Killing tensors: from spinning tops to SUSY in the sky

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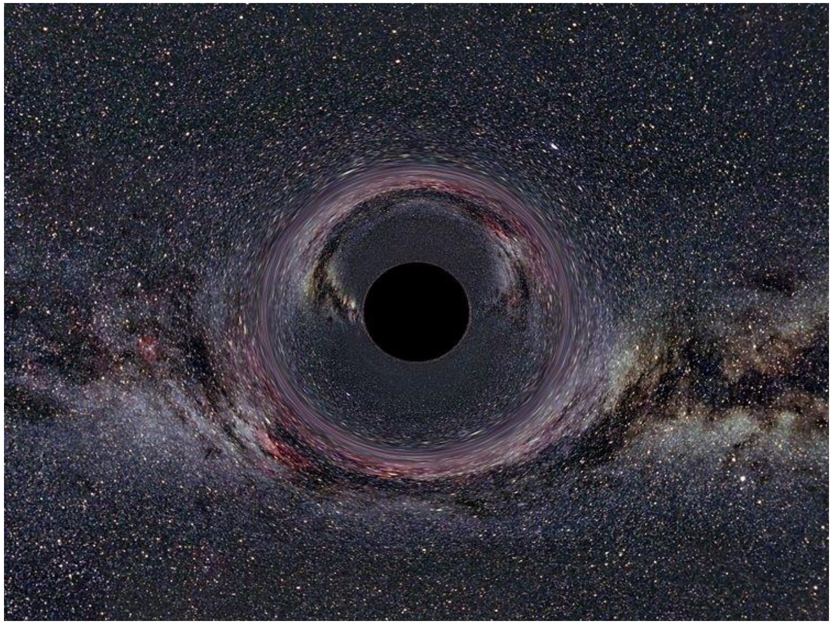


Selected topics in General Relativity 2024/25

Plan for the course

Lecture 1: *Introduction to hidden symmetries*

a) Why hidden? Basics of symplectic geometry
b) Symmetries in GR: Killing & KY tensors

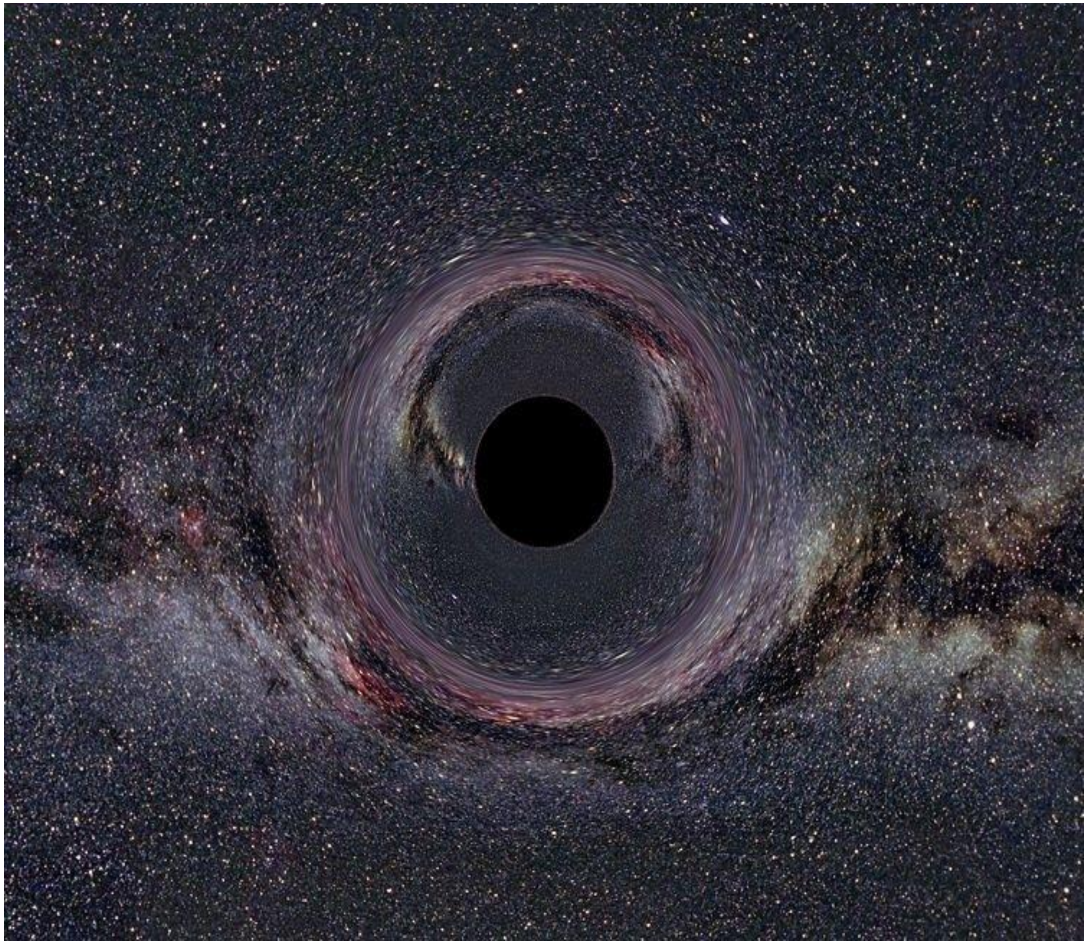
Lecture 2: *Lifting mechanical systems & their symmetries*

