Special Topic Lectures UTF
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Part II Quantum Cosmology

Methodology:

1. We write down an action and a metric (FLRW)
2. Compute and solve field equations (to derive mode functions)
3. Quantize by
   (a) Postulating a mode expansion
   (b) with commutation relations
   (c) and a normalization condition

1 Mode decomposition

Again, we consider the FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\Omega^2\right), \tag{1}$$

which has flat geometry and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. We write this in its conformal form

$$\begin{aligned} ds^2 &= a^2(t)\left[-a^{-2}dt^2 + dr^2 + r^2 d\Omega^2\right], \\ &= a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 d\Omega^2\right], \end{aligned} \tag{2}$$

by introducing a new conformal time parameter η defined via $d\eta = a^{-1}dt$. Now, consider a massless scalar field Φ with action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \xi R\Phi^2\right), \tag{3}$$

i.e. a scalar field action which (at the moment) has a generic coupling to gravity through the coupling constant ξ. Varying the action w.r.t. Φ

$$\begin{aligned}\delta_\Phi S &= \frac{1}{2}\int d^4x\sqrt{-g}\left(2g^{\mu\nu}\partial_\mu\delta\Phi\partial_\nu\Phi + 2\xi R\Phi\delta\Phi\right)\\ &= \int d^4x\left(-\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) + \sqrt{-g}\xi R\Phi\right)\delta\Phi + t.d.,\end{aligned} \tag{4}$$

leads to the field equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right] - \xi R\Phi = 0, \quad \text{i.e.} \quad (\Box - \xi R)\Phi = 0. \tag{5}$$

This is the *massless Klein-Gordon equation* coupled to gravity. Let's unpack this wave equation for the given metric to yield

$$\begin{aligned}\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right] &= \frac{1}{\sqrt{-g}}\partial_\eta\left(\sqrt{-g}g^{\eta\eta}\partial_\eta\Phi\right) + \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ij}\partial_j\Phi\right)\\ &= -\frac{1}{a^4}\partial_\eta\left(a^2\partial_\eta\Phi\right) + \frac{1}{a^2\sqrt{-\delta}}\partial_i\left(\sqrt{-\delta}\delta^{ij}\partial_j\Phi\right)\\ &= \frac{1}{a^2}\left[-\partial_\eta^2\Phi - 2\frac{a'}{a}\partial_\eta\Phi + \Delta\Phi\right],\end{aligned} \tag{6}$$

where $\sqrt{-g} = a^4r^2\sin\theta$ and $\sqrt{-\delta} = r^2\sin\theta$ so $\sqrt{-g} = a^4\sqrt{-\delta}$

Definitions:

- The Laplace operator in spherical coordinates:

$$\begin{aligned}\Delta &= \frac{1}{\sqrt{-\delta}}\partial_i\left[\sqrt{-\delta}\delta^{ij}\partial_j\Phi\right]\\ &= \frac{1}{r^2\sin\theta}\partial_r\left[r^2\sin\theta\partial_r\Phi\right] + \frac{1}{r^2\sin\theta}\partial_\theta\left[\sin\theta\partial_j\Phi\right] + \frac{1}{r^2\sin^2\theta}\partial_\phi\partial_\phi\Phi\\ &= \frac{1}{r^2}\partial_r\left(r^2\partial_r\Phi\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_j\Phi\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\Phi\end{aligned} \tag{7}$$

- Curvature scalar (in FLRW conformal coordinates)

$$R = 6\frac{a''(\eta)}{a(\eta)^3} \tag{8}$$

Thus,

$$-\partial_\eta^2\Phi - 2\frac{a'}{a}\partial_\eta\Phi + \Delta\Phi + 6\xi\frac{a''}{a}\Phi = 0. \tag{9}$$

The single derivative term is problematic so we introduce an auxiliary field $\phi = a\Phi$ so that

$$\partial_\eta\Phi = a^{-1}\partial_\eta\phi - \frac{a'}{a^2}\phi, \qquad \text{and} \quad \partial_\eta^2\Phi = -\frac{\phi a''}{a^2} - \frac{2a'}{a^2}\partial_\eta\phi + \frac{2\phi a'^2}{a^3} + \frac{1}{a}\partial_\eta^2\phi \tag{10}$$

