FACULTY
OF MATHEMATICS
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Three lectures on variational principles

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Contents

Chapter 1: Lecture 1: Variational principles in GR

1.1 Matter in a curved spacetime

- Coupling to gravity. We have seen that the action for a particle in curved spacetime can be obtained by simply replacing the Minkowski metric by the general metric. This is an example of the minimal coupling principle. The recipe is to write the new laws in the tensorial form so that they reduce to the special relativistic laws in a local inertial frame. In practice this amounts to replacing the Minkowski metric with a general metric, and partial derivatives with covariant derivatives:

$$\eta_{\mu\nu} \to g_{\mu\nu}\,, \quad \partial_\mu \to \nabla_\mu\,. \tag{1.1}$$

  Of course, this procedure is vague and not unique (as one can always add for example the curvature terms) and the final say about whether the theory is right or wrong is decided (as always in physics) by experiment.

- Action principle. To write the action for the matter fields we have to use the invariant volume element, together with the principle of minimal coupling, to write

$$\boxed{S_m = \int d^dx\sqrt{-g}\mathcal{L}_m(\phi, \nabla\phi, g)\,,} \tag{1.2}$$

  where $\mathcal{L}_m$ is the scalar Lagrangian density and ϕ stands for various fields. The variation gives

$$\delta S_m = \int d^dx \frac{\delta S_m}{\delta\phi}\delta\phi = \int d^dx\sqrt{-g}\Big(\frac{\partial\mathcal{L}_m}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}_m}{\partial\nabla_\mu\phi}\delta\nabla_\mu\phi\Big)\,. \tag{1.3}$$

  As always, we now interchange the covariant derivative with δ, and, using the Stokes theorem, $\int_\Sigma \nabla_\mu V^\mu\sqrt{-g}d^dx = \int_{\partial\Sigma}\sqrt{\gamma}V^\mu d\Sigma_\mu$, integrate by parts as follows:

$$\int A^\mu(\nabla_\mu B)\sqrt{-g}d^dx = -\int B(\nabla_\mu A^\mu)\sqrt{-g}d^dx + \text{boundary term}\,, \tag{1.4}$$

  to have

$$\delta S_m = \int d^dx\sqrt{-g}\Big[\frac{\partial\mathcal{L}_m}{\partial\phi} - \nabla_\mu\Big(\frac{\partial\mathcal{L}_m}{\partial(\nabla_\mu\phi)}\Big)\Big]\delta\phi\,. \tag{1.5}$$

  Thus we have derived the generalized Euler–Lagrange field equations:

$$\boxed{\frac{\delta S_m}{\delta\phi} = 0} \quad \Leftrightarrow \quad \boxed{\frac{\partial\mathcal{L}_m}{\partial\phi} - \nabla_\mu\Big(\frac{\partial\mathcal{L}_m}{\partial(\nabla_\mu\phi)}\Big) = 0\,.} \tag{1.6}$$

- Energy–momentum tensor. We can also define the following object:

$$\boxed{\delta_g S_m = -\frac{1}{2}\int d^d x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}\,.} \tag{1.7}$$

We call the object $T_{\mu\nu}$ the (Rosenfeld's) energy momentum tensor. It is symmetric by construction.

For diffeomorphism invariant Lagrangian densities $\mathcal{L}_m$, such an energy momentum is conserved in the following sense: *If the equations of motion for matter are satisfied, the we have*

$$\boxed{\nabla_\mu T^{\mu\nu} = 0\,.} \tag{1.8}$$

(In fact, as we shall see, the conservation of energy–momentum is in many cases equivalent to the equations of motion for the matter.)

The argument for this goes as follows. First, let us remind that an infininitesimal diffeomorphism, generated by a vector field ξ,

$$x^\mu \to x^\mu - \xi^\mu\,, \tag{1.9}$$

induces the following variation of the fields and the components of the metric:

$$\delta\phi = \mathcal{L}_\xi \phi\,, \quad \delta g^{\mu\nu} = \mathcal{L}_\xi g^{\mu\nu} = 2\nabla^{(\mu}\xi^{\nu)}\,. \tag{1.10}$$

Second, consider a general variation of the matter action:

$$\delta S_m = \int \frac{\delta S_m}{\delta g^{\mu\nu}}\delta g^{\mu\nu} + \int \frac{\delta S_m}{\delta\phi}\delta\phi = -\frac{1}{2}\int d^d x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} + \int \frac{\delta S_m}{\delta\phi}\delta\phi\,, \tag{1.11}$$

using the above definition of $T_{\mu\nu}$. If this variation is induced by a diffeomorphism, and since the action is diffeo-invariant, we have

$$\begin{aligned} 0 = \delta_\xi S_m &= -\frac{1}{2}\int d^d x\sqrt{-g}T_{\mu\nu}\mathcal{L}_\xi g^{\mu\nu} + \int \frac{\delta S_m}{\delta\phi}\mathcal{L}_\xi\phi \\ &= -\int d^d x\sqrt{-g}\,\underbrace{T_{\mu\nu}\nabla^{(\mu}\xi^{\nu)}}_{T_{\mu\nu}\nabla^\mu\xi^\nu} + \int \frac{\delta S_m}{\delta\phi}\mathcal{L}_\xi\phi \\ &= \int d^d x\sqrt{-g}\nabla^\mu T_{\mu\nu}\xi^\nu + \int \frac{\delta S_m}{\delta\phi}\mathcal{L}_\xi\phi\,, \end{aligned} \tag{1.12}$$

where in the last step we have integrated by parts and thrown away the boundary term. Obviously, and since ξ^ν (in the bulk) is arbitrary, we have

$$\nabla_\mu T^{\mu\nu} = 0 \quad \Leftrightarrow \quad \frac{\delta S_m}{\delta\phi}\mathcal{L}_\xi\phi = 0\,, \tag{1.13}$$

up to total derivatives (boundary terms). In particular, if the equation of motion for the matter are satisfied, $\frac{\delta S_m}{\delta \phi} = 0$, the energy-momentum tensor is conserved. Typically, the converse is also true, provided the 'field is sufficiently non-degenerate', see below for explicit examples.

To calculate $T_{\mu\nu}$ explicitly, we can use the following two tricks:

$$\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,, \quad \delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}\,.} \tag{1.14}$$

- Example 1: Scalar field. Using the minimal coupling principle, the Lagrangian reads

$$\boxed{\mathcal{L} = -\frac{1}{2}g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - V(\phi)\,.} \tag{1.15}$$

Using (1.6), the corresponding field equation is

$$\nabla^\mu\nabla_\mu\phi - \frac{dV}{d\phi} = 0\,. \tag{1.16}$$

To calculate the energy momentum tensor, we simply vary the corresponding action:

$$\delta_g S = \int d^d\Big[\mathcal{L}\delta\sqrt{-g} + \sqrt{-g}\delta\mathcal{L}\Big] = -\frac{1}{2}\int d^dx\sqrt{-g}\,\underbrace{\Big[g_{\mu\nu}\mathcal{L} + \nabla_\mu\phi\nabla_\nu\phi\Big]}_{T_{\mu\nu}}\delta g^{\mu\nu} \tag{1.17}$$

In the flat space limit, this is the canonical energy momentum tensor for the scalar field derived from the Noether's procedure due spacetime translation invariance.

Let us next look at what imposing $\nabla_\mu T^{\mu\nu} = 0$ yields. We have

$$\nabla_\mu T^{\mu\nu} = \underbrace{g^{\mu\nu}\nabla_\mu\mathcal{L}}_{-\nabla^\nu\nabla^\mu\phi\nabla_\mu\phi - \frac{dV}{d\phi}\nabla^\nu\phi} + \nabla^2\phi\nabla^\nu\phi + \nabla_\mu\phi\nabla^\mu\nabla^\nu\phi \tag{1.18}$$

$$= \nabla_\mu\phi\Big(\underbrace{\nabla^\mu\nabla^\nu\phi - \nabla^\nu\nabla^\mu\phi}_{0}\Big) + \nabla^\nu\phi\Big(\nabla^2\phi - \frac{dV}{d\varphi}\Big) = 0\,. \tag{1.19}$$

So, provided $\nabla\phi \neq 0$ we recover the equations of motion (1.16).

