Mathematical structures behind Lie groups

- at least from the geometrical point of view

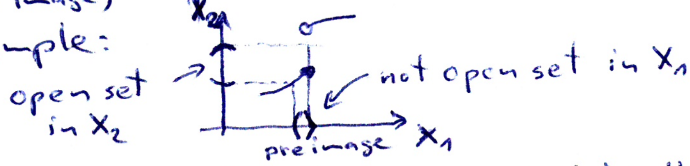
Def: A topological space is a set X with a collection τ of subsets of X (called topology τ on X) satisfying the following axioms:

1) $\emptyset \in \tau$, $X \in \tau$, that is the empty set and the whole X always belong to the topology τ on X
2) a union of arbitrary number of sets from τ belongs to τ
3) an intersection of a finite number of sets from τ belongs to τ

- sets from τ are called open sets and their complements in X are closed sets
  - notice that $\emptyset$ and X are both!
- a natural topology on $\mathbb{R}^n$ is a collection of all open sets (open intervals and their Cartesian products) and their arbitrary unions

Def: Continuity: A mapping $\phi : (X_1, \tau_1) \to (X_2, \tau_2)$ is continuous if the preimage (inverse image) of every open set in X_2 is an open set in X_1

counterexample:



Neighbourhood of a point $x \in (X, \tau)$ is a subset of X that contains an opet subset containing x, that is $U(x)$ is a neighbourhood of x, if there exists $\sigma \in \tau$ such that $x \in \sigma$ and $\sigma \subset U(x)$

Continuity in the point: A mapping ϕ is continuous at $x \in (X_1, \tau_1)$ if for each neighbourhood U_2 of the image $\phi(x)$ in X_2 there exists $U_1(x) \subset X_1$ such that $\phi(U_1(x)) \subset U_2(\phi(x))$

$\Rightarrow$ a mapping ϕ is continuous if it is continuous at every point $x \in X_1$

An interior of a set = the largest open subset of a set

A closure of a set = the smallest closed set in X containing a given set

Homeomorphism = A mapping $\phi : (X_1, \tau_1) \to (X_2, \tau_2)$ that is injective, continuous and has a continuous inverse mapping ϕ^{-1}

- if there exists a homeomorphism between (X_1, τ_1) and (X_2, τ_2) then such topological spaces are homeomorphic
  $\Rightarrow$ equivalence classes of topological spaces

- note: homeomorphism is used in topology for „spaces of similar shape"
  homomorphism is used in algebra for „sets with similar structure" in algebraic sense

A cover of a set: A set $M \subset X$ is covered by a system of sets $D \subset X$ if $M \subset \bigcup_{Y \in D} Y$

- if all sets in D are open then it is an open cover
- if D_0 is a part of D that still covers M then D_0 is a subcover of M

Compactness: A subset M of X is compact if from any open cover D of M, we can select a finite subcover $D_0 \subset D$ that is D_0 contains a finite number of open sets

- in Euclédean spaces with standard topology (open intervals) such subsets are bounded and closed
- if $M = X$ then the whole topological space is called compact
  - notice that an open interval $(0,1)$ as a topological space with standard topology of open subintervals is not compact even though it is bounded (because we can map continuously $\mathbb{R}$ on it)
  - but $SO(2)$ (as a circle) and $SO(3)$ are compact
- continuous image of a compact set is compact

- A topological space is locally compact if its every point has a compact neighbourhood.

Connectedness: A set $S \subset (X,\tau)$ is connected if there exist no non-empty open sets A and B from τ such that

$S = A \cup B$ and $A \cap B = \emptyset$ (they have no intersection)

that is S is not formed by two separate open sets

- a topological space is connected if it is its own connected subset
  - for example: - $SO(3)$ is connected (all proper rotations, $\det R = 1$)
    but $O(3)$ is not connected (2 seperate components, $\det R = \pm 1$)
    - both parts (components) are at the same time open and closed subsets
- a topological space is locally connected at x if for each open set V such that $x \in V$ there exists a connected open set U such that $x \in U$ and $U \subset V$.
  - the whole space is locally connected if it is locally connected at each point

Path-connectedness

- a path from a point x to a point y in a topological space (X,τ) is a continuous mapping $\gamma : \langle 0,1 \rangle \to (X,\tau)$ such that $\gamma(0) = x$ and $\gamma(1) = y$
- a path-connected space is a topological space where every two points can be connected by a path
- note: path-connectedness implies connectedness but not vice versa (look up for example „topologist's sine curve"
  $T = \{(x, \sin\frac{1}{x}) \text{ for } x \in (0,1\rangle\} \cup \{(0,0)\}$)
- a topological space is simply connected if it is path-connected and each path between two points can be transformed continuously into an arbitrary else path between these two points
  or in other words, every closed path ($\gamma : S^1 \to X$) can be contracted continuously into a point, that is
  there exists a continuous mapping $\phi : D^2 \text{ (a disc in } E^2) \to X$
  such that $\phi \downarrow S^1 = \gamma$ (restriction of ϕ on S^1 gives γ)

