Tutorial: Double cover of $SO(3)$ by $SU(2)$ or more generally L^0 by $SL(2,\mathbb{C})$

T-LG1

- introduction:
  - $SO(3)$ = the group of all orthogonal matrices 3×3 satisfying
    $$A^T A = A A^T = 1\,,\quad \det A = 1 \quad \text{(preserving norms)}$$
    isomorphic to the group of all proper rotations in $\mathbb{R}^3$
  - $SU(2)$ = the group of all complex unitary matrices 2×2 satisfying
    $$A^\dagger A = A A^\dagger = 1\,,\quad \det A = 1$$
    isomorphic to the group of all linear transformations of the complex plane preserving distances (no reflections)
  - note: both these groups are 3-parametric, but we will see later that $SO(n)$ is $\frac{n(n-1)}{2}$-param. and $SU(n)$ is (n^2-1)-par.

    thus for $n>3$, $SO(n)$ cannot be covered by $SU(n-1)$
  - L = the group of all transformations Λ of Minkowski spacetime M^4 with the metric $\|x\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$

    satisfying $\|\Lambda x\|^2 = \|x\|^2$ for all $x\in M^4$
    - it is called Lorentz group
  - L^0 = the proper Lorentz group = the subgroup of L with Λ satisfying $\det\Lambda = 1$ and $\Lambda^0{}_0 > 0$
    - these are proper Lorentz transf. (no reflections or inversion)
  - $SL(2,\mathbb{C})$ = the group of all complex matrices 2×2 with $\det A = 1$

    isomorphic to all linear transf. of 2-dim. complex space without reflections
  - note: both L^0 and $SL(2,\mathbb{C})$ are 6-parametric groups
    - L^0 consists of 3 rotations and 3 boosts (and their compositions)
    - in $SL(2,\mathbb{C})$ we restrict an 8-par. general complex matrix 2×2 by two conditions $\det A = 1$ (real and imaginary part)

T-LG2

- <u>Goal of the tutorial</u>: Show by construction that there exists

a homomorphism $\phi: SL(2,\mathbb{C}) \xrightarrow{\text{on}} L^{\circ}$ (or into L)

and for subgroups $\phi: SU(2) \xrightarrow{\text{on}} SO(3)$ (or into $O(3)$)

$\qquad SU(2) \subset SL(2,\mathbb{C}), \quad SO(3) \subset L^{\circ}$

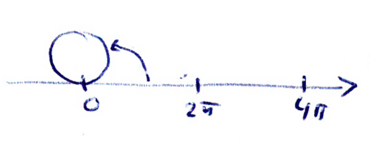
that is the image of ϕ is the whole L° and $SO(3)$

and the kernel $\operatorname{Ker}\phi = \{\mathbb{1}, -\mathbb{1}\}$ giving the double cover.

notes: 1) in general, if $\phi: G \xrightarrow{\text{on}} G'$ is a homomorphism between two (Lie) groups and $\#\operatorname{Ker}\phi = n$ then we call it n-to-1 homomorphism or n-fold cover of G' by G

- it can be shown that if G is simply connected (all closed curves can be contracted to a point) then it is the <u>universal covering group</u> that is unique (up to isomorphism) for any connected group G' and the corresponding Lie algebras are isomorphic

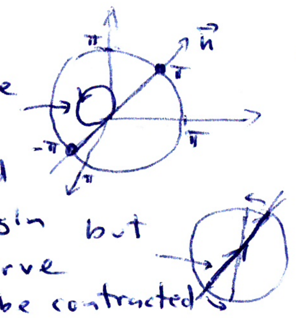
Example: the universal covering group of $SO(2)$ is $(\mathbb{R},+)$



$\Rightarrow$ ∞-fold cover

but there exist also 2-fold, 3-fold, etc covers of $SO(2)$ but these are not simply connected

2) $SO(3)$ is not simply connected:



SO(3) is a ball in 3D for which we identify the opposite points (rotation around $\vec{n}$ by π and $-\pi$ is the same)

but if we take a „double" curve

we can contract it

$\Rightarrow$ „double cover"

3) $SU(2)$ is simply connected and it is the universal covering group of $SO(3)$

we can write $A \in SU(2)$ as

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

with $|a|^2 + |b|^2 = 1$

by substitution $a = y_1 - iy_2$, $b = y_3 - iy_4$ we set condition

$$\det A = y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1$$

$\Rightarrow$ $SU(2)$ is a 3-dim. sphere S^3 in 4D space that is simply connected

Construction of the homomorphism $\phi : SL(2,\mathbb{C}) \overset{2:1}{\rightarrow} L^{\circ}$ (T-LG3)