- Example 2: Electromagnetic field. The Lagrangian reads

$$\boxed{\mathcal{L}[A_\mu, g^{\alpha\beta}] = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{16\pi}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\,, \quad F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]}\,.} \tag{1.20}$$

It gives rise to the Maxwell equations in curved spacetime:

$$\nabla_\mu F^{\mu\nu} = 0\,, \quad dF = 0\,, \tag{1.21}$$

where the latter is automatically satisfied from the definition of the field strength. Varying the action w.r.t. the metric, we have

$$\delta_g S = -\frac{1}{16\pi}\int d^d x\Big[F^2\delta\sqrt{-g} + 2\sqrt{-g}F_{\mu\delta}F_\nu{}^\delta\delta g^{\mu\nu}\Big]\,. \tag{1.22}$$

This immediately yields the (automatically symmetric and gauge invariant) energy–momentum tensor:

$$\boxed{T_{\mu\nu} = \frac{1}{4\pi}\Big(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2\Big)\,.} \tag{1.23}$$

It is this electromagnetic energy momentum tensor that couples to gravity (not the canonical one, which is not symmetric and not gauge invariant). Moreover, we find

$$4\pi\nabla_\mu T^\mu{}_\nu = \nabla_\mu F^{\mu\alpha}F_{\nu\alpha} + \underbrace{F^{\mu\alpha}\nabla_\mu F_{\nu\alpha}}_{\frac{1}{2}F^{\beta\alpha}(\nabla_\beta F_{\nu\alpha}+\nabla_\alpha F_{\beta\nu})} - \frac{1}{2}\nabla_\nu F_{\alpha\beta}F^{\alpha\beta} = \nabla_\mu F^{\mu\alpha}F_{\nu\alpha}\,, \tag{1.24}$$

where the last three terms vanished upon using the second set of Maxwell's equations.[1] Obviously, if $F_{\nu\alpha}$ has an inverse, then conservation of the energy momentum tensor implies the first set of Maxwell's equations.

1.2 Einstein–Hilbert action

- Action. Let us now think about how to construct the variational principle for the gravitational field itself. To get the 2nd-order equations of motion for $g_{\mu\nu}$, we want a scalar invariant which depends on the metric and its first derivatives, $I = I(g, \partial g)$. Unfortunately there is no such thing—why? So we have to give up and take an invariant that depends also on the second derivatives of the metric. The simplest one is the Ricci scalar. This leads to the following Einstein–Hilbert action:

$$\boxed{S_{\rm EH}[g] = \frac{1}{16\pi G}\int \sqrt{-g}R(g,\partial g,\partial^2 g)\,.} \tag{1.26}$$

- Variation. Varying this action, and using the dirty trick that $R = R_{\mu\nu}g^{\mu\nu}$, we get

$$\delta S_{\rm EH} = \frac{1}{16\pi G}\int \delta(\sqrt{-g}R_{\alpha\beta}g^{\alpha\beta}) = \frac{1}{16\pi G}\int (R\delta\sqrt{-g} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu})\,. \tag{1.27}$$

[1] We could have derived this directly using the result (1.13). Namely, using the Cartan's lemma $\mathcal{L}_\xi\omega = \xi\cdot(d\omega) + d(\xi\cdot\omega)$, valid for any p-form ω, we have

$$\frac{\delta S_m}{\delta A_\mu}\mathcal{L}_\xi A_\mu \propto \sqrt{-g}\nabla_\nu F^{\nu\mu}\mathcal{L}_\xi A_\mu = \sqrt{-g}\nabla_\nu F^{\nu\mu}(dA)_{\mu\alpha}\xi^\alpha + \sqrt{-g}\nabla_\nu F^{\nu\mu}\nabla_\mu(\xi\cdot A)\,. \tag{1.25}$$

However, by integrating the second term by parts (we are under an integral really), we turn it into $\nabla_\mu\nabla_\nu F^{\mu\nu}(\xi\cdot A) = 0$. Thus we arrive at the same conclusion as above.

The first two terms are easy, they give

$$R\delta\sqrt{-g}+\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}=\sqrt{-g}\Big(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\Big)\delta g^{\mu\nu}=\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}\,. \tag{1.28}$$

upon using the first identity (1.14). On the other hand, the last term, $g^{\mu\nu}\delta R_{\mu\nu}$, seems horrible. Fortunately it can be show that it only gives a boundary term:[2]

$$g^{\mu\nu}\delta R_{\mu\nu}=\nabla_\mu V^\mu\,,\quad V_\mu=\nabla^\beta(\delta g_{\mu\beta})-g^{\alpha\beta}\nabla_\mu(\delta g_{\alpha\beta})\,, \tag{1.38}$$

[2]If you want to know that bloody details, here they are, see also [1]. We have Palatini lemma:

$$\boxed{\delta R_{\mu\nu}=(\delta\Gamma^\lambda{}_{\mu\lambda})_{;\nu}-(\delta\Gamma^\lambda{}_{\mu\nu})_{;\lambda}}\,. \tag{1.29}$$

To prove this lemma, we shall use 'local inertial frame' equipped with Riemann normal coordinates, that is $\Gamma^\mu{}_{\nu\lambda}=0$ at a point, but $\partial\Gamma$ non-vanishing ($g_{\mu\nu,\lambda}=0$ but $g_{\mu\nu,\lambda\delta}$ is not) together with the fact that if the tensor identity is proved in one frame, it is valid in any frame. So,

$$\begin{aligned}\delta R_{\alpha\beta}&=\delta R^\gamma{}_{\alpha\gamma\beta}=\delta[\Gamma^\gamma{}_{\alpha\beta,\gamma}-\Gamma^\gamma{}_{\alpha\gamma,\beta}+\Gamma\Gamma-\Gamma\Gamma]\doteq\delta[\Gamma^\gamma{}_{\alpha\beta,\gamma}-\Gamma^\gamma{}_{\alpha\gamma,\beta}]\\&=\nabla_\gamma(\delta\Gamma)^\gamma{}_{\alpha\beta}-\nabla_\beta(\delta\Gamma)^\gamma{}_{\alpha\gamma}=\delta\Gamma^\gamma{}_{\alpha\beta;\gamma}-\delta\Gamma^\gamma{}_{\alpha\gamma;\beta}\,,\end{aligned} \tag{1.30}$$

where we have used the fact that in normal coordinates ∂ and ∇ are the same and that $\delta\Gamma$ is a tensor. We also have

$$\begin{aligned}\delta\Gamma^\mu{}_{\nu\lambda}&=\frac{1}{2}g^{\mu\sigma}(\delta g_{\sigma\nu,\lambda}+\delta g_{\sigma\lambda,\nu}-\delta g_{\nu\lambda,\sigma})+\frac{1}{2}\delta g^{\mu\sigma}(g_{\sigma\nu,\lambda}+g_{\sigma\lambda,\nu}-g_{\nu\lambda,\sigma})\\&=\frac{1}{2}g^{\mu\sigma}(\delta g_{\sigma\nu;\lambda}+\delta g_{\sigma\lambda;\nu}-\delta g_{\nu\lambda;\sigma})+\frac{1}{2}\delta g^{\mu\sigma}(g_{\sigma\nu;\lambda}+g_{\sigma\lambda;\nu}-g_{\nu\lambda;\sigma})\\&=\frac{1}{2}g^{\mu\sigma}(\delta g_{\sigma\nu;\lambda}+\delta g_{\sigma\lambda;\nu}-\delta g_{\nu\lambda;\sigma})\,,\end{aligned} \tag{1.31}$$

which also yields

$$\delta\Gamma^\alpha{}_{\alpha\beta}=\frac{1}{2}g^{\alpha\sigma}(\nabla_\beta\delta g_{\sigma\alpha}+\nabla_\alpha\delta g_{\sigma\beta}-\nabla_\sigma\delta g_{\alpha\beta})=\frac{1}{2}g^{\alpha\sigma}\nabla_\beta\delta g_{\alpha\sigma}\,. \tag{1.32}$$

