Symmetrization (projection) operators for finite groups

- if we describe some system with a symmetry group G, for example in quantum mechanics, in a general basis that is not suited to the symmetry of the system (basis functions are typically combinations of functions from different invariant subspaces with respect to the action of G), it can be very useful to have a tool which would enable us to construct a symmetrized basis, the functions of which would be from the invariant subspaces, usually corresponding to some IR of G
- to achieve this, we can use symmetrization (or projection) operators (only some of them will be true projectors, $\hat{P}^2 = \hat{P}$, therefore ~~the~~ term „symmetrization operator" is more suitable
- let us consider a vector space V and its invariant subspace $W \subset V$ under the action of G, that corresponds to a certain IR Γ^μ of G, that is we have, for the basis $\{e_i^\mu\}_{i=1}^{d_\mu} \in W$,

$$\underbrace{T(g)}\, e_i^\mu = \sum_{k=1}^{d_\mu} e_k^\mu D_{ki}^\mu(g)$$

(this is an operator representing $g \in G$ on V and W is preserved under this action)

that is e_i^μ forms a basis of the representation Γ^μ and we say that e_i^μ belongs to the i-th column of D^μ

- from this equation and using the great orthogonality relations for $D^\mu(g)$ we get for unitary representations (for simplicity)

$$\sum_{g\in G} D_{rs}^\mu(g)^* T(g) e_i^\mu = \sum_{k=1}^{d_\mu} \sum_{g\in G} D_{rs}^\mu(g)^* D_{ki}^\mu(g) e_k^\mu =$$

$$= \sum_{k=1}^{d_\mu} \frac{\#G}{d_\mu} \delta_{rk} \delta_{si} e_k^\mu = \frac{\#G}{d_\mu} \delta_{si} e_r^\mu$$

and for $s=i$ we have

$$e_r^\mu = \frac{d_\mu}{\#G} \sum_{g\in G} D_{ri}^\mu(g)^* T(g) e_i^\mu = P_{ri}^\mu e_i^\mu$$

where we defined the symmetrization operator

$$\boxed{P^{\mu}_{rs} = \frac{d_\mu}{\# G} \sum_{g \in G} D^{\mu}_{rs}(g)^* \, T(g)}$$

and in general we get $\quad P^{\mu}_{rs} e^{\mu}_{i} = \delta_{si} \, e^{\mu}_{r}$

- notice that it is not, in general, a projection operator as for example $P^{\mu}_{12} e^{\mu}_{2} = e^{\mu}_{1}$, but $P^{\mu}_{12} e^{\mu}_{1} = 0$

- applying this operator to a general vector v we get

$$P^{\mu}_{jk} \, v = \frac{d_\mu}{\# G} \sum_{g \in G} D^{\mu}_{jk}(g)^* \, T(g) \, v = v^{\mu}_{jk}$$

and if these vectors are non-zero vectors for a certain k, then they form a basis $\{v^{\mu}_{jk}\}_{j=1}^{d_\mu}$ of IR Γ^{μ} in a subspace $W^{\mu} = \mathcal{L}(\{v^{\mu}_{jk}\}_{j=1}^{d_\mu})$ since

$$T(h) \, v^{\mu}_{jk} = \frac{d_\mu}{\# G} \sum_{g \in G} D^{\mu}_{jk}(g)^* \underbrace{T(h) T(g)}_{T(hg)} \, v = \quad \text{substitution } hg = g' \text{ or } g = h^{-1} g'$$

$$= \frac{d_\mu}{\# G} \sum_{g' \in G} \sum_{r=1}^{d_\mu} D^{\mu}_{jr}(h^{-1})^* D^{\mu}_{rk}(g')^* \, T(g') \, v =$$

$$= \sum_{r=1}^{d_\mu} D^{\mu}_{jr}(h^{-1})^* \, v^{\mu}_{rk} = \sum_{r=1}^{d_\mu} D^{\mu}_{rj}(h) \, v^{\mu}_{rk} \quad \leftarrow \text{a linear combination over } r$$

unitarity of D^{μ}

(as it should be for a basis of IR Γ^{μ})

- we get the whole basis of W^{μ} from a single vector $v \in V$, but only if v contains a contribution from W^{μ}, otherwise we get zero vectors

- unfortunately, we usually do not have the matrices but only tabulated characters of IRs of G, therefore we often work with the incomplete symmetrization operators

$$\boxed{P^{\mu} = \sum_{j=1}^{d_\mu} P^{\mu}_{jj} = \sum_{j=1}^{d_\mu} \sum_{g \in G} \frac{d_\mu}{\# G} D^{\mu}_{jj}(g)^* \, T(g) = \frac{d_\mu}{\# G} \sum_{g \in G} \chi^{\mu}(g)^* \, T(g)}$$

- these are projection operators, but we do not construct the basis directly, we get only a certain vector from W^{μ}

$$P^{\mu} v = \sum_{j=1}^{d_\mu} v^{\mu}_{jj}$$

and to get the basis we have to apply P^{μ} to different vectors and orthogonalize them