Putting this together leaves

$$-\partial_\eta^2\phi + \Delta\phi + (1-6\xi)\frac{a''}{a}\phi = 0 \tag{11}$$

Now, we see that the equation simplifies for the choice $\xi = 1/6$ which is precisely the value of the *conformal coupling* in four dimensions. This choice leaves the action conformally invariant. Thus, taking $\xi = 1/6$ leaves

$$(-\partial_\eta^2 + \Delta)\phi(\eta, x) = 0. \tag{12}$$

Now, we transform the field ϕ into *Fourier space* using

$$\phi(x) = \int \frac{d^3\vec{k}}{(2\pi)}\phi_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} \tag{13}$$

so that the field equation becomes

$$\begin{aligned}\int \frac{d^3\vec{k}}{(2\pi)}\left(-\partial_\eta^2\phi_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} + \phi_{\vec{k}}(\eta)\Delta e^{i\vec{k}\cdot\vec{x}}\right) &= \int \frac{d^3\vec{k}}{(2\pi)}\left(-\phi''_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} + \phi_{\vec{k}}(\eta)\partial^2 e^{ik^\mu x_\mu}\right)\\ &= \int \frac{d^3\vec{k}}{(2\pi)}\left(-\phi''_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} + \phi_{\vec{k}}(\eta)(ik^\mu)(ik_\mu)e^{i\vec{k}\cdot\vec{x}}\right)\\ &= \int \frac{d^3\vec{k}}{(2\pi)}e^{i\vec{k}\cdot\vec{x}}\left(-\phi''_{\vec{k}}(\eta) - k^2\phi_{\vec{k}}(\eta)\right). \end{aligned}\tag{14}$$

We can then write

$$\phi''_{\vec{k}}(\eta) + k^2\phi_{\vec{k}}(\eta) = 0 \quad \text{or} \quad \phi''_{\vec{k}}(\eta) + \omega_{\vec{k}}^2\phi_{\vec{k}}(\eta) = 0, \tag{15}$$

with $\omega_{\vec{k}} = |k|$ which has the general solution

$$\phi_{\vec{k}} = \frac{1}{\sqrt{2}}\left(a^-_{\vec{k}}v^*_k + a^+_{-\vec{k}}v_k\right). \tag{16}$$

Here, $a^\pm$ are complex constants of integration dependent only on the vector $\vec{k}$ satisfying $a^+_{\vec{k}} = (a^-_{\vec{k}})^*$, while the mode functions v_k are normalised such that

$$\phi_{\vec{k}}\partial_\eta\phi^*_{\vec{k}} - \phi^*_{\vec{k}}\partial_\eta\phi_{\vec{k}} = \frac{1}{2}\left[v'_k v^*_k - v_k v^{*\prime}_k\right] = i. \tag{17}$$

Note: In writing down the normalization condition, one must take note of the following identities: As ϕ is real, $\phi^* = \phi$; from the Fourier transform, we have $\phi^*_{\vec{k}} = \phi_{-\vec{k}}$; and as v_k depends on $|k|$, we have $v_k = v_{-k}$. This, along with the identity $a^+_{\vec{k}} = (a^-_{\vec{k}})^*$, gives $a^+_{\vec{k}}a^-_{\vec{k}} = a^-_{-\vec{k}}a^+_{-\vec{k}}$, which is required to write down the normalisation condition.

Note: Here, $d\Sigma$ is the volume element of a spacelike hypersurface Σ (assumed to be a Cauchy surface) and we define $d\Sigma^\mu \equiv n^\mu d\Sigma$ for a future-directed unit vector n^μ orthogonal to the hypersurface.

This normalisation condition is simply a result of defining the scalar product

$$(\phi_1, \phi_2) = -i \int_\Sigma d\Sigma^\mu \sqrt{-g_\Sigma}\, \phi_1 \overleftrightarrow{\partial_\mu} \phi_2^*, \tag{18}$$

and noting that there exists a complete set of mode solutions v_k satisfying

$$(v_k, v_{k'}) = \delta_{kk'}, \quad (v_k^*, v_{k'}^*) = -\delta_{kk'}, \quad (v_k, v_{k'}^*) = 0. \tag{19}$$

Substitution of (17) into (13) allows us to write down the mode decomposition

$$\hat{\phi}(x) = \frac{1}{\sqrt{2}} \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \left(\hat{a}_{\vec{k}}^- v_k^* e^{i\vec{k}\cdot\vec{x}} + \hat{a}_{\vec{k}}^+ v_k e^{-i\vec{k}\cdot\vec{x}} \right), \tag{20}$$

where the constants $a^\pm$ have been elevated to creation and annihilation operators satisfying the usual commutation relations

$$[\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}'}^+] = \delta_{\mathbf{k}\mathbf{k}'}, \qquad [\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}'}^-] = [\hat{a}_{\mathbf{k}}^+, \hat{a}_{\mathbf{k}'}^+] = 0, \tag{21}$$

following canonical quantization, Refs. [Birell, Mukhanov].