Hence we found

$$\delta R_{\mu\nu}=\frac{1}{2}\big[\nabla_\mu\nabla^\lambda\delta g_{\lambda\nu}+\nabla_\nu\nabla^\lambda\delta g_{\lambda\mu}-\nabla^2\delta g_{\mu\nu}-\nabla_\mu\nabla_\nu(g^{\lambda\sigma}\delta g_{\lambda\sigma})\big]\,. \tag{1.33}$$

Everywhere above "δg" means variation of $g_{\mu\nu}$. We also need variation w.r.t. $g^{\mu\nu}$, which is given by

$$\delta g^{\mu\nu}=-g^{\sigma\mu}g^{\lambda\nu}\delta g_{\lambda\sigma}\,. \tag{1.34}$$

So we have

$$g^{\mu\nu}\delta R_{\mu\nu}=-\nabla_\mu\nabla_\nu\delta g^{\mu\nu}+\nabla^2\delta g=\nabla_\mu V^\mu\,,\quad V^\mu=-\nabla_\nu\delta g^{\mu\nu}+\nabla^\mu\delta g^{-1}\,, \tag{1.35}$$

where

$$\delta g^{-1}=g_{\mu\nu}\delta g^{\mu\nu}\,, \tag{1.36}$$

Note also a useful identity

$$\delta\Gamma^\alpha{}_{\alpha\beta}=-\frac{1}{2}\nabla_\beta\delta g^{-1}\,. \tag{1.37}$$

which does not contribute to the equations of motion. Thus we have the following Einstein equations in the absence of matter:

$$\boxed{G_{\mu\nu} = 0\,.} \tag{1.39}$$

Note that, by contracting the equation with $g^{\mu\nu}$, we find that in d-dimensions we have $(1 - \frac{1}{2}d)R = 0$ and thence for $d \neq 2$ we must also have $R = 0$ and the vacuum equations can be written as: $R_{\mu\nu} = 0$. Do you know what happens in 2d?

- Let us make 4 remarks.
  - Remark 1. Obviously, the Einstein-Hilbert action is diffeomorphism invariant. We can thus repeat the argument leading to the conservation of the energy-momentum tensor, to arrive at the conclusion that even off-shell (that is for any metric) we have the following Bianchi identity:

$$\boxed{\nabla_\mu G^{\mu\nu} = 0\,.} \tag{1.40}$$

    The same conclusion will be true for 'generalized Einstein tensor' of any diffeomorphism invariant (higher-curvature) theory of gravity.
  - Remark 2. The obtained equation is second order PDE for the metric, $G_{\mu\nu} = G_{\mu\nu}(g, \partial g, \partial^2 g)$. How is this possible? We started from the action, given by R, which already is of second order in derivatives. We should thus have received 4th order equations of motion. However, this is not the case and the reason is simple. It can be shown that the Lagrangian density can be split to a piece that depends only on the first derivatives, plus a piece that is a total derivative (none of which is a tensor):

$$\sqrt{-g}R(g, \partial g, \partial^2 g) = \sqrt{-g}\tilde{R}(g, \partial g) + \partial_\mu \hat{R}^\mu(g, \partial g)\,, \tag{1.41}$$

    where $\delta\hat{R}^\mu = \sqrt{-g}V^\mu$.[3] The latter term does not contribute to the equations of motion, provided we impose the corresponding boundary conditions. These, however, do not give rise to the standard Dirichlet problem. To achieve that, one needs to add the so called York–Gibbons–Hawking term, as we shall see in the next lecture.
  - Remark 3. In the above variation we used the so called second-order formalism: the action was treated as a function of the metric $g_{\mu\nu}$ and contained its first and second derivatives.

[3]In fact, one has

$$\tilde{R} = g^{\iota\kappa}(\Gamma^\lambda{}_{\mu\kappa}\Gamma^\mu{}_{\lambda\iota} - \Gamma^\lambda{}_{\iota\kappa}\Gamma^\mu{}_{\lambda\mu})\,, \quad \hat{R}^\mu = \sqrt{-g}(g^{\iota\kappa}\Gamma^\mu{}_{\iota\kappa} - g^{\iota\mu}\Gamma^\kappa{}_{\iota\kappa})\,. \tag{1.42}$$

Moreover, there is a 'holographic relation' between the bulk and surface part of the Lagrangian, with the surface terms obtainable from the bulk on by differentiation, [2].

Perhaps more useful is the first-order (Palatini) formalism where the action is treated as action for two fields: the metric $g_{\mu\nu}$ and the connection $\Gamma^\alpha{}_{\beta\gamma}$:

$$S_{\text{Palatini}}[g,\Gamma] = \frac{1}{16\pi G}\int \sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma)\,, \tag{1.43}$$

where we used the fact that $R_{\mu\nu}$ can be entirely written in terms of the connection and its derivatives, $R_{\mu\nu} = R_{\mu\nu}(\Gamma)$. Thus the variation w.r.t. the metric yields immediately the Einstein equations (1.39). It can then be shown that the variation w.r.t. the connection establishes that the connection is given by Christoffel symbols (this was an input in the second order formalism.)

Note that in this spirit, one can also write down the purely connection dependent action for gravity with cosmological constant Λ, known as the Eddington's action:

$$S_{\text{Eddington}}[\Gamma] = \frac{1}{8\pi G\Lambda}\int d^4x\sqrt{-\det R_{\mu\nu}(\Gamma)}\,. \tag{1.44}$$

If you want to, please show the equivalence with the Einstein-Hilbert action in the presence of Λ.

– Remark 4: Einstein equations. Recovering now the full Einstein equations is easy. We simply add the corresponding matter Lagrangian density:

$$S = S_{\text{EH}}[g] + \int d^4x\sqrt{-g}\mathcal{L}_m\,. \tag{1.45}$$

The variation w.r.t. the metric and throwing away the boundary terms then yields

$$\delta S = \int d^4x\sqrt{-g}\Big(\frac{1}{16\pi G}G_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu}\Big) \tag{1.46}$$

Thus we recover the famous 1915 Einstein's field equations:[4]

$$\boxed{G^{\mu\nu} = 8\pi G T^{\mu\nu}\,.} \tag{1.47}$$

1.3 Higher-curvature gravities

- Other curvature invariants. One might think that the Ricci scalar we chose is simply one possibility for the action, but that we have other choices, such as:

$$R^2\,,\quad R_{\mu\nu}R^{\mu\nu}\,,\quad R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}\,,\quad R^3\,,\quad \nabla_\mu R\nabla^\mu R\,,\dots\,. \tag{1.48}$$

[4]Poor Einstein was scooped by Hilbert by a couple of months, after giving a seminar to him about what he is trying to do. Well, perhaps an appropriate punishment for leaving Prague before finishing his theory of gravitation :).

However, one can show that this is not the case, as we have the following [3]:

Lovelock theorem. *In four dimensions, the Einstein–Hilbert action is the only local action (apart from the cosmological constant and topological terms) that leads to the second order differential equations for the metric.*

In other words, in four dimensions Einstein's theory is the unique theory we can obtain from the action principle that yields 2nd-order EOM for the metric. In higher dimensions this is no longer true – we have a possibility of the so called Lovelock gravities. The corresponding Lagrangian is given in terms of the Euler densities, a certain combination of curvature invariants constructed from the powers of the Riemann tensor. They lead to the second order equations of motion for the metric, naturally generalizing the Einstein equations to higher dimensions.