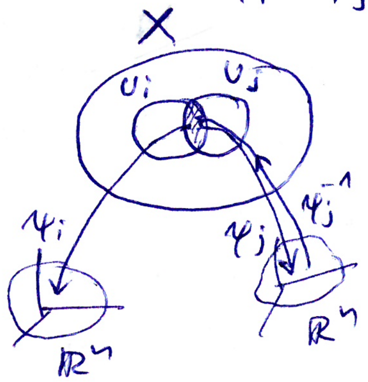
Def: A topological manifold of dimension n is a topological space which is a Hausdorff space with a countable basis and is locally homeomorphic to the Euclidean space $\mathbb{R}^n$, that is there exists a neighbourhood to each x in the manifold which is homeomorphic with an open set in $\mathbb{R}^n$ with the standard topology

- a Hausdorff space is a topological space (X,τ) if for each $x, y \in X$ there exist $U(x)$ and $V(y)$ such that $U(x) \cap V(y) = \emptyset$ or in other words, points are sufficiently separate by open sets.
- a basis B of a topological space X is a collection of open sets such that every open set in X can be obtained as some union of sets from B
- for example: $\mathbb{R}$ has a countabe basis of open intervals with lenghts and midpoints given by rational numbers
  - similarly, $\mathbb{R}^n$ has a countable basis (balls with rational radii and centers)
- a homeomorphis of an open neighbourhood of $x \in X$ into $\mathbb{R}^n$ is usually called a map and their collection an atlas if it covers the whole space X

Def: A C^k-differentiable manifold of dimension n is a topological manifold X of dim. n, for which all transition functions (maps)

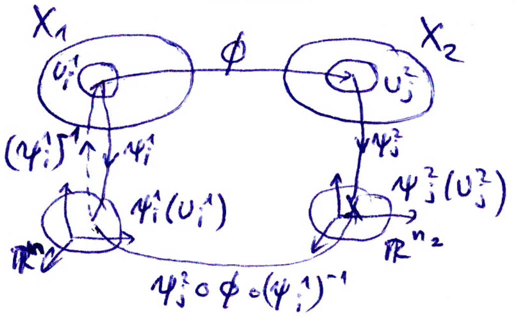
$$\varphi_i \circ \varphi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n \text{ defined on sets } \varphi_j(U_i \cap U_j)$$

have smooth derivatives up to the order k where (U_i, φ_i) are maps on X.



- A smooth manifold is a C^∞-diff. manifold
- an analytic manifold is a smooth manifold for which $\varphi_i \circ \varphi_j^{-1}$ are an analytic mappings (Taylor series is absolutely convergent on a certain open neighbourhood)

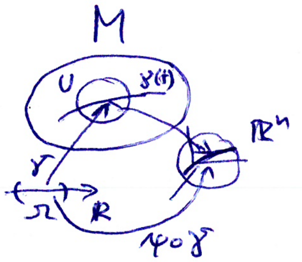
- a mapping $\phi : X_1 \to X_2$ beetween two smooth manifolds is smooth if it is continuous (in a topological sense) and the mapping $\psi_j^2 \circ \phi \circ (\psi_i^1)^{-1}$ is a smooth mapping from $\mathbb{R}^{n_1}$ to $\mathbb{R}^{n_2}$ for all maps (U_i^1, ψ_i^1) on X_1 and (U_j^2, ψ_j^2) on X_2



- a diffeomorphism between two smooth manifolds is a smooth homeomorphism ϕ, for which both ϕ and ϕ^{-1} are smooth and bijective.