– we do that in 6 steps, but the last one is given as a homework

1) existence of such a homomorphism

- to each 4-vector $x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$ from M^4 we assign a Hermitian $(X^+ = X)$ matrix 2×2

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = x_0 \mathbb{1} + \sum_{i=1}^{3} x_i \sigma_i$$

where σ_i are standard Pauli matrices

- notice that $\det X = x_0^2 - x_1^2 - x_2^2 - x_3^2 = \|x\|^2$

- we know that the Lorentz group L acts on M^4 as $\tilde{x} = \Lambda x$, similarly $SL(2,\mathbb{C})$ acts on the space $\mathcal{X}$ of all hermitian matrices X by $\tilde{X} = AXA^+$ where $X, \tilde{X} \in \mathcal{X}$ and $A \in SL(2,\mathbb{C})$

- it is really a group action as $\tilde{X}^+ = (AXA^+)^+ = AX^+A^+ = \tilde{X}$ and $\mathbb{1}X\mathbb{1}^+ = X$ and $A(BXB^+)A^+ = (AB)X(AB)^+$

- using the correspondence between $x \in M^4$ and $X \in \mathcal{X}$ we can define an action of $SL(2,\mathbb{C})$ on M^4, that is, to each $A \in SL(2,\mathbb{C})$ we assign a Lorentz transf. $\phi(A)$ such that $\tilde{x} = \phi(A)x$ where $\tilde{x}$ corresponds to $\tilde{X} = AXA^+$

to show that it is actually a Lorentz transf. we write

$$\|\phi(A)x\|^2 = \|\tilde{x}\|^2 = \det \tilde{X} = \det AXA^+ = \det X = \|x\|^2$$

which has to be satisfied by any Lorentz transf.

- thus $\phi : A \to \phi(A)$ is a homomorphism from $SL(A,\mathbb{C})$ to L

2) restriction to $SU(2)$ and $SO(3)$

- if $A \in SU(2) \subset SL(2,\mathbb{C})$ satisfying $AA^+ = \mathbb{1}$ then from $A\mathbb{1}A^+ = \mathbb{1}$ we get $\phi(A)e_0 = e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

and thus $\phi(A) = \begin{pmatrix} 1 & a_1\, a_2\, a_3 \\ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & R \end{pmatrix}$ but using $\|\phi(A)x\|^2 = \|x\|^2$ we get $a_1 = a_2 = a_3 = 0$ (compare terms with x_0x_1, x_0x_2 and x_0x_3)

$\Rightarrow$ $\phi(A)$ for $A \in SU(2)$ must be an orthogonal transf. of $\mathbb{R}^3$ from $O(3)$

3) SL(2,ℂ) is connected

- every matrix $A \in SL(2,\mathbb{C})$ can be transformed using a similarity transformation into an upper triangular matrix $\begin{pmatrix} & \\ 0 & \end{pmatrix}$ which is the Jordan form of A,

  as for any 2×2 matrix with $\det A = 1$ we have

$$A = B \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} B^{-1} \quad \text{for a certain } 2\times 2 \text{ matrix } B$$

- now, consider parametrization of A given by

$$A(t) = B \begin{pmatrix} a(t) & b(t) \\ 0 & \frac{1}{a(t)} \end{pmatrix} B^{-1} \quad \text{with fixed } B$$

  and $a(t)$ and $b(t)$ chosen in such a way that

  for $t=0$ we have $A(0) = \mathbb{1} = B \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B^{-1}$, i.e., $a(0) = 1$, $b(0) = 0$

  and for $t=1$ we have $A(1) = A = B \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} B^{-1}$, i.e., $a(1) = a$, $b(1) = b$

- as $a(t)$ and $b(t)$ can be chosen continuous

  (for example $a(t) = 1 - t + ta$, $b(t) = bt$)

  $A(t)$ will be also continuous and thus

  every $A \in SL(2,\mathbb{C})$ can be connected with $\mathbb{1}$

  or, in other words, $SL(2,\mathbb{C})$ is connected

- because only L° component of the whole Lorentz group L is connected, ϕ maps all $A \in SL(2,\mathbb{C})$ into L°, and also it maps

  $A \in SU(2)$ into $SO(3) \subset L^{\circ}$

  (because $O(3)$ is not connected, it has two components with $\det = +1$ or -1)

4) matrices from $SL(2,\mathbb{C})$ corresponding to rotations around x_2 and x_3 and boost in x_3

(T-LG5)