- Gauss–Bonnet gravity. To give and example, let us consider the so called 2nd-order Lovelock gravity, also know as the Gauss–Bonnet gravity. The corresponding Lagrangian reads:

$$S = -\frac{1}{16\pi G_N}\int d^dx\sqrt{-g}\,(R+\alpha\mathcal{G})\,, \quad \mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\,, \tag{1.49}$$

where α is a coupling constant with the dimension $[\alpha] = L^2$. This extension of General Relativity naturally appears in the low energy effective action of heterotic string theory. It yields the following equations of motion:

$$G_{\alpha\beta} + \alpha H_{\alpha\beta} = 0\,, \tag{1.50}$$

where the Gauss–Bonnet modification amounts to

$$H_{\alpha\beta} = -\frac{1}{2}g_{\alpha\beta}\mathcal{G} + 2RR_{\alpha\beta} - 4R_{\alpha\gamma}R_{\beta}{}^{\gamma} + 4R_{\gamma\alpha\beta\delta}R^{\gamma\delta} + 2R_{\alpha}{}^{\gamma\delta\kappa}R_{\beta\gamma\delta\kappa}\,. \tag{1.51}$$

As argued above, it has to satisfy $\nabla_\mu H^{\mu\nu} = 0$. Note that, despite the appearance of higher derivative curvature terms in the action, the equations of motion for the metric remain of the second order. Surprisingly, the modification (1.51) is non-trivial only in $d \geq 5$ dimensions; $\mathcal{G}$ is topological (a total derivative) in $d = 4$, and vanishes for $d < 4$.

- Quite recently, people have been thinking about taking a $d \to 4$ limit of the above theory. This leads to the following scalar-tensor theory with a peculiar kinetic term [4, 5, 6]:

$$S = -\frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\Big[R + \alpha\Big(\phi\mathcal{G} + 4G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 4(\partial\phi)^2\Box\phi + 2(\nabla\phi)^4\Big)\Big]\,, \tag{1.52}$$

which is a special case of Horndeski theory [7] (the most general scalar-tensor theory whose equations of motion remain second-order).

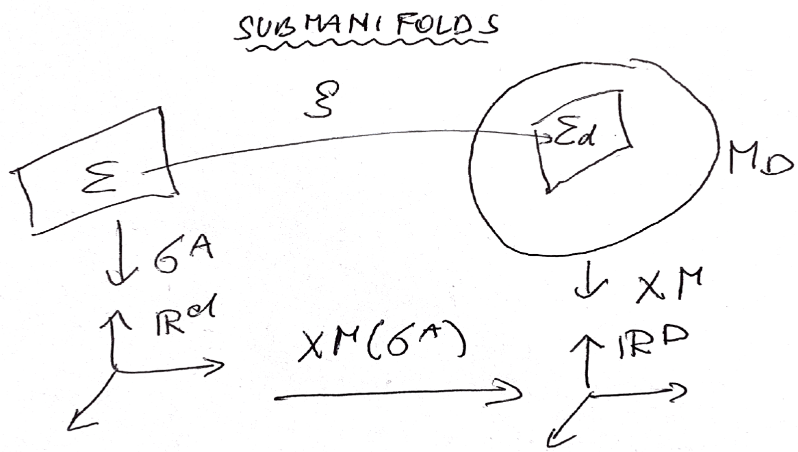
Chapter 2: Lecture 2: Submanifolds & York–Gibbons–Hawking term

2.1 Gauss–Codazzi formalism

- Motivation. i) To describe extended low-dimensional objects such as domain walls, cosmic strings, and branes. ii) To make sense of the variational principle for gravity.

  A boundary is a hypersurface in spacetime, that is, a surface with 1 dim less than M. Lower-dimensional surfaces in spacetime can be submanifolds.

- A submanifold $\Sigma \in M$ is a subset of M that is also a manifold in its own right, see picture:



  Here and in what follows we use the following notation: we assume

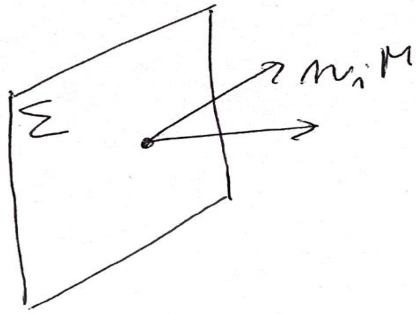
$$\dim(M) = D\,, \quad \dim(\Sigma) = d = D - n\,, \tag{2.1}$$

  and denote indices on Σ by $A, B, \ldots$, and indices on M by $\mu, \nu, \ldots$; coordinates on Σ are σ^A and coordinates on M are x^μ.

  The co-dimension of Σ is $n = D - d$, that is, there are n linearly independent normal directions to Σ, that is vectors n_i in T_pM $(p \in \Sigma, M)$ such that

$$n_i(\sigma^A) = 0 \quad i = 1, \ldots, n\,, \tag{2.2}$$

for all coordinate functions σ^A on Σ:



We will take

$$n_i \cdot n_j = \pm\delta_{ij}\,, \tag{2.3}$$

where $-$ is for timelike n and $+$ for spacelike n.

We now have two ways to describe things. i) spacetime point of view (in terms of spacetime objects) and ii) submanifold point of view (in terms of objects on a submanifold). Let us start with the first:

- Spacetime point of view. In this description we define the first fundamental form, or induced metric of Σ as

$$\boxed{h_{\mu\nu} = g_{\mu\nu} + \sum_{j=1}^{n} \epsilon_j n_{j\mu} n_{j\nu}\,,} \tag{2.4}$$

where $\epsilon_j = +$ for timelike and $-$ for spacelike n_j. $h_{\mu\nu}$ is the metric Σ inherits from M (but lies in $T^*M \otimes T^*M$) At the same time

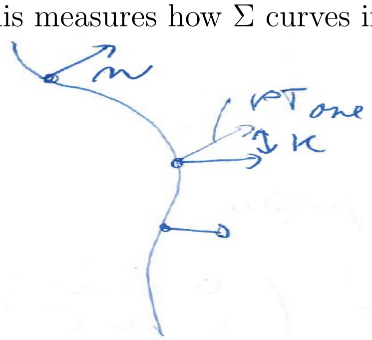
$$h^{\mu}{}_{\nu} \tag{2.5}$$

is a projector onto Σ: obeys $h^{\mu}{}_{\nu} h^{\nu}{}_{\kappa} = h^{\mu}{}_{\kappa}$ and $h^{\mu}{}_{\nu} n_i^{\nu} = 0$. It follows that $h_{\mu\nu}$ is degenerate from the spacetime point of view.

We also define the second fundamental form or extrinsic curvature of Σ as

$$\boxed{K_{i\mu\nu} = h_{(\mu}{}^{\sigma} h_{\nu)}{}^{\lambda} \nabla_\sigma n_{i\lambda}\,.} \tag{2.6}$$

This measures how Σ curves in M.



In what follows we focus on a co-dimension 1 hypersurface. We also define the corresponding extrinsic curvature scalar as

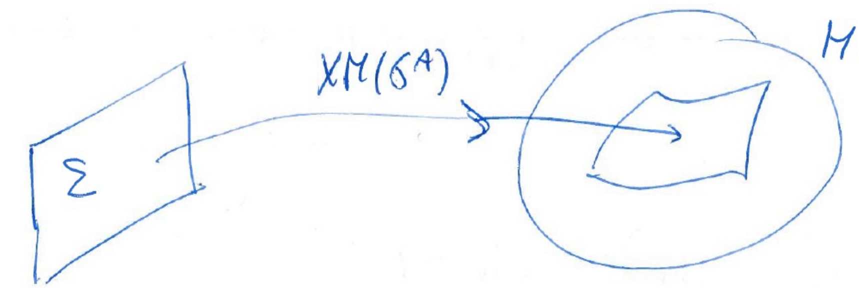
$$\boxed{K = g^{\mu\nu} K_{\mu\nu} = \nabla_\mu n^\mu\,.} \tag{2.7}$$