Tangent space of a smooth manifold M at a point $p = T_pM$

- it can be difined either using tangent vectors to curves or, equivalently, using differential operators on functions defined on the manifold (directional derivaties)

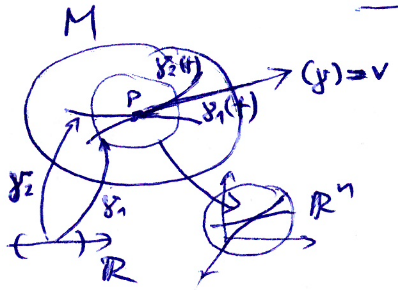


- a curve $\gamma(t)$ on a smooth manifold M is a smooth mapping $\gamma : \Omega \to M$ where $\Omega \subset \mathbb{R}$ is an open interval, in coordinates we can write (for $\dim M = n$)

$$\underbrace{\psi(\gamma(t))}_{\text{a point in } M} = \underbrace{(x^1(\gamma(t)), \dots, x^n(\gamma(t)))}_{\text{a point in } \mathbb{R}^n} = (x^1(t), \dots x^n(t))$$

↑ simplified notation

- two curves are tangent at a point $p \in M$



if 1) $\gamma_1(0) = \gamma_2(0) = p$

2) in coordinates on the neighbourhood $U(p)$ they satisfy

$$\left.\frac{dx^\mu(\gamma_1(t))}{dt}\right|_{t=0} = \left.\frac{dx^\mu(\gamma_2(t))}{dt}\right|_{t=0}$$

for all $\mu = 1, \dots, n$

or in a compact form

$$\dot{\gamma}_1(0) = \dot{\gamma}_2(0) \quad \text{where } \dot{\gamma}(t) = (\dot{x}^1(t), \dots, \dot{x}^n(t))$$

- a tangent vector v at a point $p \in M$ is usually defined as an equivalence class of curves (γ) mutually tangent at the point p and it can be determined using an arbitrary curve $\gamma(t)$ from (γ) as above

$$v = \dot{\gamma}(0)$$

- <u>all tangent vectors</u> at a point $p \in M$ (defined using curves going through p in different directions)
<u>form a vector space</u> in which we define $(\gamma) = \alpha(\gamma_1) + \beta(\gamma_2)$ as a tangent vector corresponding to the class of curves equivalent to the curve $\gamma(t)$ given, using a local map ψ in the neighbourhood $U(p)$ satisfying $\psi(p) = (0, 0, \dots 0)$, as

$$\gamma(t) = \psi^{-1}\left[\alpha\, \underbrace{\psi \circ \gamma_1(t)} + \beta\, \underbrace{\psi \circ \gamma_2(t)}\right]$$

and with $\gamma(0) = p$

curves in $\mathbb{R}^n$ going through $\psi(p)$ where we know how to add vectors and multiply them by a number

we thus have

$$\left.\frac{d x^\mu(\gamma(t))}{dt}\right|_{t=0} = \alpha \left.\frac{d x^\mu(\gamma_1(t))}{dt}\right|_{t=0} + \beta \left.\frac{d x^\mu(\gamma_2(t))}{dt}\right|_{t=0}$$

- a <u>tangent space T_pM</u> at a point $p \in M$ is thus a vector space of the above defined tangent vectors at p
- a <u>differentiable function $f: M \to \mathbb{R}$</u> on a neighbourhood $U(p) \subset M$ with a local map $\psi: M \to \mathbb{R}^n$ is such a function for which $f \circ \psi^{-1}: \mathbb{R}^n \to \mathbb{R}$ is differentiable (up to a certain order)
- a <u>directional derivative of f along the tangent vector $v = (\gamma)$</u> an a point $p \in M$ is defined as

i.e. it is a change of f along the curve $\gamma(t)$ with a tangent vector v

$$v(f) = \left.\frac{d f(\gamma(t))}{dt}\right|_{t=0} = \lim_{\Delta t \to 0} \frac{f(\gamma(t+\Delta t)) - f(\gamma(t))}{\Delta t}$$

and in coordinates we can write

$$v(f) = \left.\frac{d\left(f \circ \psi^{-1}(x^1(t), \dots x^n(t))\right)}{dt}\right|_{t=0} = \sum_{\mu=1}^{n} \left.\frac{d x^\mu(t)}{dt}\right|_{t=0} \frac{\partial f \circ \psi^{-1}}{\partial x^\mu} =$$

$$= \sum_{\mu=1}^{n} v^\mu \frac{\partial}{\partial x^\mu} f$$

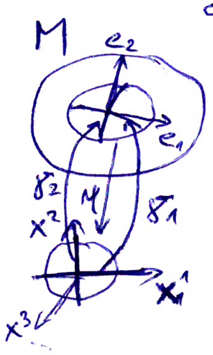
v^μ - a component of a tangent vector in the direction x^μ