Quantization of the field $\hat{\phi}$ is achieved by postulating the (1) mode expansion along with the (2) commutation relations and (3) the normalisation condition, while the mode functions v_k are specific to the theory. As $v_k \equiv v_k(\eta)$ form a basis of solutions to field equations, we can find explicit expressions for these by solving the differential equation

$$v_k'' + k^2 v_k = 0 \qquad \text{to find} \qquad v_k = \frac{e^{i|k|\eta}}{\sqrt{|k|}}. \tag{22}$$

Exercise: From the differential equation $v_k''+k^2 v_k = 0$, it is clear that $v_k = n_k e^{i|k|\eta}$ for some unknown constant n_k is a solution. Take this solution and insert it into the inner product (18) to arrive at the appropriate form for the normalisation constant n_k.

Difficult Exercise: Try and solve the differential equation $(-\partial_\eta^2 + \Delta)\phi(\eta, x) = 0$ by the 'separation of variables method', i.e. separate the variables according to the Ansatz $\phi(\eta, x) = \varphi_k(\eta) u(r) v(\theta) w(\phi)$ and substitute into the differential equation.

Hint: consider the expression $\phi^{-1}(\eta, x)(-\partial_\eta^2 + \Delta)\phi(\eta, x) = 0$ and isolate functions of the same variable.

Further hint: begin by focusing on the angular coordinates and note that $\frac{w''}{w}$ must be equal to a constant. Choose $w''/w = -l^2$,solve and then follow the same methodology focusing on the $\theta-$dependent terms.

2 Wightman two point function

Next, we turn our attention to the two-point Wightman function

$$W(x; x') \equiv \langle 0|\Phi(x)\Phi(x')|0\rangle, \tag{23}$$

which will be instrumental in understanding the interaction of the detector with the quantum field. Substitution of the mode decomposition, while noting that we have defined the field $\hat{\Phi}$ in terms of the auxiliary field $\hat{\phi}$ like so $\hat{\Phi} \equiv a^{-1}\hat{\phi}$, gives

$$\begin{aligned}
W(x;x') &= \langle 0|\Phi(x)\Phi(x')|0\rangle \\
&= \langle 0|\frac{\hat{\phi}(x)\hat{\phi}(x')}{a(\eta)a(\eta')}|0\rangle \\
&= \langle 0|\frac{1}{a(\eta)a(\eta')}\frac{1}{\sqrt{2}}\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\left(\hat{a}^-_{\vec{k}}v^*_k e^{i\vec{k}\cdot\vec{x}} + \hat{a}^+_{\vec{k}}v_k e^{-i\vec{k}\cdot\vec{x}}\right) \\
&\times \frac{1}{\sqrt{2}}\int\frac{d^3\vec{k'}}{(2\pi)^{3/2}}\left(\hat{a}^-_{\vec{k'}}v^*_{k'}e^{i\vec{k'}\cdot\vec{x'}} + \hat{a}^+_{\vec{k'}}v_{k'}e^{-i\vec{k'}\cdot\vec{x'}}\right)|0\rangle \\
&= \langle 0|\frac{1}{a(\eta)a(\eta')}\frac{1}{2}\int\frac{d^3\vec{k}}{(2\pi)^{3/2}}\int\frac{d^3\vec{k'}}{(2\pi)^{3/2}}\left([\hat{a}^-_{\vec{k}},\hat{a}^+_{\vec{k'}}]v^*_k v_{k'}e^{i(\vec{k}\cdot\vec{x}-\vec{k'}\cdot\vec{x'})}\right)|0\rangle \\
&= \langle 0|\frac{1}{a(\eta)a(\eta')}\frac{1}{2(2\pi)^3}\int d^3\vec{k}\int d^3\vec{k'}\delta(k-k')v^*_k v_{k'}e^{i(\vec{k}\cdot\vec{x}-\vec{k'}\cdot\vec{x'})}|0\rangle \\
&= \int\frac{d^3\vec{k}}{2(2\pi)^3}\frac{v^*_k(\eta)v_k(\eta')}{a(\eta)a(\eta')}e^{i\vec{k}\cdot(\vec{x}-\vec{x'})} \qquad (24)
\end{aligned}$$