Indeed, we have

$$\begin{aligned} K &= g^{\alpha\beta} K_{\alpha\beta} = g^{\alpha\beta} h^\gamma{}_\alpha h^\delta{}_\beta \nabla_\gamma n_\delta = h^{\gamma\beta}(\delta^\delta_\beta + \epsilon n^\delta n_\beta)\nabla_\gamma n_\delta \\ &= h^{\gamma\beta}\nabla_\gamma n_\beta = (g^{\gamma\beta} + \epsilon n^\gamma n^\beta)\nabla_\gamma n_\beta = \nabla_\beta n^\beta + \epsilon n^\beta \nabla_n n_\beta = \nabla_\beta n^\beta\,. \end{aligned} \tag{2.8}$$

- Manifold point of view. Since Σ is a manifold in its own right, we can also consider quantities intrinsic to Σ. To this purpose, can think of Σ as a map $\Sigma \to M$, given by:

$$\boxed{\Sigma : x^\mu(\sigma^A)\,.} \tag{2.9}$$



The corresponding pull-back allows one to define the following projection to Σ:

$$\boxed{e^\mu{}_A = \frac{\partial x^\mu}{\partial \sigma^A}: \quad T^*_p(M) \to T^*_p(\Sigma)\,.} \tag{2.10}$$

For example,

$$\omega_\mu \to e^\mu{}_A \omega_\mu = \omega_A\,. \tag{2.11}$$

In particular, we define the intrinsic (pull-back) metric (element of $T^*\Sigma \otimes T^*\Sigma$):

$$\boxed{h_{AB} = e^\mu{}_A e^\nu{}_B g_{\mu\nu}\,.} \tag{2.12}$$

One can verify the following 'completeness relation':

$$g^{\alpha\beta} = h^{AB} e^\alpha_A e^\beta_B - \epsilon n^\alpha n^\beta\,. \tag{2.13}$$

The simplest example is the intrinsic metric for a particle

$$h_{\tau\tau} = \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} = \left(\frac{ds}{d\tau}\right)^2, \tag{2.14}$$

where $x^\mu(\tau)$ is the particle's worldline.

Note also that we can think of $e_A = e^\mu{}_A \partial_\mu$ as tangent vectors to Σ:

$$e_A \cdot n_i = n_{i\mu} \frac{\partial x^\mu}{\partial \sigma^A} = 0\,, \quad i = 1, \ldots, n\,. \tag{2.15}$$

Can also define the corresponding extrinsic curvature (from the intrinsic point of view)

$$\boxed{K_{AB} = e^\mu{}_A e^\nu{}_B \nabla_\mu n_\nu = -n_\mu D_A e^\mu{}_B\,,} \tag{2.16}$$

where D is the intrinsic covariant derivative (again Levi-Civita) inherited from the covariant derivative ∇:

$$\boxed{D_\mu V_\nu = h_\mu{}^\lambda h_\nu{}^\sigma \nabla_\lambda V_\sigma\,,} \tag{2.17}$$

for any V tangent to Σ ($V \cdot n_i = 0$). We also have

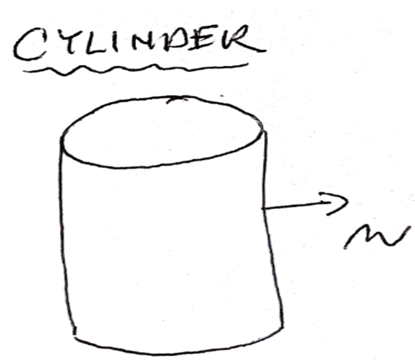
$$K = h^{AB} K_{AB} \tag{2.18}$$

That this is the same as before can be seen as follows:

$$K = h^{AB} e^\alpha_A e^\beta_B \nabla_\alpha n_\beta = (g^{\alpha\beta} + \epsilon n^\alpha n^\beta) \nabla_\alpha n_\beta = \nabla_\alpha n^\alpha + \epsilon n^\beta \nabla_n n_\beta = \nabla_\alpha n^\alpha = K\,. \tag{2.19}$$

- For example: consider a cylinder $\{x^2 + y^2 = a^2\} \subset \mathbb{R}^3$, using the Cartesian, $x^\alpha = (x, y, z)$, and cylindrical, $x^\alpha = (r, \theta, z)$, coordinates:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + dz^2\,. \tag{2.20}$$



We have $n_\mu \propto \partial_\mu \Sigma$, and upon normalizing, we find

$$n = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \tag{2.21}$$

with the first expression valid in Cartesian and the latter in cylindrical coordinates. We then have

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.22}$$

The extrinsic curvature in polar coordinates is then

$$K_{\mu\nu} = -\Gamma^r_{\mu\nu} = r\delta^\theta_\mu\delta^\theta_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.23}$$

and $K = \frac{1}{a}$. Intrinsic picture involves coordinates $\sigma^A = (\theta, z)$. This yields

$$\gamma_{AB} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_{AB} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.24}$$

- Curvature of Σ is related to curvature of M and extrinsic curvature through the Gauss equations:

$$\boxed{{}^{(d)}R^\alpha{}_{\beta\gamma\delta} = {}^{(D)}R^{\alpha'}{}_{\beta'\gamma'\delta'}h^\alpha{}_{\alpha'}h^{\beta'}{}_\beta h^{\gamma'}{}_\gamma h^{\delta'}{}_\delta \mp \left(K^\alpha{}_\gamma K_{\beta\delta} - K^\alpha{}_\delta K_{\beta\gamma}\right),} \tag{2.25}$$

where the minus sign applies to spacelike normal.

Cylinder example: because ${}^{(3)}R^\alpha{}_{\beta\gamma\delta} = 0$ (we are in $\mathbb{R}^3$) and $K_{\alpha\beta}$ has only one component we have $K^\alpha{}_\gamma K_{\beta\delta} - K^\alpha{}_\delta K_{\beta\gamma} = 0$ and therefore intrinsic Riemann, ${}^{(2)}R^\alpha{}_{\beta\gamma\delta} = 0$.

2.2 York–Gibbons–Hawking boundary term

- In the previous section we have shown that

$$\delta S_{\text{EH}} = -\frac{1}{16\pi G}\int_M d^4x\sqrt{-g}\; G_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{16\pi G}\underbrace{\int_M d^4x\sqrt{-g}\nabla_\mu V^\mu}_{\int_{\partial M} d^3x\sqrt{h}V^\mu n_\mu}, \tag{2.26}$$

where

$$V = V^\mu n_\mu = \nabla_n\,\delta g^{-1} - n_\alpha\nabla_\beta\,\delta g^{\alpha\beta}, \quad \delta g^{-1} = g_{\alpha\beta}\delta g^{\alpha\beta}. \tag{2.27}$$

The second term does not yield the standard Dirichlet boundary conditions. In what follows we would like to eliminate it, by adding a proper boundary term to the Einstein–Hilbert action, to get the Dirichlet variational principle for gravity.

- To this purpose we use the Gauss–Codazzi formalism introduced above, thinking of ∂M as Σ. That is, we consider co-dimension one Σ – a "wall". To simplify the calculations, we take ∂M to be a timelike boundary, that is n^μ spacelike, and (w.l.o.g.) extend n^μ geodesically into the bulk. Thus we have

$$n^2 = 1 \,, \quad h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \,, \quad \nabla_n n^\mu = 0 \,. \tag{2.28}$$

Lemma: We can show that

$$\delta\Gamma^\alpha{}_{\alpha\beta} = -\frac{1}{2}\nabla_\beta \delta g^{-1} \,, \quad \delta n^\mu = \frac{1}{2} n_\nu \delta g^{\mu\nu} \,. \tag{2.29}$$

The first one was shown in a Footnote 2 of the previous lecture. The second one is proved here.[1]