- in other words, we expressed v as a differential operator of the first order

$$v = \sum_{\mu=1}^{n} v^\mu \frac{\partial}{\partial x^\mu}$$

on functions f on M

- a basis of the tangent space T_pM can be taken



as $e_\mu = \dot{\gamma}_\mu(0)$ where $\gamma_\mu(t)$ are coordinate curves

$$\gamma_\mu(t) = (x^1(p), \dots, x^\mu(p)+t, \dots, x^n(p))$$

for which we set

$$e_\mu = \dot{\gamma}_\mu(0) = (0, \dots, 0, \underset{\text{for } \mu}{1}, 0, \dots, 0) \quad \text{and} \quad v = \sum_{\mu=1}^{n} v^\mu e_\mu$$

- this basis can be identified with $\frac{\partial}{\partial x^\mu}$ as

$$e_\mu(f)(p) = \frac{d f(\gamma_\mu(t))}{dt}\Big|_{t=0} = \sum_{\alpha=1}^{n} \frac{\partial f \circ \varphi^{-1}(x)}{\partial x^\alpha} \underbrace{\frac{d x^\alpha(\gamma_\mu(t))}{dt}\Big|_{t=0}}_{\delta^\alpha_\mu} = \frac{\partial}{\partial x^\mu} f(p)$$

- we thus obtained a bijective mapping (isomorphism) between the tangent space T_pM and a vector space of the differential operators of the first order D_pM, the dimensions of whose are both equal to n

Vector fields on the manifold M

- if we assign to each point $p \in M$ one of the vectors tangent to M we get a vector field V on M which is a mapping

$$V : M \to TM \quad \text{where } TM = \bigcup_{p \in M} T_pM = \text{tangent bundle on the manifold } M$$

- because tangent vectors can be interpreted as differential operators on functions on the manifold, the vector field will produce a new function Vf by acting on a function f on M

$$Vf : M \to \mathbb{R} \quad \text{or} \quad Vf(p) = V_p(f) \in \mathbb{R}$$

operator, new function, point, tangent vector at p i.e. $V_p \in T_pM$

- a smooth vectorfield is a vector field on M which gives smooth functions by acting on smooth functions that is $(Vf) \circ \varphi^{-1}$ is a smooth function of coordinates given by a local map φ on $U(p)$

- all vector fields that are smooth on M form an ∞-dimensional vector space

- in coordinates using a local map ψ on $U(p)$ we can write

$$V = \sum_{\mu=1}^{n} \underbrace{V^\mu(x)}_{\substack{\text{smooth functions}\\ \text{for a smooth vector field}}} \frac{\partial}{\partial x^\mu} \quad \left(\begin{array}{l}\text{as for ~~one~~ tangent vector but with}\\ V^\mu \text{ being a function of } x \text{ now}\end{array}\right)$$

and thus

$$(Vf)(x) = \sum_{\mu=1}^{n} V^\mu(x) \frac{\partial f(x)}{\partial x^\mu} \quad \left(\begin{array}{l}\text{this is a usual way how to write}\\ \text{the result without a local map } \psi\text{,}\\ \text{more accurately we should use } f\circ\psi^{-1}\end{array}\right)$$

- when changing the coordinates

$$x \to x' = x'(x)$$

the components $V^\mu(x)$ of the vector field are transformed as

$$V'^\mu(x'(x)) = J^\mu{}_\nu(x) V^\nu(x) \quad \text{where} \quad J^\mu{}_\nu(x) = \frac{\partial x'^\mu(x)}{\partial x^\nu}$$

(using the chain rule for

$$\frac{\partial f(x'(x))}{\partial x^\nu} = \sum_{\mu=1}^{n} \frac{\partial f}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\nu} \quad)$$

is the Jacobian of the change of the coordinates

Integral curves of a vector field V on M

- an integral curve of a smooth vector field V going through a point $p \in M$ is a curve

$$\gamma_V : (-\varepsilon, \varepsilon) \to M \quad \text{where } (-\varepsilon,\varepsilon) \text{ is an open interval in } \mathbb{R}$$

satisfying $\gamma_V(0) = p$ which has tangent vectors at each $t \in (-\varepsilon, \varepsilon)$ given by

$$V_{\gamma_V(t)} = \text{a tangent vector from the vector field } V \text{ at } \gamma_V(t)$$

that is we can write $\dot{\gamma}_V(t) = V_{\gamma_V(t)}$

or in coordinates using a local map ψ

$$\frac{d x^\mu(\gamma_V(t))}{dt} = V^\mu(\gamma_V(t)) = V^\mu(x(\psi(\gamma_V(t))))$$