where $\langle 0|a^+ = 0$ and $a^-|0\rangle = 0$ and $\int_{-\infty}^{\infty} f(x)\delta(x-x')dx = f(x')$. Now, $v_k = \frac{e^{i|k|\eta}}{\sqrt{|k|}}$ so that

$$W(x,x') = \int\frac{d^3\vec{k}}{2(2\pi)^3}\frac{1}{|k|}\frac{e^{-i|k|(\eta-\eta'-i\epsilon)+i\vec{k}\cdot(\vec{x}-\vec{x'})}}{a(\eta)a(\eta')}, \qquad (25)$$

where we have inserted the small parameter $\epsilon > 0$ to ensure that the expression is a distribution.

How do we compute this integral? Consider

$$\mathcal{I} = \int\frac{d^3\vec{k}}{|k|}e^{-i|k|\eta+i\vec{k}\cdot\vec{x}}. \qquad (26)$$

Uusing dimensionsal regularisation, we find

$$\begin{aligned}
d^3\vec{k} &= \int d\Omega\int dk\ k^2 \\
&= \int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\int dk\ k^2 \\
&= 4\pi\int dk\ k^2. \qquad (27)
\end{aligned}$$

We also note that we can rewrite the expression $e^{i\vec{k}\cdot\vec{x}}$ without vectors like so

$$e^{i\vec{k}\cdot\vec{x}} = \int_{-1}^{1}du\ e^{i|k|ux}. \qquad (28)$$

Putting this together and we find

$$\begin{aligned}\mathcal{I} &= 4\pi \int_0^\infty dk|k|e^{-i|k|\eta}\frac{\sin(|k|x)}{|k|x} \\ &= 4\pi \int_0^\infty dke^{-ik\eta}\frac{\sin(kx)}{x} \\ &= \frac{4\pi}{-|\eta|^2 + |x|^2} \end{aligned} \tag{29}$$

which we have computed using contour integration or the theorem of residues. That is,

$$W_\epsilon(x; x') = \frac{1}{(2\pi)^2}\frac{1}{a(\eta)a(\eta')}\frac{1}{-|\eta - \eta' - i\epsilon|^2 + |\vec{x} - \vec{x}'|^2}. \tag{30}$$

We can express these in terms of proper time like so

$$W_\epsilon(\tau; s) = \frac{1}{(2\pi)^2}\frac{1}{a(\tau)a(\tau - s)}\frac{1}{-|\eta(\tau) - \eta(\tau - s) - i\epsilon|^2 + |\vec{x}(\tau) - \vec{x}(\tau - s)|^2}, \tag{31}$$

where now the coordinates have been elevated to trajectories, i.e. functions of proper time. This two point function allows us to encode the history and trajectory of a detector as it propagates through a classical spacetime. In the next lecture we will try and understand conceptually what this means in terms of quantum phenomena and particle detection.

A Transition rate in de Sitter space (additional notes for Lecture 3)

Consider the transition rate

$$\dot{\mathcal{F}}_\tau(\omega) = 2\int_0^{\Delta\tau} ds \left(\cos\omega s\, W_\epsilon(\tau, s) + \frac{1}{4\pi^2 s^2}\right) + \frac{1}{2\pi^2\Delta\tau} - \frac{\omega}{4\pi}. \tag{32}$$

