Tutorial: Covering of SU(2) and SO(3) by exponential maps (T-LG10)

1) SO(3) ≡ a group of 3x3 orthogonal matrices satisfying

$$A^T A = A A^T = \mathbb{1} \quad \text{and} \quad \det A = 1$$

- it is 3-dimensional Lie group parametrized, for example, by 3 Euler angles, or the direction of rotational axis $\vec{n}$ and angle φ

- it is a subgroup of $GL(3,\mathbb{R})$, thus its LA of left-invariant v. fields is isomorphic to some LA of 3x3 matrices C that form a Lie subalgebra of all real 3x3 matrices, i.e. $so(3) \subset gl(3,\mathbb{R})$

- to find matrices C, we write a one-parametric subgroup as

$$A(t) = e^{tC} = 1 + tC + \frac{t^2}{2!}C^2 + \dots$$

and substitute into the condition $A^T A = \mathbb{1}$ getting

$$\left.\begin{array}{l}(\mathbb{1} + tC^T + \dots)(\mathbb{1} + tC + \dots) = \mathbb{1} \\ \mathbb{1} + t\underbrace{(C^T + C)}_{0} + O(t^2) = \mathbb{1}\end{array}\right\} \Rightarrow \boxed{C = -C^T}$$

- thus, LA $so(3)$ is formed by all antisymmetric real matrices 3x3 with zeroes on the diagonal, therefore the other condition $\det A = \det e^{tC} = e^{t\,\mathrm{Tr}\,C} = 1 \Rightarrow \mathrm{Tr}\,C = 0$ does not add another restriction on C

- as elements of C below the diagonal are given by those above the diagonal, we have only 3 independent elements of $C \Rightarrow$ $so(3)$ and also $SO(3)$ are 3-dimensional

(note: a similar result is true for $SO(n)$ and $so(n)$, the condition $C = -C^T$ is the same and we have $\dim so(n) = \frac{n(n-1)}{2}$)

- as a standard basis of $so(3)$, we can choose the following matrices

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad (L_i)_{kl} = -\varepsilon_{ikl}$$

with the commutators

$$[L_i, L_j] = \sum_{k=1}^{3} \varepsilon_{ijk} L_k, \quad \text{that is} \quad c_{ij}^k = \varepsilon_{ijk}$$

(structure constants)

- one-parametric subgroups are determined by a general element of the LA $so(3)$ that can be parametrized as $C = \varphi\, \vec{n}\cdot\vec{L}$ where $\vec{L} = (L_1, L_2, L_3)$ is the basis and $\vec{n} = (n_1, n_2, n_3)$ is a unit vector $\left(\sum_{i=1}^{3} n_i^2 = 1\right)$
- the angle φ can play the role of a general parameter t in e^{tC} therefore to simplify the expressions below we do not use t
- by exponential mapping we get

$$A(\vec{n},\varphi) = e^{\varphi\,\vec{n}\cdot\vec{L}} = 1\!\!1 + \varphi\,\vec{n}\cdot\vec{L} + \frac{\varphi^2}{2!}(\vec{n}\cdot\vec{L})^2 + \dots$$

and using $(L_k)_{ij} = -\varepsilon_{kij}$ and $\sum_{s=1}^{3} \varepsilon_{ijs}\varepsilon_{kes} = \delta_{ik}\delta_{je} - \delta_{ie}\delta_{jk}$

we obtain

$$(\vec{n}\cdot\vec{L})_{ij} = -\sum_{k=1}^{3} n_k \varepsilon_{kij}$$

$$(\vec{n}\cdot\vec{L})^2_{ij} = \sum_{k,e}\sum_{s=1}^{3}(-n_k\varepsilon_{kis})(-n_e\varepsilon_{esj}) = -\delta_{ij} + n_i n_j$$

$$(\vec{n}\cdot\vec{L})^3_{ij} = \sum_{s=1}^{3}(-\delta_{is} + n_i n_s)\left(-\sum_{k=1}^{3} n_k\varepsilon_{ksj}\right) = \sum_{k=1}^{3} n_k\varepsilon_{kij} = -(\vec{n}\cdot\vec{L})_{ij}$$

$$(\vec{n}\cdot\vec{L})^4_{ij} = \delta_{ij} - n_i n_j = -(\vec{n}\cdot\vec{L})^2_{ij}$$

$$(\vec{n}\cdot\vec{L})^5_{ij} = (\vec{n}\cdot\vec{L})_{ij} \qquad \text{etc.}$$

and finally

$$A(\vec{n},\varphi) = \delta_{ij}\overbrace{\left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \dots\right)}^{\cos\varphi} + n_i n_j \overbrace{\left(\frac{\varphi^2}{2!} - \frac{\varphi^4}{4!} + \dots\right)}^{1-\cos\varphi} - \sum_{k=1}^{3} n_k\varepsilon_{kij}\overbrace{\left(\varphi - \frac{\varphi^3}{3!} + \dots\right)}^{\sin\varphi}$$