- thus, the integral curve is given by a set of ordinary differential equations

$$\frac{dx^1(t)}{dt} = V^1(x^1(t), \dots, x^n(t))$$
$$\vdots$$
$$\frac{dx^n(t)}{dt} = V^n(x^1(t), \dots, x^n(t))$$

with the initial condition

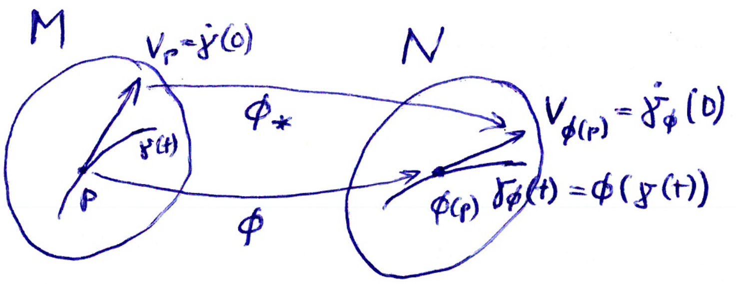
$$x^\mu(0) = x^\mu(p)$$

notes:

- we say that a vector field is complete
  if every integral curve of this field going through any point $p \in M$
  can be extended for all $t \in \mathbb{R}$
- for compact manifolds, every smooth vectorfield is complete
- integral curves of a complete vector field V cover the whole
  manifold M as they form a congruence of infinitely many curves
  that do not cross up to singular points where $V = 0$

Push-forward mapping of a vector field

- it is a mapping from the tangent space T_pM to a manifold M
  at a point p into the tangent space $T_{\phi(p)}N$ to a manifold N
  at a point $\phi(p)$ given by a certain mapping $\phi: M \to N$



- it is denoted as
$$\phi_* : T_pM \to T_{\phi(p)}N$$
and it is given by
$$\phi_*(\dot{\gamma}(0)) = \dot{\gamma}_\phi(0)$$

where $\dot{\gamma}_\phi(0)$ is the tangent vector from $T_{\phi(p)}N$ for the curve
$$\gamma_\phi(t) = \phi(\gamma(t)) \quad \text{with} \quad \gamma_\phi(0) = \phi(p)$$
where $\gamma(t)$ is a curve on M with $\gamma(0) = p$

- in coordinates, using a local map φ_x in the neighbourhood $U(p) \subset M$
  with coordinates x^μ and a local map φ_y in the neighbourhood
  $U(\phi(p)) \subset N$ with coordinates y^ν we get
  for ϕ simply $y^\nu = y^\nu(x)$ and for vector fields
$$V = V^\mu \frac{\partial}{\partial x^\mu} \quad \text{and} \quad \phi_* V = W^\nu \frac{\partial}{\partial y^\nu} \quad \left(\text{using Einstein notation without sums}\right)$$

- if we act on a function f on the manifold N by the vector field $\phi_* V$ we get

$$(\phi_* V) f = \frac{d f(\overbrace{\phi(\gamma(t))}^{\text{points in } N \text{ with coordinates } y^\nu})}{dt} = \frac{\partial f}{\partial y^\nu} \frac{\partial y^\nu}{\partial x^\mu} \underbrace{\frac{d x^\mu(\gamma(t))}{dt}}_{V^\mu} =$$

$$= V^\mu \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial f}{\partial y^\nu} = W^\nu \frac{\partial f}{\partial y^\nu}$$

or we have $W^\nu = J^\nu_\mu V^\mu$ where $J^\nu_\mu = \frac{\partial y^\nu}{\partial x^\mu}$ is the Jacobian of the mapping ϕ in the coordinates $y^\nu(x)$

Lie bracket of two fields V and W on M

- it is a vector field denoted by $[V,W]$ (a commutator) which is given at a point $p \in M$ by

$$[V,W]_p(f) = V_p(\underbrace{Wf}) - W_p(\underbrace{Vf})$$

functions on M

- if this is written in coordinates, the second derivatives subtract and we get again a tangent vector from T_pM and $[V,W]$ is thus a vector field on M
- components are given in local coordinates x^μ as

$$[V,W]^\mu(x) = V^\nu(x) \frac{\partial W^\mu}{\partial x^\nu} - W^\nu(x) \frac{\partial V^\mu(x)}{\partial x^\nu}$$

- it can be shown that if ϕ is a diffeomorphism (a smooth and bijective mapping) between M and N then the correspoing push-forward (mapping) trasfers the Lie bracket from M to N that is

$$\phi_*[V,W] = [\phi_* V, \phi_* W]$$

- the Lie bracket is antisymmetric $[V,W] = -[W,V]$ (it follows directly from the definition) and it satisfies Jacobi's identity

$$[V,[W,Z]] + [W,[Z,V]] + [Z,[V,W]] = 0$$

and thus the vector space of all smooth vector fields with the Lie bracket as a commutator forms an ∞-dimensional Lie algebra