We interpret this as the rate of particle detection for the detector per unit τ. The counter term is added to ensure regularity. $\Delta\tau = \tau - \tau_0$ is the detection time and $\omega = E - E_0$ is the energy gap. Let's consider a comoving detector, i.e. $t = \tau$ and $x = x'$. Express like so

$$\begin{aligned}\dot{\mathcal{F}}_\tau(\omega) &= 2\int_0^{\infty} ds \left(\cos\omega s\, W_\epsilon(\tau, s) + \frac{1}{4\pi^2 s^2}\right) \\ &\quad - 2\int_{\Delta\tau}^{\infty} ds \left(\cos\omega s\, W_\epsilon(\tau, s) + \frac{1}{4\pi^2 s^2}\right) + \frac{1}{2\pi^2\Delta\tau} - \frac{\omega}{4\pi} \\ &= 2\int_0^{\infty} ds \left(\cos\omega s\, W_\epsilon(\tau, s) + \frac{1}{4\pi^2 s^2}\right) - J_\tau + \underbrace{\left[-2\int_{\Delta\tau}^{\infty} \frac{ds}{4\pi^2 s^2} + \frac{1}{2\pi^2\Delta\tau}\right]}_{\text{cancel}} - \frac{\omega}{4\pi}\end{aligned} \tag{33}$$

for fluctuating tail

$$J_\tau \equiv 2\int_{\Delta\tau}^{\infty} ds \left(\cos\omega s\, W_\epsilon(\tau, s)\right). \tag{34}$$

Next

$$\begin{aligned}\dot{\mathcal{F}}_\tau(\omega) &= 2\int_0^{\infty} ds \left(\cos\omega s\, W_\epsilon(\tau, s) + \frac{\cos\omega s}{4\pi^2 s^2}\right) - J_\tau + \frac{1}{2\pi^2}\int_0^{\infty} ds \left(\frac{1 - \cos\omega s}{s^2}\right) - \frac{\omega}{4\pi} \\ &= 2\int_0^{\infty} ds \left(\cos\omega s\, W_\epsilon(\tau, s) + \frac{\cos\omega s}{4\pi^2 s^2}\right) - J_\tau + \frac{|\omega| - \omega}{4\pi}\end{aligned} \tag{35}$$

Recall that our conformal time trajectory was defined via

$$d\eta = a^{-2} dt \quad \Longrightarrow \quad \eta(\tau) = \int \frac{d\tau}{a(\tau)}. \tag{36}$$

For de Sitter space, we have $a(\tau) = e^{H\tau}$ where H is the (constant Hubble parameter) and

$$\eta(\tau) = -H^{-1} e^{-H\tau} \tag{37}$$

while

$$W_0(\tau;s) = -\frac{1}{(2\pi)^2}\frac{1}{a(\tau)a(\tau-s)}\frac{1}{|\eta(\tau)-\eta(\tau-s)|^2}$$
$$= -\frac{1}{(2\pi)^2}\frac{1}{e^{H\tau}e^{H(\tau-s)}}\frac{1}{|H^{-1}e^{-H\tau}(-1+e^{+Hs})|^2}$$
$$= -\frac{1}{(2\pi)^2}\frac{H^2}{e^{-Hs}(-1+e^{+Hs})^2}$$
$$= -\frac{1}{(2\pi)^2}\frac{H^2}{2+e^{-Hs}+e^{Hs}}$$
$$= -\frac{1}{(2\pi)^2}\frac{H^2}{2[-1+\cosh(Hs)]}$$
$$= -\frac{1}{(2\pi)^2}\frac{H^2}{4\sinh^2(Hs/2)} \tag{38}$$

as $2\sinh^2 x = \cosh(2x) - 1$. Thus

$$\dot{\mathcal{F}}_\tau(\omega) = \frac{1}{4\pi^2}\int_0^\infty ds \cos\omega s \left(-\frac{H^2}{4\sinh^2(Hs/2)} + \frac{1}{s^2}\right) - J_\tau + \frac{|\omega|-\omega}{4\pi} \tag{39}$$

Define $S \equiv Hs/2$ then

$$\dot{\mathcal{F}}_\tau(\omega) = \frac{H^2}{8\pi^2}\int_{-\infty}^\infty dS e^{-2i\omega S/H}\left(-\frac{1}{\sinh^2 S} + \frac{1}{S^2}\right) - J_\tau + \frac{|\omega|-\omega}{4\pi}. \tag{40}$$

Evaluating at $\tau \to \infty$ gives

$$\dot{\mathcal{F}}_\tau(\omega) = \frac{\omega}{2\pi}\left(\frac{1}{e^{\omega/T_{dS}}-1}\right), \quad T_{dS} = 2\pi, \tag{41}$$

where T_{dS} is the de Sitter temperature, i.e. the temperature a thermalised observer in de Sitter space will read on their thermometer.