- Consider next

$$\begin{aligned} \delta K &= \delta(\nabla_\mu n^\mu) = \delta(\partial_\alpha n^\alpha + \Gamma^\alpha{}_{\alpha\beta} n^\beta) = \nabla_\alpha \delta n^\alpha + \underbrace{\delta\Gamma^\alpha{}_{\alpha\beta}}_{-\frac{1}{2}\nabla_\beta \delta g^{-1}} n^\beta \\ &= \nabla_\alpha \delta n^\alpha - \frac{1}{2}\nabla_n \delta g^{-1} = \nabla_\alpha (\frac{1}{2} n_\beta \delta g^{\alpha\beta}) - \frac{1}{2}\nabla_n \delta g^{-1} \\ &= \frac{1}{2}\underbrace{\left[n_\beta \nabla_\alpha \delta g^{\alpha\beta} - \nabla_n \delta g^{-1}\right]}_{-V} + \frac{1}{2}\nabla_\alpha n_\beta \delta g^{\alpha\beta} \,. \end{aligned} \tag{2.30}$$

However, we have (using geodesicity of n)

$$K_{\alpha\beta} = h^\gamma{}_\alpha h^\delta{}_\beta \nabla_\gamma n_\delta = \nabla_\alpha n_\beta \,, \quad \delta g^{\alpha\beta} \nabla_\alpha n_\beta = \delta h^{\alpha\beta} K_{\alpha\beta} \,, \tag{2.31}$$

and so

$$V = -2\delta K + K_{\alpha\beta} \delta h^{\alpha\beta} \,. \tag{2.32}$$

- The boundary term in (2.26) is thus

$$\begin{aligned} \delta S_{\partial M} &= \frac{1}{16\pi G}\int d^3x \sqrt{h}\left[2\delta K - K_{\mu\nu}\delta h^{\mu\nu}\right] \\ &= \frac{1}{8\pi G}\int d^3x \delta(\sqrt{h}K) - \frac{1}{16\pi G}\int d^3x \sqrt{h}(K_{\mu\nu} - K h_{\mu\nu})\delta h^{\mu\nu} \,. \end{aligned} \tag{2.33}$$

The first term is a (minus) variation of the famous York–Gibbons–Hawking boundary term:

$$\boxed{S_{\rm GH} = -\frac{1}{8\pi G}\int d^3x \sqrt{h} K \,,} \tag{2.34}$$

whose variation cancels the unwanted boundary term in (2.26).

[1] Under construction – or up to you :)

- To summarize, we should consider the following gravitational action:

$$\boxed{S_g = S_{\mathrm{EH}} + S_{\mathrm{YGH}} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R - \frac{1}{8\pi G}\int d^3x\sqrt{h}K\,.} \tag{2.35}$$

The first terms gives the E-L equations plus the boundary derivative term. The latter is cancelled by the second term. However, we still get the "$K_{\mu\nu} - Kh_{\mu\nu}$" term:

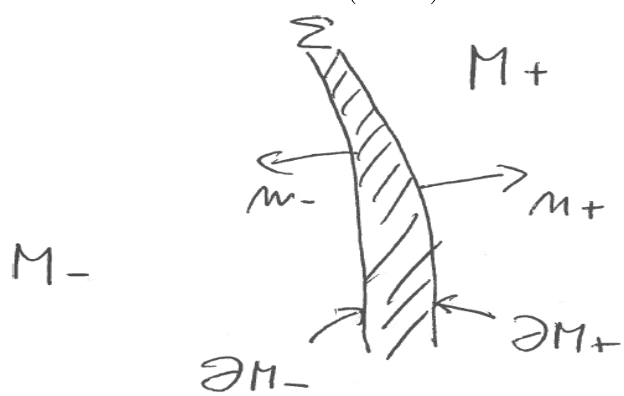
$$\delta S_g = -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}G_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{16\pi G}\int d^3x\sqrt{h}(K_{\mu\nu} - Kh_{\mu\nu})\delta h^{\mu\nu}\,. \tag{2.36}$$

For fixed boundary and Dirichlet boundary conditions we have $\delta h^{\mu\nu} = 0$, which yields a well defined variational principle.

- Dynamical boundary. However, we could also have a "dynamical boundary". If $h_{\mu\nu}$ is no longer fixed on ∂M, we have

$$\frac{\delta S_{\mathrm{GH}}}{\delta h^{\mu\nu}} = \frac{K_{\mu\nu} - Kh_{\mu\nu}}{-16\pi G}\,. \tag{2.37}$$

In particular, consider a thin wall/brane-world. We can think about it as a combination of 2 manifolds (bulks) with boundary on each side:



$\Sigma = \partial M_+ = \partial M_-$ sourced by wall energy-momentum tensor

$$T_{\mu\nu} \sim S_{\mu\nu}\delta\big(x^\mu - x^\mu(\sigma^A)\big)\,. \tag{2.38}$$

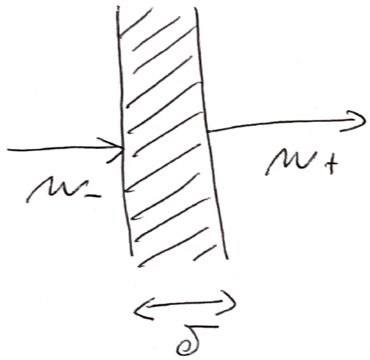
Boundary description gives

$$\boxed{h_{\mu\nu}|_{\partial M_+} = h_{\mu\nu}|_{\partial M_-}\,, \quad \langle K_{\mu\nu} - Kh_{\mu\nu}\rangle = 8\pi G S_{\mu\nu}\,,} \tag{2.39}$$

where $\langle\cdot\rangle$ stands for averaging of "$+$" $+$"$-$" quantities, e.g. $K(n_+) + K(n_-)$. These are the Israel junction conditions.

- To derive these, Israel [8] took the limit $\delta \to 0$ of the Einstein equations, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, in the following setup (note that in this case we have a continuous normal across Σ):

ISRAEL EQUATIONS



$$\boxed{\Delta h_{\mu\nu} = 0}$$

$$\boxed{\Delta K_{\mu\nu} - \Delta K h_{\mu\nu} = 8\pi G S_{\mu\nu}}$$

$$\Delta T \equiv T(M^+)|_\Sigma - T(M^-)|_\Sigma$$

He then used these to study the gravitational collapse of thin matter shells.

- Alternatively, one could consider a certain kind of the Neumann boundary conditions, see e.g. [9].

Chapter 3: Lecture 3: Black hole thermodynamics

3.1 Motivational foreplay

- Charged AdS black hole. Let us consider the following solution:

$$\boxed{ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}\,, \quad A = -\frac{Q}{r}dt\,,} \tag{3.1}$$

  where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. This is a solution of Einstein Maxwell-AdS equations

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi T^{(EM)}_{\mu\nu}\,, \quad \nabla_\mu F^{\mu\nu} = 0\,, \quad F = dA\,, \tag{3.2}$$

  where the cosmological constant is given by $\Lambda = -3/l^2$. Identifying its contribution in Einstein equations with the energy-momentum tensor of a perfect fluid:

$$-\Lambda g_{\mu\nu} = 8\pi\Big((\rho + P)u_\mu u_\nu + P g_{\mu\nu}\Big)\,, \tag{3.3}$$

  we may identify the corresponding 'cosmological pressure' as

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2} = -\rho\,. \tag{3.4}$$

- Basic properties. The above solution is characterized by its mass M and charge Q. For given M, Q, l, its horizon is located at the horizon radius r_+, given by the largest root of

$$f(r_+, M, Q, l) = 0\,. \tag{3.5}$$

  It is a *Killing horizon*: a null surface generated by Killing field $k = \partial_t$. Associated with it is the concept of surface gravity κ:

$$k^\mu \nabla_\mu k^\nu\big|_H = \kappa\, k^\nu\big|_H$$

surface gravity

It can be shown that for a spherically symmetric metric characterized by a single metric function f, as above, we have

$$\kappa = \frac{1}{2}\frac{\partial f}{\partial r}\Big|_{r=r_+} . \tag{3.6}$$

We may also calculate the horizon area. Taking $dt = 0 = dr$, the induced spatial metric 'on the horizon' is $d\gamma^2 = r_+^2 d\Omega^2$. The area then reads

$$A = \int \sqrt{\det \gamma} d\theta d\varphi = \int r_+^2 \sin\theta d\theta d\varphi = 4\pi r_+^2 . \tag{3.7}$$

