Relations among representations of groups and their subgroups

1) from a group G to its subgroup H or restriction

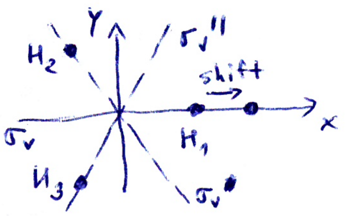
- if we restrict a certain representation Γ of the group G only to elements of its subgroup H, we get again a representation but this time of the subgroup H
- such a representation is called a restricted representation $\Gamma \downarrow H$ from G to H
- if Γ is one of IRs of G, i.e. Γ_G^{μ} with $\chi_G^{\mu}(g)$, then $\Gamma_G^{\mu} \downarrow H$ is, in general, a reducible repr. of H with the character $\boxed{\chi_H^{\mu \downarrow H}(h) = \chi_G^{\mu}(h) \text{ for all } h \in H}$

and we can decompose it into IRs of H using the formula

$$\boxed{\alpha_{\nu}^{\mu \downarrow H} = \frac{1}{\# H} \sum_{h \in H} \chi_H^{\nu}(h)^* \chi_G^{\mu}(h)}$$

where $\alpha_{\nu}^{\mu \downarrow H}$ denotes the number how many times the IR Γ_H^{ν} is contained in the restricted repr. $\Gamma_G^{\mu} \downarrow H$.

Examples: a) restriction from C_{3v} onto a subgroup $H = \{E, \sigma_v\} \sim C_s$



when the symmetry of the molecule NH_3 (ammonia) is broken by prolongation of one of the N-H bonds

- using the character tables we find

C_{3v}	E	$2C_3$	$3\sigma_v$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

$H \sim C_s$	E	σ_v	
A'	1	1	characters of IRs of $H \sim C_s$
A''	1	-1	
$A' = A_1 \downarrow H$	1	1	characters of restricted representations
$A'' = A_2 \downarrow H$	1	-1	
$A' \oplus A'' = E \downarrow H$	2	0	

decomposition either by using the formula for $\alpha_{\nu}^{\mu \downarrow H}$, for example $\alpha_{A'}^{E \downarrow H} = \frac{1}{2}(1 \cdot 2 + 1 \cdot 0) = 1$ etc.

or simply by looking at IRs of $H \sim C_s$ and adding them up to get $E \downarrow H$

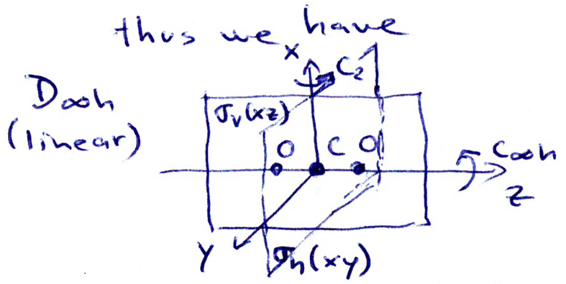
- we see that doubly generated states belonging to E will become two non-generate states belonging to A' and A''

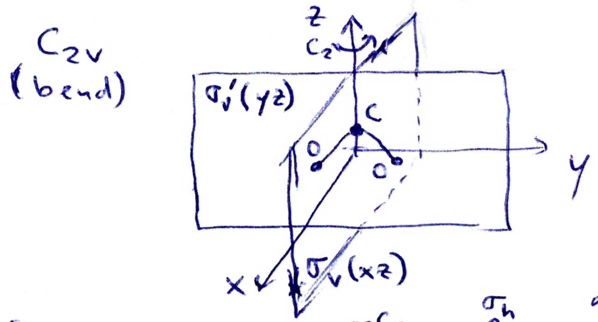
b) restriction from $D_{\infty h}$ (linear geometry) to C_{2v} (bend geometry)

- it describes what happens with states of CO_2 molecule when the symmetry is broken by bending
- it is necessary to be careful with the orientation of coordinate systems
- usual orientation is that the z-axis is the main rotational axis which is different for $-O=C=O \xrightarrow[z]{} C_\infty$

and for

thus we have





- standard character tables

$D_{\infty h}$	E	$2C_\infty^\varphi(z)$	...	$C_2(z)$	$\infty\sigma_v$	$\sigma_h(xy)$	$2S_\infty^\varphi(z)$	...	i	∞C_2
Σ_g^+	1	1	...	1	1	1	1	...	1	1
$R_z \to \Sigma_g^-$	1	1	...	1	-1	1	1	...	1	-1
$(R_x,R_y) \to \Pi_g$	2	$2\cos\varphi$	...	-2	0	-2	$-2\cos\varphi$	...	2	0
Δ_g	2	$2\cos 2\varphi$	...	2	0	2	$2\cos 2\varphi$	...	2	0
⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮
$z \to \Sigma_u^+$	1	1	...	1	1	-1	-1	...	-1	-1
Σ_u^-	1	1	...	1	-1	1	1	...	1	-1
$(x,y) \to \Pi_u$	2	$2\cos\varphi$	...	-2	0	2	$2\cos\varphi$	...	-2	0
Δ_u	2	$2\cos 2\varphi$	...	2	0	-2	$-2\cos 2\varphi$	...	-2	0