- consider

$$U_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in SL(2,\mathbb{C}) \text{, we get}$$

$$U_\theta X U_\theta^\dagger = \begin{pmatrix} x_0 + x_3 & e^{-2i\theta}(x_1 - i x_2) \\ e^{2i\theta}(x_1 + i x_2) & x_0 - x_3 \end{pmatrix} = \begin{pmatrix} \tilde{x}_0 + \tilde{x}_3 & \tilde{x}_1 - i\tilde{x}_2 \\ \tilde{x}_1 + i\tilde{x}_2 & \tilde{x}_0 - \tilde{x}_3 \end{pmatrix}$$

or

$$\left.\begin{aligned} \tilde{x}_1 &= x_1 \cos 2\theta - x_2 \sin 2\theta \\ \tilde{x}_2 &= x_1 \sin 2\theta + x_2 \cos 2\theta \end{aligned}\right\} \text{ rotation around } x_3 \text{ by } 2\theta$$

- we can observe that if $\theta \in \langle 0, 2\pi)$ we rotate (x_1, x_2) twice

as $U_0 = \mathbb{1}$, $U_\pi = -\mathbb{1}$ but $\phi(U_0) = \phi(\underset{-U_0}{U_\pi}) = \mathbb{1}_{4\times 4}$

- this is actually true in general because for $-A$ we get

$$(-A) X (-A)^\dagger = A X A^\dagger \text{ and thus } \phi(A) = \phi(-A)$$

- in a similar way, one can easily check that

$V_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in SL(2,\mathbb{C})$ corresponds to the rotation around x_3 by 2α

and $M_{e^\tau} = \begin{pmatrix} e^{\tau} & 0 \\ 0 & e^{-\tau} \end{pmatrix} \in SL(2,\mathbb{C})$ corresponds to the Lorentz transf. (or boost) in the direction x_3 with velocity $v = \tanh 2\tau$

as in this last case

$$\tilde{x}_0 = \tfrac{1}{2}\left(e^{2\tau} + e^{-2\tau}\right) x_0 + \tfrac{1}{2}\left(e^{2\tau} - e^{-2\tau}\right) x_3 = x_0 \cosh 2\tau + x_3 \sinh 2\tau$$

and $\tilde{x}_3 = x_0 \sinh 2\tau + x_3 \cosh 2\tau$

5) every Lorentz transformation can be written as $\Lambda = R_1 \Lambda_v^{x_3} R_2$

where R_1 and R_2 are some rotations, decomposable

as $R = R_\phi^{x_3} R_\theta^{x_2} R_\psi^{x_3}$ where (ϕ, θ, ψ) are standard Euler angles

and thus for each $\Lambda \in L^0$ we know that there exists $A \in SL(2,\mathbb{C})$ such that $\phi(A) = \Lambda$ (see section 3))

and ϕ is a homomorphism that is onto (surjective)

T-L66

- to show that $\Lambda = R_1 \Lambda_v^{x_3} R_2$ consider its action on e_0:

$$\Lambda e_0 = \begin{pmatrix} x_0 \\ \vec{x} \end{pmatrix}$$ and use a rotation S taking $\vec{x}$ into x_3-axis

some general $x \in M^4$

$$S\Lambda e_0 = \begin{pmatrix} x_0 \\ 0 \\ 0 \\ \|\vec{x}\| \end{pmatrix} \overset{\text{correspondence}}{\longleftrightarrow} \begin{pmatrix} x_0 + \|\vec{x}\| & 0 \\ 0 & x_0 - \|\vec{x}\| \end{pmatrix} = X$$

- now we apply $M_r = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \in SL(2,\mathbb{C})$ (boost in x_3 for $r = e^{\tau}$, see section 3))

to X and set $r^2 = x_0 - \|\vec{x}\|$ which is greater than 0

because $\|e_0\|^2 = 1 = \|\Lambda e_0\|^2 = x_0^2 - \|\vec{x}\|^2 = \underbrace{(x_0 + \|\vec{x}\|)}_{\geq 1 \text{ for } \Lambda \in L^0} \overbrace{(x_0 - \|\vec{x}\|)}^{>0}$

we get

$$M_r X M_r^{\dagger} = \begin{pmatrix} x_0^2 - \|\vec{x}\|^2 & 0 \\ 0 & \frac{x_0 - \|\vec{x}\|}{x_0 - \|\vec{x}\|} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and thus

$\phi(M_r) S \Lambda e_0 = e_0$ showing that $\phi(M_r) S \Lambda$ is

some rotation, let us denote it as R_2

- finally, we get for Λ expression

$$\Lambda = \underbrace{S^{-1}}_{R_1} \underbrace{[\phi(M_r)]^{-1}}_{\Lambda_v^{x_3}} R_2 = R_1 \Lambda_v^{x_3} R_2$$

6) it remains to show that $\mathrm{Ker}\,\phi = \{\mathbb{1}, -\mathbb{1}\}$,

that is that there is no other matrix $A \in SL(2,\mathbb{C})$

satisfying $\phi(A) = \mathbb{1}_{4\times 4}$

- see the first problem of the homework on Lie groups