- First law of black hole mechanics. Let's now consider a physical process under which the black hole spacetime is perturbed and eventually settles to a new charged AdS black hole spacetimes with modified spacetime parameters, $\{M + \delta M, Q + \delta Q, l + \delta l\}$. Consequently, the horizon radius also modifies to $r_+ + \delta r_+$, determined from $f(r_+ + \delta r_+, M + \delta M, Q + \delta Q, l + \delta l) = 0$. Using the Taylor expansion to linear order in perturbation, together with (3.5), we thus have

$$0 = \frac{\partial f}{\partial M}\delta M + \frac{\partial f}{\partial r}\delta r_+ + \frac{\partial f}{\partial Q}\delta Q + \frac{\partial f}{\partial l}\delta l\Big|_{r=r_+} . \tag{3.8}$$

Re-arranging this equation, we thus have

$$\begin{aligned} \delta M &= -\Big(\frac{\partial f}{\partial M}\Big)^{-1}\Big(\frac{\partial f}{\partial r}\delta r_+ + \frac{\partial f}{\partial Q}\delta Q + \frac{\partial f}{\partial l}\Big)_{r=r_+} \\ &= \frac{r_+}{2}\Big(2\kappa\delta r_+ + \frac{2Q}{r_+^2}\delta Q - \frac{2r_+^2}{l^3}\delta l\Big) , \end{aligned} \tag{3.9}$$

where in the second line we used (3.6) and the specific form of f for the charged AdS black hole. We thus find

$$\delta M = \frac{\kappa}{2\pi}\frac{\delta A}{4} + \frac{Q}{r_+}\delta Q + \frac{4}{3}\pi r_+^3 \delta P , \tag{3.10}$$

treating $M = M(A, Q, P)$. Let's now make the following definitions of electrostatic potential and black hole volume:

$$\phi = \Big(\frac{\partial M}{\partial Q}\Big)_{A,P} = \frac{Q}{r_+} , \quad V = \Big(\frac{\partial M}{\partial P}\Big)_{A,Q} = \frac{4}{3}\pi r_+^3 . \tag{3.11}$$

The first is indeed the electrostatic potential on the horizon, $\phi = -k \cdot A|_{r=r_+}$. Similarly, there exists a geometric definition for V [10]. This yields the first law of black hole mechanics:

$$\delta M = \frac{\kappa}{2\pi}\frac{\delta A}{4} + \phi\delta Q + V\delta P . \tag{3.12}$$

However, this looks a lot like a 1st law of (black hole) thermodynamics (especially because of the work terms), provided we identify M with enthalpy and

$$\boxed{T = \frac{\hbar\kappa}{2\pi k_B}\,, \quad S = \frac{A}{4\hbar G_N}\,.} \tag{3.13}$$

While this seems strange for classical black holes, as shown by Hawking in 1974, [11, 12], when quantum effects are taken into account, black holes radiate away as black body with these characteristics. Derivation used QFT in curved space. Hawking basically showed "stimulated emission". The problem with his derivation is that due to the bluehift near the horizon, the test field approximation breaks down and we cannot really trust the result. However, since then the same result has been reproduced by many other approaches, e.g: Euclidean path integral, tunneling, string theory, LQG. Let's exploit our knowledge of variational principles to 'derive' these results.

3.2 Euclidean Trick

- Let us sue the following fact. Thermal Green functions have periodicity in imaginary Euclidean time $\boxed{\tau = it}$:

$$\boxed{G(\tau) = G(\tau + \beta)\,, \quad \beta = 1/T\,.} \tag{3.14}$$

Conversely, periodicity of G defines a thermal state. Green functions of quantum fields in the vicinity of black holes have this property (as seen by a static observer). What about gravitational field itself?

- The Euclideanized spherical black hole solution ($\tau = it$) is

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,. \tag{3.15}$$

Near the horizon we may expand

$$f = \underbrace{f(r_+)}_{0} + \underbrace{(r - r_+)}_{\Delta r}\underbrace{f'(r_+)}_{2\kappa} + \cdots = 2\kappa\Delta r\,. \tag{3.16}$$

Therefore, the near horizon limit of the Euclidean solution takes the following form:

$$ds^2 = 2\kappa\Delta r d\tau^2 + \frac{dr^2}{2\kappa\Delta r} + r_+^2 d\Omega^2\,. \tag{3.17}$$

We can now introduce a new coordinate ρ by

$$d\rho^2 = \frac{dr^2}{2\kappa\Delta r} \quad \Leftrightarrow \quad d\rho = \frac{dr}{\sqrt{2\kappa\Delta r}} \quad \Leftrightarrow \quad \Delta r = \frac{\kappa}{2}\rho^2\,, \tag{3.18}$$

getting

$$ds^2 = \kappa^2 \rho^2 d\tau^2 + d\rho^2 + r_+^2 d\Omega^2 = \rho^2 d\varphi^2 + d\rho^2 + r_+^2 d\Omega^2\,, \tag{3.19}$$

upon introducing a new angle coordinate, $\varphi = \kappa\tau$. This looks like a flat space written in polar coordinates, provided the angle φ has a period 2π, otherwise there is a conical singularity at $\rho = 0$, which corresponds to the original black hole horizon. The reasoning now goes as follows: since the original black hole was originally non-singular (there is no matter there), we expect it to be non-singular again. This is achieved by setting (we want to avoid conical singularity)

$$\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \tau \sim \tau + \underbrace{2\pi/\kappa}_{\beta} \quad \Leftrightarrow \quad \boxed{T = \frac{\kappa}{2\pi}\,,} \tag{3.20}$$

which is the famous Hawking temperature. In particular, for Schwarzschild black hole we recover

$$T = \frac{1}{8\pi G M k_B} \to \frac{\hbar c^3}{8\pi G_N M k_B} \sim 6 \times 10^{-8} \frac{M_\odot}{M} K\,. \tag{3.21}$$

Temperature for Kerr*

- The Euclidean trick described above can also be used to determine the temperature of the Kerr black hole. In order to get real Euclidean geometry, we have to rotate both the time and the rotation parameter:

$$t \to i\tau\,, \quad a \to ib\,, \tag{3.22}$$

which yields

$$ds_E^2 = \frac{\Delta}{\Sigma}(d\tau - b\sin^2\theta d\varphi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\big((r^2 - b^2)d\varphi + bd\tau\big)^2\,, \tag{3.23}$$

where $\Delta = r^2 - b^2 - 2Mr$ and $\Sigma = r^2 - b^2\cos^2\theta$. When zooming on the horizon $r \to r_+$ given by the largest root of $\Delta(r_+) = 0$, we have to eliminate the last term in the metric, setting

$$d\varphi = -\frac{bd\tau}{r_+^2 - b^2}\,. \tag{3.24}$$

When this is plugged back, we have $d\tau - b\sin^2\theta d\varphi \approx \Sigma_+/(r_+^2 - b^2)d\tau$. Thus we can write

$$ds_E^2 \approx \frac{\Sigma_+}{r_+^2 - b^2}\Big(f d\tau^2 + \frac{dr^2}{f} + \dots\Big)\,, \quad f = \frac{\Delta}{r_+^2 - b^2}\,. \tag{3.25}$$

So, up to an overall constant conformal factor (which does not matter) we are back to the spherical case and derive the following temperature:

$$T = \frac{\Delta'(r_+)}{4\pi(r_+^2 - b^2)} = \frac{r_+ - M}{2\pi(r_+^2 + a^2)}\,, \tag{3.26}$$

where in the last step we have Wick-rotated back the rotation parameter $b \to -ia$.