		∞C_2 ↕	σ_h ↕	$\infty\sigma_v$ ↕	
C_{2v}	E	$C_2(z)$	$\sigma_v(xz)$	$\sigma_v'(yz)$	
A_1	1	1	1	1	z
A_2	1	1	-1	-1	R_z
B_1	1	-1	1	-1	x, R_y
B_2	1	-1	-1	1	y, R_x
$\Sigma_g^- \downarrow C_{2v}$	1	-1	1	-1	$= B_1$
$\Pi_g \downarrow C_{2v}$	2	0	-2	0	$= A_2 \oplus B_2$
$\Sigma_u^+ \downarrow C_{2v}$	1	-1	-1	1	$= B_2$
$\Pi_u \downarrow C_{2v}$	2	0	2	0	$= A_1 \oplus B_1$

- ! assignment of x, y, z and R_x, R_y, R_z to IRs of $D_{\infty h}$ and C_{2v} can be confusing, it is not true in our case that $\underset{\substack{(x,y)\\ \text{in } D_{\infty h}}}{\Pi_u} = \underbrace{\underset{x}{B_1} \oplus \underset{y}{B_2}}_{\text{in } C_{2v}}$
- to decompose the restricted repr. correctly it is necessary to use the following correspondence

in $D_{\infty h}$: $x \leftrightarrow z$ in C_{2v}; $y \leftrightarrow x$ $\Big\} \Rightarrow \underset{(x,y)}{\Pi_u} = \underset{(z)}{A_1} \oplus \underset{(x)}{B_1}$

better to write $\Pi_u \downarrow C_{2v} = A_1 \oplus B_1$ etc.

$z \leftrightarrow y \Rightarrow \underset{(z)}{\Sigma_u^+} = \underset{(y)}{B_2}$

$R_x \leftrightarrow R_z$; $R_y \leftrightarrow R_x$ $\Big\} \Rightarrow \underset{(R_x,R_y)}{\Pi_g} = \underset{(R_z)}{A_2} \oplus \underset{(R_x)}{B_2}$

$R_z \leftrightarrow R_y \Rightarrow \underset{(R_z)}{\Sigma_g^-} = \underset{(R_y)}{B_1}$

in agreement with decompositions below the character table for C_{2v}

2) Induction from a subgroup to the group

- this is more complicated than restriction, but doable
- let G be a group and H its subgroup
  then G can be decomposed to left cosets $G = \overset{e}{\overset{\shortparallel}{g_1}} H + g_2 H + \ldots + g_M H$
  where g_i can be chosen arbitrarily from the left cosets and $M = \frac{\#G}{\#H}$
- let $D_H(h)$ be an arbitrary matrix repr. of the subgroup H
  and let $\phi_1, \ldots, \phi_d$ be a basis of this repr. - it is a basis of some
  invariant irred. subspace under H-action, but not necessarily under G
  that is we have $$T(h)\,\phi_i = \sum_{j=1}^{d} \phi_j\, D_H(h)_{ij} \quad \text{for all } h \in H$$
- let us define formally (we assume that in this way we get linearly independent functions)
  $\phi_{ti} = T(g_t)\,\phi_i$ where g_t's define left cosets of H in G, $t = 1, \ldots, M$, $i = 1, \ldots, d$
  and show that these functions form a basis of $M \times d$-dimensional
  repr. of G, i.e., they satisfy
  $$T(g)\,\phi_{ti} = \sum_{sj} \phi_{sj}\, D_G(g)_{sj,ti}$$
- by the action of g we get
  $$T(g)\,\phi_{ti} = T(g g_t)\,\phi_i = T(g_s) T(g_s^{-1} g g_t)\,\phi_i =$$
  $$= T(g_s) \sum_{j=1}^{d} \phi_j\, D_H(g_s^{-1} g g_t)_{ji} =$$
  $$= \sum_j \phi_{sj}\, D_H(g_s^{-1} g g_t)_{ji}$$
  ↑ here we use that $g g_t$ belongs to some leftcoset, e.g. $g_s H$ and we have $g_s^{-1} g g_t \in H$

  and thus $T(g)\,\phi_{ti}$ is a linear combination of ϕ_{sj} and
  moreover, for such s that $g g_t \in g_s H$, therefore, the matrix
  $D_G(g)$ can be obtained by adding $\delta_{st}(g) = \begin{cases} 1 & \text{for } g g_t \in g_s H \\ 0 & \text{otherwise} \end{cases}$

