1)\otimes(2)}(g)$ is a direct (tensor) product of $D^{(1)}(g)$ and $D^{(2)}(g)$

$$D^{(1)\otimes(2)}(g) = D^{(1)}(g) \otimes D^{(2)}(g) = \begin{pmatrix} D_{11}^{(1)}(g)\, D^{(2)}(g) & D_{12}^{(1)}(g)\, D^{(2)}(g) & \cdots & \\ D_{21}^{(1)}(g)\, D^{(2)}(g) & D_{22}^{(1)}(g)\, D^{(2)}(g) & \cdots & \\ \vdots & \vdots & \ddots & \\ & & & D_{n_1 n_1}^{(1)}(g)\, D^{(2)}(g) \end{pmatrix}$$

when the basis in $V^{(1)} \otimes V^{(2)}$ is ordered as

$$e_1^{(1)} \otimes e_1^{(1)},\ e_1^{(1)} \otimes e_2^{(2)}, \ldots,\ e_2^{(1)} \otimes e_1^{(2)}, \ldots \quad ,\ e_{n_1}^{(1)} \otimes e_{n_2}^{(2)}$$

- because, in general, for arbitrary matrices A, B, A', B', we have

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB')$$

we get also

$$D_{ik,j\ell}^{(1)\otimes(2)}(g_1 g_2) = \sum_{r,s} D_{ik,rs}^{(1)\otimes(2)}(g_1) \cdot D_{rs,j\ell}^{(1)\otimes(2)}(g_2)$$

and $D^{(1)\otimes(2)}(g)$ forms a new representation of G of dim. $n_1 n_2$.

- its character is simply

$$\chi^{(1)\otimes(2)}(g) = \sum_{i,k} D_{ik,ik}^{(1)\otimes(2)}(g) = \sum_{i=1}^{n_1} D_{ii}^{(1)}(g) \sum_{k=1}^{n_2} D_{kk}^{(2)}(g) = \chi^{(1)}(g) \cdot \chi^{(2)}(g)$$

- in general, it is reducible representation and we can decompose it into irreducible ones
- in the case of the direct product of two IR of the group G, such decomposition is called Clebsch-Gordan series (later)
- sometimes, this product is called a tensor product

- for general vectors $\vec{x} = \sum_{j=1}^{n_1} x_j e_j^{(1)}$ and $\vec{y} = \sum_{\ell=1}^{n_2} y_\ell e_\ell^{(2)}$

we set

$$\left[T^{(1)}(g) \otimes T^{(2)}(g)\right](\vec{x} \otimes \vec{y}) = \sum_{j,\ell=1}^{n_1 n_2} x_j y_\ell \sum_{i=1}^{n_1} e_i^{(1)} D_{ij}^{(1)}(g) \otimes \sum_{k=1}^{n_2} e_k^{(2)} D_{k\ell}^{(2)}(g) =$$

$$= \sum x_i' y_k' \left(e_i^{(1)} \otimes e_k^{(2)}\right)$$

or

$$x_i' y_k' = \sum_{j=1}^{n_1} \sum_{\ell=1}^{n_2} D_{ij}^{(1)}(g) D_{k\ell}^{(2)}(g) x_j y_\ell = \sum_{j,\ell} D_{ik,j\ell}^{(1)\otimes(2)}(g)\, x_j y_\ell$$

<u>decomposition to a symmetric and antisymmetric parts for the same representation $\Gamma^{(1)} = \Gamma^{(2)} = \Gamma$</u>

- in this case, the superscripts (1) and (2) are not needed and we write (similarly we could do everything in the basis)

$$x_i' y_k' = \sum_{j,\ell} D_{ij}(g) D_{k\ell}(g) x_j y_\ell = \sum_{\ell,j} D_{i\ell}(g) D_{kj}(g) x_\ell y_j$$

$$x_k' y_i' = \sum_{j,\ell} D_{kj}(g) D_{i\ell}(g) x_j y_\ell = \sum_{\ell,j} D_{k\ell}(g) D_{ij}(g) x_\ell y_j$$

only change in indices

- by adding up all four expressions, we get a <u>symmetric subrepresentation</u>

we finally divided by 2

$$x_i' y_k' + x_k' y_i' = \sum_{j,\ell} \underbrace{\frac{1}{2}\left[D_{ij}(g) D_{k\ell}(g) + D_{i\ell}(g) D_{kj}(g)\right]}_{\text{matrix representation } [\Gamma\otimes\Gamma]} (x_j y_\ell + x_\ell y_j)$$

matrix representation $[\Gamma \otimes \Gamma] =$

$=$ symmetric part of $\Gamma \otimes \Gamma$

- its dimension is $\frac{1}{2} n(n+1)$ if n is the dimension of Γ

and the character is

$$\chi^{[\Gamma\otimes\Gamma]}(g) = \frac{1}{2}\left[\chi(g)^2 + \chi(g^2)\right]$$

since we set $i=j$ and $k=\ell$ and the second term is a product of two matrices $D(g) \cdot D(g) = D(g^2)$

- by subtracting the expressions, we get an <u>antisymmetric subrepresentation</u>

$$x_i' y_k' - x_k' y_i' = \sum_{j,\ell} \underbrace{\frac{1}{2}\left[D_{ij}(g) D_{k\ell}(g) - D_{i\ell}(g) D_{kj}(g)\right]}_{\text{matrix repr. } \{\Gamma\otimes\Gamma\} = \text{antisym. part}} (x_j y_\ell - x_\ell y_j)$$

with the dimension $\frac{1}{2} n(n-1) = \chi^{\{\Gamma\otimes\Gamma\}}(e)$

and character $\chi^{\{\Gamma\otimes\Gamma\}}(g) = \frac{1}{2}\left[\chi(g)^2 - \chi(g^2)\right]$