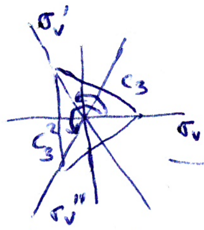
Tutorial: induced representations of C_{3v} from $H=\{E,\sigma_v\}$ (T11)

- the group C_{3v} is not a direct product of its subgroups, but only a semidirect product of C_3 and $C_s \sim H=\{E,\sigma_v\}$
- still, we can find IRs of C_{3v} by induction from H if we start with a trivial representation of C_{3v} and decompose obtained induced representations using this trivial repr.
- the subgroup $H=\{E,\sigma_v\}$ has two IRs

H	E	σ_v
A'	1	1
A''	1	-1

and C_{3v} can be decomposed into left cosets as



$$C_{3v} = \underset{g_1}{E}\cdot H + \underset{g_2}{C_3}\cdot H + \underset{g_3}{C_3^2}\cdot H$$

$$= \{E,\sigma_v\} + \{C_3,\sigma_v''\} + \{C_3^2,\sigma_v'\}$$

- to get characters of induced representations $A'\uparrow C_{3v}$ and $A''\uparrow C_{3v}$, it is not necessary to construct matrices for these repr., we can use directly

$$\chi^{M\uparrow G}(g) = \sum_{s=1}^{M} \delta_{ss}(g)\,\chi_H^M(g_s^{-1} g g_s)$$

where $M = \frac{\#G}{\#H} = 3$, $g_1=E$, $g_2=C_3$, $g_3=C_3^2$ and

$$\delta_{ss}(g) = \begin{cases} 1 & \text{for } g_s^{-1} g g_s \in H \\ 0 & \text{otherwise} \end{cases}$$

- now, if $g=E$, we set the dimension of the induced repr.
in our case $\dim \Gamma^{M\uparrow C_{3v}} = M = 3$
as both A' and A'' are one-dimensional

if $g=C_3$ (or $g=C_3^2$) we cannot get by conjugation $g_s^{-1} g g_s$ any element of $H=\{E,\sigma_v\}$, thus $\chi^{M\uparrow C_{3v}}(C_3)=0$

and if $g=\sigma_v$ (or $g=\sigma_v'$ or $g=\sigma_v''$) we have

$$E\sigma_v E = \sigma_v \in H$$
$$C_3^{-1}\sigma_v C_3 = \sigma_v'' \notin H$$
$$(C_3^2)^{-1}\sigma_v C_3^2 = \sigma_v' \notin H$$

and thus $\chi^{A'\uparrow C_{3v}}(\sigma_v) = \chi_H^{A'}(\sigma_v) = 1$

and $\chi^{A''\uparrow C_{3v}}(\sigma_v) = \chi_H^{A''}(\sigma_v) = -1$

- we can write

C_{3v}	E	$2C_3$	$3\sigma_v$	
$A'\uparrow C_{3v}$	3	0	1	result: $= A_1 \oplus E$
$A''\uparrow C_{3v}$	3	0	-1	$= A_2 \oplus E$
A_1	1	1	1	
E	2	-1	$0 = A'\uparrow C_{3v} \ominus A_1$	
A_2	1	1	$-1 = A''\uparrow C_{3v} \ominus E$	

we know (trivial) A_1

for this repr. we get $\alpha_{A_1}^{A'\uparrow C_{3v}} = \frac{1}{6}(3+0+3) = 1$

and $\alpha_{A_1}^{A''\uparrow C_{3v}} = \frac{1}{6}(3+0-3) = 0$

by Frobenius we can easily check that it is an irred. repr. of C_{3v}

and E denotes a two-dimensional repr. of any point group

- moreover, we get $\alpha_E^{A''\uparrow C_{3v}} = \frac{1}{6}(6+0+0) = 1$

and we can subtract E from $A''\uparrow C_{3v}$ as it is done on the last row of the table

- the result is in agreement with Frobenius reciprocity theorem:
  - we found earlier that

$$A_1 \downarrow H = A'$$
$$A_2 \downarrow H = A''$$
$$E \downarrow H = A' \oplus A''$$

and thus for all Γ_H^{ν} and $\Gamma_{G=C_{3v}}^{\mu}$ we have

$$\alpha_{\mu}^{\nu\uparrow C_{3v}} = \alpha_{\nu}^{\mu\downarrow H}$$