3.3 Euclidean action calculation

- Consider the partition function of a system at temperature $1/\beta$:

$$Z = \mathrm{Tr} e^{-\beta H} = e^{-\beta F} \sim e^{-\beta H[g_c]}\,, \tag{3.27}$$

using the WKB approximation, in which the sum is dominated by classical (stationary) solutions g_c. For those we can then write:

$$H \sim \int d^3x \mathcal{H} = \frac{1}{\beta}\int d^4x \mathcal{L}_E = \frac{I_E}{\beta}\,. \tag{3.28}$$

Thus, we can identify the Euclidean action with the free energy

$$\boxed{F = \frac{I_E}{\beta} = -\frac{1}{\beta}\log Z\,.} \tag{3.29}$$

Our goal is thus to calculate the Euclidean action for the given black hole solution. We then find the corresponding entropy by the standard thermodynamic relation:

$$S = -\frac{\partial F}{\partial T} = \beta^2 \partial_\beta F\,. \tag{3.30}$$

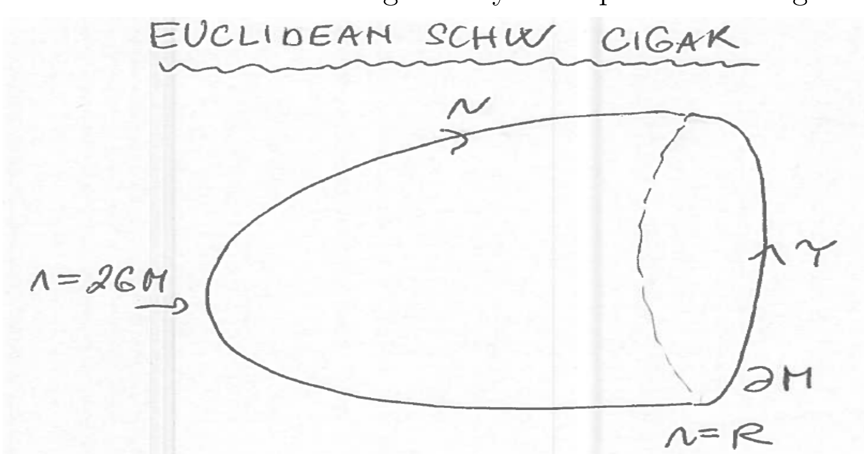
- Calculation for Schwarzschild. Recall that we need to calculate

$$I_E = -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}R - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}K \tag{3.31}$$

for the Euclidean Schwarzschild solution

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2M}{r}\,. \tag{3.32}$$

The Euclidean Schwarzschild geometry corresponds to a "cigar":



Even after the Wick rotation, this is a vacuum solution of Einstein equations, $R_{\mu\nu} = 0$ and thence

$$S_{\mathrm{EH}} = 0\,. \tag{3.33}$$

- However, we have a boundary at large $r = R$ and so the Gibbons–Hawking term may contribute. The induced metric is

$$d\gamma^2 = f(R)d\tau^2 + R^2 d\Omega^2 \,. \tag{3.34}$$

The corresponding normal in M is $n \propto \partial_r$, and after normalization reads

$$n = \sqrt{f}\partial_r\Big|_{r=R} \,. \tag{3.35}$$

- Extrinsic curvature is

$$K = \nabla_\mu n^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}n^\mu) = \frac{1}{r^2}(r^2 n^r)'\Big|_{r=R} = \frac{2}{R}\sqrt{f(R)} + \frac{1}{2}\frac{f'(R)}{\sqrt{f(R)}} \,. \tag{3.36}$$

The volume factor is

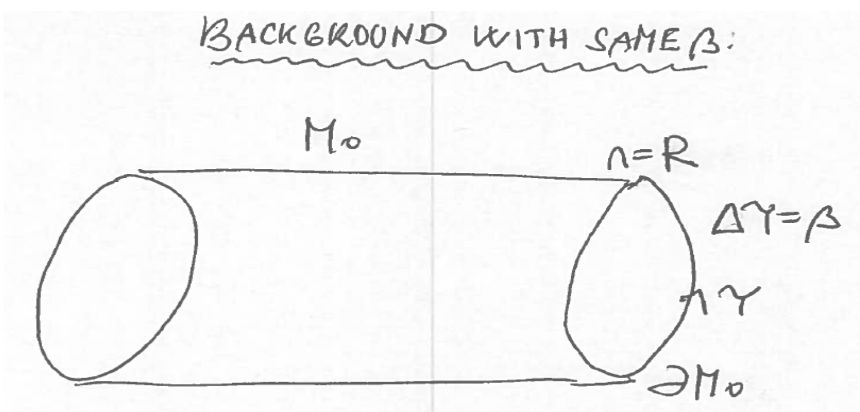
$$\sqrt{h}d^3x = \sqrt{f(R)}R^2 \sin\theta d\tau d\theta d\phi \,. \tag{3.37}$$

Putting together, we have

$$\int K\sqrt{h}d^3x = \underbrace{4\pi\beta}_{\int d\tau d\theta d\phi} \; R^2\Big[\frac{2}{R}f(R) + \frac{1}{2}f'(R)\Big] = 4\pi\beta(2R - 3M) \,. \tag{3.38}$$

This diverges as $R \to \infty$! However, note that this is also divergent for $M = 0$, that is flat spacetime has divergent action!

- To deal with this divergence we consider the appropriate background that has the same time periodicity β and matches our spacetime exactly at the boundary:



This 'cylinder' corresponds to 'thermal flat space' filled with radiation of $T = 1/\beta$. Its metric and induced metric are

$$ds_0^2 = f_0 d\tau^2 + dr^2 + r^2 d\Omega^2 \,, \quad d\gamma_0^2 = f_0 d\tau^2 + R^2 d\Omega^2 \,. \tag{3.39}$$

To match ∂M_0 and ∂M, we take $f_0 = f(R)$. Then we have

$$n = \partial_r \,, \quad K_0 = \frac{2}{R} \,, \quad \sqrt{h_0} d^3x \to \sqrt{f(R)} R^2 \sin\theta d\theta d\phi d\tau \,, \tag{3.40}$$

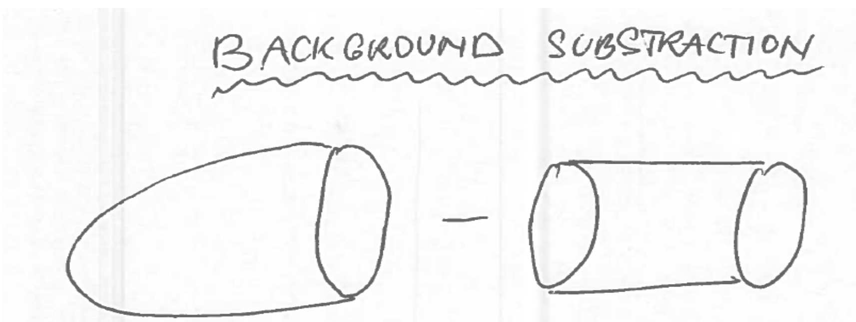
and so

$$\int d^3x \sqrt{h_0} K_0 = 8\pi\beta R \sqrt{f(R)} = 8\pi\beta R \Big(1 - \frac{M}{R} + O(1/R^2)\Big) \,. \tag{3.41}$$

- We now use the 'renormalization' subtraction procedure and write the total gravitational action as:

$$\boxed{I_E = -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} (K - K_0) \,,} \tag{3.42}$$

where the first boundary term, with K, is called the York term, and its purpose is to yield a well posed Dirichlet variational principle, and the second term, with K_0, is called the Gibbons-Hawking term, and its purpose is to 'tune' the value of the action.



Thus we have

$$I_{\text{Sch}} = \frac{\beta M}{2} \,. \tag{3.43}$$

The corresponding free energy is thus

$$F = \frac{I_{\text{Sch}}}{\beta} = \frac{M}{2} = \frac{\beta}{16\pi} \,, \tag{3.44}$$

and the corresponding entropy reads:

$$S = \beta^2 \partial_\beta F = \frac{\beta^2}{16\pi} = 4\pi M^2 = \pi r_+^2 = \frac{A}{4} \,, \tag{3.45}$$

deriving the Bekenstein formula. At the same time, we can check that

$$F = M - TS \,, \tag{3.46}$$

as it must for the (Gibbs) free energy.

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