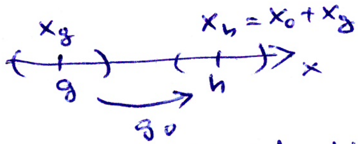
Tutorial: left-invariant vector fields

T-LG7

1) Let us consider the group $G = (\mathbb{R}, +)$ with the global coordinate x. We will show that vector field $V_1 = \frac{\partial}{\partial x}$ is left-invariant on this group, but $V_2 = x\frac{\partial}{\partial x}$ is not.

- left multiplication on G: $L_{g_0}: g \mapsto h = g_0 g$
  can be written as $x_h = x_0 + x_g$ in the coordinates



- for a vector field $V = V(x)\frac{\partial}{\partial x}$ to be left-invariant, it must satisfy $(L_g)_* V = V$ or in the coordinates

$$V(x_h(x_g)) = J(x_g)\, V(x_g)$$

where the Jacobian is $J(x_g) = \dfrac{\partial x_h(x_g)}{\partial x_g} = 1$

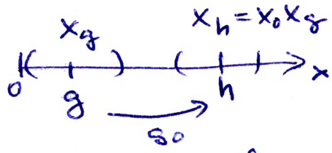
and thus we have the condition

$$V(x_0 + x_g) = V(x_g)$$

which is clearly satisfied for $V_1(x) = 1$, but not for $V_2(x) = x$

2) <u>Left-invariant vector field on $G = (\mathbb{R}^+, \cdot)$</u>

- now we have $L_{g_0}: g \mapsto h = g_0 g$
  in coordinates given as



and thus $J(x_g) = \dfrac{\partial x_h}{\partial x_g} = x_0$

- the condition for a vector field $V = V(x)\frac{\partial}{\partial x}$ to be left-invariant is

$$V(x_h(x_g)) = V(x_0 x_g) = J(x_g)V(x_g) = x_0 V(x_g) \quad (*)$$

- now, we can „guess" the solution to be $V(x) = cx$
  or we can choose $g = e$ $(x_g = 1)$ and $V_e = c\frac{\partial}{\partial x}$ ($c \in \mathbb{R}$ can be arbitrary)
  and get (we transfer a vector at the identity e to any point g_0)

$$V(x_0 x_g) \underset{x_g = 1}{=} V(x_0) \underset{\text{condition } (*)}{=} x_0 V(x_e) = c\, x_0$$

- thus, the general left-invariant vector field is

$$V = cx\frac{\partial}{\partial x}$$

3) Left-invariant vector fields of $GL(n,\mathbb{R})$

- $GL(n,\mathbb{R})$ is a group of all real $n\times n$ matrices A with $\det A \neq 0$ (invertible matrices) with the standard matrix multiplication as a group operation
- for $n>1$, $GL(n,\mathbb{R})$ is non-abelian and it has dimension n^2 (for $n=1$, it is $\sim (\mathbb{R}^{\neq},\cdot)$)
- it is isomorphic with $GL(V)$, the group of all automorphisms of the real n-dimensional vector space V (we just choose a suitable basis on V)
- it is not connected (2 components, $GL^+(n,R)$ with $\det A>0$ and the second with $\det A<0$) and not compact (the largest compact component is $O(n)$)
- left-invariant field - we use global parametrization $x^i{}_j = a_{ij}$ (elements of A) and we use Einstein summation rule and omit sums if there is the same index up and down
  - for left multiplication $L_{g_0}: g \mapsto g_0 g = h$ we have $(x_h)^i{}_j = (x_0)^i{}_k (x_g)^k{}_j$ (sum over k)
  - the Jacobian is
$$\frac{\partial (x_h)^i{}_j}{\partial (x_g)^l{}_m} = (x_0)^i{}_k \delta^k{}_l \delta^m{}_j = (x_0)^i{}_l \delta^m{}_j$$
giving the condition for a left-invariant vector field
$$V(x_h)^i{}_j = V(x_0 x_g)^i{}_j = (x_0)^i{}_l \delta^m{}_j V(x_g)^l{}_m$$
  - by the choice $g=e$, i.e. $x_g = \mathbb{1}_{n\times n}$, and $V(x_e)^l{}_m = C^l{}_m$

    a matrix of parameters for a general vector $V_e = C^l{}_m \frac{\partial}{\partial x^l{}_m}$

    we finally get
$$V(x_0)^i{}_j = (x_0)^i{}_l C^l{}_j$$
thus the left-invariant vector fields on $GL(n,\mathbb{R})$ are in general given by
$$V_C = x^i{}_l C^l{}_j \frac{\partial}{\partial x^i{}_j} = C^l{}_j V^j{}_l \quad \text{where } V^j{}_l = x^i{}_l \frac{\partial}{\partial x^i{}_j}$$
form a basis of n^2 independent left-inv. vector fields

- Lie algebra of $GL(n,\mathbb{R})$

- it is formed by left-invariant vector fields with the basis $V^j{}_e = x^i{}_e \frac{\partial}{\partial x^i{}_j} = X^i{}_e \partial^j{}_i$

and the commutator of the basis vectors

$$[V^i{}_j, V^k{}_l] f = x^r{}_j \partial^i{}_r (x^s{}_l \partial^k{}_s f) - x^s{}_l \partial^k{}_s (x^r{}_j \partial^i{}_r f) =$$

the second derivatives subtract

$$= x^r{}_j \delta^s{}_r \delta^i{}_l \partial^k{}_s f - x^s{}_l \delta^k{}_j \delta^r{}_s \partial^i{}_r f =$$

$$= (\delta^i{}_l V^k{}_j - \delta^k{}_j V^i{}_l) f$$

these deltas determine the structure constants of the Lie algebra „$c^k{}_{ij}$" but with 3 pairs of indices

- for a general commutator, we get

$$[V_C, V_D] = [c^i{}_j V^j{}_i, d^k{}_l V^l{}_k] = c^i{}_j d^k{}_l (\delta^j{}_k V^l{}_i - \delta^l{}_i V^j{}_k) =$$

$$= c^i{}_k d^k{}_l V^l{}_i - d^k{}_i c^i{}_j V^j{}_k = [C,D]^i{}_l V^l{}_i = V_{[C,D]}$$

and we see that the commutator of two left-invariant vector fields given by general matrices C and D corresponds the left-invariant vector field given by $[C,D]$

- as we also have (trivially) $V_{\alpha C + \beta D} = \alpha V_C + \beta V_D$ for all $\alpha, \beta \in \mathbb{R}$, the Lie algebra of real $n \times n$ matrices with the commutator $[C,D] = CD - DC$ is thus isomorphic with the Lie algebra of the left-invariant vector fields on $GL(n,\mathbb{R})$

- therefore, we consider them as the same Lie algebra of the Lie group $GL(n,\mathbb{R})$ denoted $gl(n,\mathbb{R})$

- in other words we have isomorphisms

$$\mathcal{L}(GL(n,\mathbb{R})) = gl(n,\mathbb{R}) \sim T_e(GL(n,\mathbb{R})) \sim End(\mathbb{R}^n)$$

left-invariant vector fields $[V,W]f = V(Wf) - W(Vf)$

the tangent space at identity $e \in GL(n,\mathbb{R})$ with the commutator $[V_e, W_e] = [V,W]_e$

matrices $n \times n$ $[C,D] = CD - DC$