- this is exactly the matrix representing rotation around the axis $\vec{n}$ by the angle φ (see vector representation) and thus, every rotation can be written as $e^{\varphi\,\vec{n}\cdot\vec{L}}$ and $SO(3)$ is covered by the exponential map

2) $SU(2)$ = a group of complex, unitary 2×2 matrices satisfying $A^\dagger A = 1\!\!1$ and $\det A = 1$

- the most general element of this group can be decomposed as

$$A = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} = 1\!\!1\, x_0 + i(\vec{x}\cdot\vec{\sigma}) = \begin{pmatrix} x_0 + i x_3 & x_2 + i x_1 \\ -x_2 + i x_1 & x_0 - i x_3 \end{pmatrix}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_2)$ are Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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- determinant is then expressed as

$$\det A = |a|^2 + |b|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$$

and we can see that it is a 3-parametric real Lie group

- SU(2) is a subgroup of GL(2, $\mathbb{C}$) and itis LA will be a Lie subalgebra of all complex 2x2 matrices
- using $A = e^{tC} = 1 + tC + \dots$ we set the condition on C (as for SO(3))

$$C^+ + C = 0 \quad \text{and} \quad \mathrm{Tr}\, C = 0$$

thus $su(2)$ is formed by all antihermitian 2x2 matrices $(C = -C^+)$ with zero trace (again, the same is true for SU(n))

- even though, the matrices are complex we consider SU(2) as a real Lie group and $su(2)$ as a real Lie algebra that is we consider only real linear combinations of the antihermitian matrices C
- we have

$$\overbrace{2 \cdot \underbrace{\frac{n(n-1)}{2}}_{\text{\# elements above the diagonal}} + n}^{M(n)} - \underset{\mathrm{Tr}\,C = 0}{1} = n^2 - 1 \quad \text{independent elements in } su(n)$$

real and imag. part

pure imaginary elements on the diagonal

- as a basis of $su(2)$ we can choose $L_i = -\frac{i}{2}\sigma_i \Rightarrow [L_i, L_j] = \varepsilon_{ijk} L_k$ because $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$, as for $so(3) \sim su(2)$
- a general element is

$$C = \varphi \vec{n}\cdot\vec{L} = -\frac{i}{2}\varphi \vec{n}\cdot\vec{\sigma} = \begin{pmatrix} -\frac{i}{2}\varphi n_1 & -\frac{1}{2}\varphi n_2 - \frac{i}{2}\varphi n_3 \\ \frac{1}{2}\varphi n_2 - \frac{i}{2}\varphi n_3 & \frac{i}{2}\varphi n_1 \end{pmatrix} \Rightarrow \mathrm{Tr}\, C = 0$$

- one-parameter subgroups (again φ plays the role of t)

$$e^C = e^{-\frac{i}{2}\varphi \vec{n}\cdot\vec{\sigma}} = \mathbb{1}\cos\frac{\varphi}{2} - i(\vec{n}\cdot\vec{\sigma})\sin\frac{\varphi}{2}$$

since $\sigma_i\sigma_j = i\sum_{k=1}^{3}\varepsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{1}$

and $\left(\sum_{i=1}^{3} n_i\sigma_i\right)\left(\sum_{j=1}^{3} n_j\sigma_j\right) = \sum_{i,j} n_i n_j (\underbrace{i\varepsilon_{ijk}\sigma_k}_{0} + \delta_{ij}\mathbb{1}) = \sum_i n_i n_i \mathbb{1} = \mathbb{1}$

it shoud be equal to $= \begin{pmatrix} x_0 + i x_3 & x_2 + i x_1 \\ -x_2 + i x_1 & x_0 - i x_3 \end{pmatrix}$

if we choose $x_0 = \cos\frac{\varphi}{2}$, $x_i = -n_i \sin\frac{\varphi}{2}$

we satisfy $\det e^C = 1 = x_0^2 + x_1^2 + x_2^2 + x_3^2$

and thus also SU(2) is covered by the exponential map