  or $$\boxed{D_G(g)_{sj,ti} = \delta_{st}(g)\, D_H(g_s^{-1} g g_t)_{ji}}$$

  this is the induced representation of G from the repr. $D_H(h)$
  of H denoted by $D_G = D_{H \uparrow G}$
- we can easily check that it is really repr. of G:
  a) for $g = e$ we get $g_t = g_s \Rightarrow D_G(e) = \mathbb{1}_{Md \times Md}$

b) $D_G(g)D_G(g') \Leftrightarrow \sum_{r,k} D_G(g)_{sj,rk} D_G(g')_{rk,ti} =$

$$= \sum_{r,k} \underbrace{\delta_{sr}(g)\,\delta_{rt}(g')}_{\substack{1 \text{ for } gg_r = g_s h \text{ for } h\in H \\ g'g_t = g_r h' \text{ for } h'\in H}} D_H(g_s^{-1} g g_r)_{jk}\, D_H(g_r^{-1} g' g_t)_{ki} =$$

$$g g' g_t = g_s \underbrace{hh'}_{\in H}$$

$$= \delta_{st}(gg')\, D_H(g_s^{-1} gg' g_t)_{ji} = D_G(gg')_{sj,ti} \Leftrightarrow D_G(gg')$$

- if $D_H(h)$ is unitary then $D_G(g)$ is also unitary

$$[D_G(g)^{-1}]_{sj,ti} = \delta_{st}(g^{-1})\, D_H(g_s^{-1} g^{-1} g_t)_{ji} \underset{\uparrow}{=} \delta_{ts}(g)\, D_H\left((g_t^{-1} g g_s)^{-1}\right)_{ji} =$$

Unitarity of D_H $\qquad$ $g^{-1} g_t = g_s h \Rightarrow g g_s = g_t h^{-1}$, $h \in H$

$$= \delta_{ts}(g)\, D_H(g_t^{-1} g g_s)^{*}_{ij} = D_G(g)^{*}_{ti,sj} = [D_G(g)^{\dagger}]_{sj,ti}$$

- the character of this induced repr. is

$$\boxed{\chi_G(g) = \sum_{s,j} \delta_{ss}(g)\, D_H(g_s^{-1} g g_s)_{jj} = \sum_s \delta_{ss}(g)\, \chi_H(g_s^{-1} g g_s)}$$

- induced representations are, in general, reducible, even if we start with an irreducible repr. Γ_H^{ν} of H, thus we set deco-position

$$\chi^{\nu\uparrow G}(g) = \sum_{\mu} \alpha_{\mu}^{\nu\uparrow G}\, \chi_G^{\mu}(g)$$

character of the induced repr. from IR Γ_H^{ν} of H $\qquad$ characters of IRs Γ_G^{μ} of G

which can be also written as $\qquad$ here, we replace the sum over M elem. g_s by the sum over the whole G

$$\chi^{\nu\uparrow G}(g) = \sum_s \delta_{ss}(g)\, \chi_H^{\nu}(g_s^{-1} g g_s) = \frac{1}{\# H} \sum_{\substack{g'\in G \\ g'^{-1} g g' \in H}} \chi_H^{\nu}(g'^{-1} g g')$$

we divide by $\# H$ because we can choose $\# H$ different g_s $\qquad$ necessary condition to count only correct terms

(since for any $a \in g_s H$ we get $a = g_s h$ and $a^{-1} g a = h^{-1} g_s^{-1} g g_s h \in H$ if $g_s^{-1} g g_s \in H$)

- thus $\alpha_{\mu}^{\nu\uparrow G}$ denotes the number how many times the IR Γ_G^{μ} is contained in the induced repr. $\Gamma_H^{\nu}\uparrow G$

- $\alpha_{\mu}^{\nu\uparrow G}$ is related to $\alpha_{\nu}^{\mu\downarrow H}$ for the restricted representations via Frobenius reciprocity theorem

Frobenius reciprocity theorem

Let $\alpha_\mu^{\nu\uparrow G}$ denotes the number how many times the IR Γ_G^μ of the group G is contained in the induced repr. $\Gamma_H^\nu\uparrow G$ from the IR Γ_H^ν of its subgroup

and $\alpha_\nu^{\mu\downarrow H}$ denotes the number how many times the IR Γ_H^ν of H is contained in the restricted repr. $\Gamma_G^\mu\downarrow H$ from Γ_G^μ of G,

then $\boxed{\alpha_\mu^{\nu\uparrow G} = \alpha_\nu^{\mu\downarrow H}}$ for all Γ_G^μ and Γ_H^ν.

Proof: we have

$$\alpha_\mu^{\nu\uparrow G} = \frac{1}{\#G}\sum_{g\in G}\chi_G^\mu(g)^*\chi^{\nu\uparrow G}(g) = \frac{1}{\#G}\sum_{g\in G}\chi_G^\mu(g)^*\frac{1}{\#H}\sum_{\substack{g'\in G\\ g'^{-1}gg'\in H}}\chi_H^\nu(g'^{-1}gg') =$$

$$= \frac{1}{\#G}\frac{1}{\#H}\sum_{h\in H}\sum_{g'\in G}\chi_G^\mu(\underbrace{g'hg'^{-1}})^*\chi_H^\nu(h)$$

conjugate element to h $\Rightarrow$ the same character as h

h ← substitution for $h = g'^{-1}gg'$ gives $g = g'hg'^{-1}$

$$= \frac{\overset{\#G}{\overset{\|}{\sum_{g'\in G}1}}}{\#G}\frac{1}{\#H}\sum_{h\in H}\chi_G^\mu(h)^*\chi_H^\nu(h) = \alpha_\nu^{\mu\downarrow H} = \frac{1}{\#H}\sum_{h\in H}\chi_H^\nu(h)^*\chi_G^\mu(h)$$

Basic examples: For trivial subgroups of G we get

1) if $H=G$ then $M=1$, $g_1=e$ and $D_H\uparrow G = D_H$ (the same repr.)

2) if $H=\{e\}$ then $M=\#G$, $gH=\{g\}$ and H has only one IR which is the trivial one $D_H(e)=1$

and thus, $D_G(g)_{st} = \delta_{st}(g) = \begin{cases} 1 & \text{for } gg_t = g_s \\ 0 & \text{for } gg_t \neq g_s \end{cases}$

that is, we get the regular representation

Notes: 1) Notice that from each IR Γ^ν of H we can make a repr. of G that has decomposition into IRs of G and thus every IR of H must be contained in a certain decomposition of $\Gamma_G^\mu\downarrow H$.

2) And similarly every IR of G must be contained in a certain decomposition of $\Gamma_H^\nu\uparrow G$.

3) Irreducible representations of a group $G = H_1 \otimes H_2$

- let us have a group G that is a direct product of its two subgroups H_1 and H_2, that is we can express any $g \in G$ as a product of $h_1 \in H_1$ and $h_2 \in H_2$: $g = h_1 h_2$
- then from arbitrary $\Gamma^{\mu}_{H_1}$ and $\Gamma^{\nu}_{H_2}$ (IRs of H_1 and H_2) we set IR of G as a direct product of corresponding matrices

$$D^{\mu\times\nu}_{k\ell,ij}(g=h_1h_2) = D^{\mu}_{ki}(h_1)\,D^{\nu}_{\ell j}(h_2)$$

with the character

$$\chi^{\mu\times\nu}(g=h_1h_2) = \chi^{\mu}(h_1)\,\chi^{\nu}(h_2)$$

- we can easily check that this repr. is irreducible using the Frobenius criterion of irreducibility: $\Gamma^{\mu}_{H_1}$ and $\Gamma^{\nu}_{H_1}$ are irreducible

$$\sum_{g\in G}|\chi^{\mu\times\nu}(g)|^2 \overset{g=h_1h_2}{=} \sum_{h_1\in H_1}\sum_{h_2\in H_2}|\chi^{\mu}(h_1)|^2|\chi^{\nu}(h_2)|^2 = \#H_1\cdot\#H_2 = \#G$$

- moreover, in this way, we get all IRs of G, since

$$\sum_{\mu,\nu}(d_\mu\cdot d_\nu)^2 = \sum_{\mu} d_\mu^2\cdot\sum_{\nu} d_\nu^2 = \#H_1\cdot\#H_2 = \#G$$

where d_μ and d_ν are dimensions of $\Gamma^{\mu}_{H_1}$ and $\Gamma^{\nu}_{H_2}$.

Example: group $D_{3h} = C_{3v}\otimes C_s$

where $C_{3v} = \{E, C_3, C_3^2, \sigma_v, \sigma_v', \sigma_v''\}$ and $C_s = \{E, \sigma_h\}$

as $\sigma_h C_{3v} = \{\sigma_h, S_3, S_3^5, C_2, C_2', C_2''\}$

the character table of D_{3h} can be obtained from

C_{3v}	E	$2C_3$	$3\sigma_v$
A_1	1	1	1
A_2	1	1	−1
E	2	−1	0

and

C_s	E	σ_h
A'	1	1
A''	1	−1

by making „a product" of these tables (C$_{3v}$ table is 3x multiplied by 1 and 1x by −1, see the table for D_{3h} on page T9