a) More on Killing and KY tensors
b) Eisenhart lift & other geometrizations of mechanics
c) From spinning tops to spacetimes with higher-rank KTs

Lecture 3: *SUSY in the sky*

a) More on Killing-Yano tensors
b) Classical spinning particle & its “fake SUSY”
c) Connections with other descriptions of spin

Lecture 1: Introduction to hidden symmetries

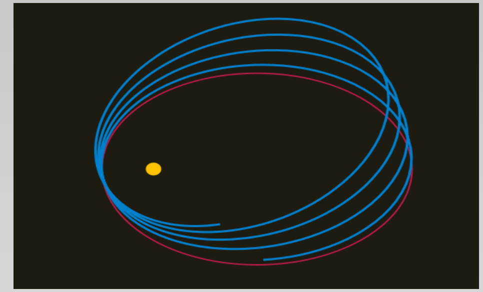


I) Why hidden? Basics of symplectic geometry

Laplace-Runge-Lenz vector

Central force:

$$E \, , \; \vec{L}$$

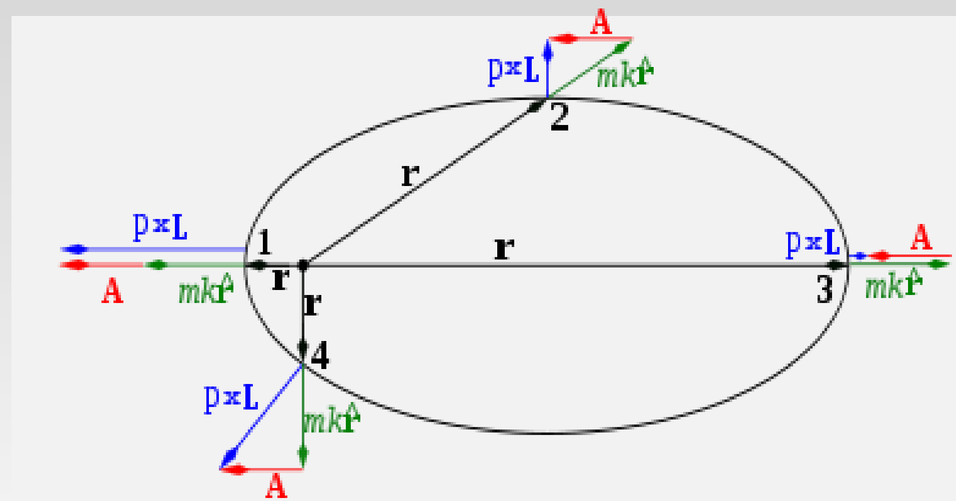


Kepler problem:

$$\boxed{\vec{F} = -\frac{k}{r^2}\hat{r}}$$

$$\boxed{\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}}$$

Laplace-Runge-Lenz vector



Wikipedia

- motion maximally superintegrable

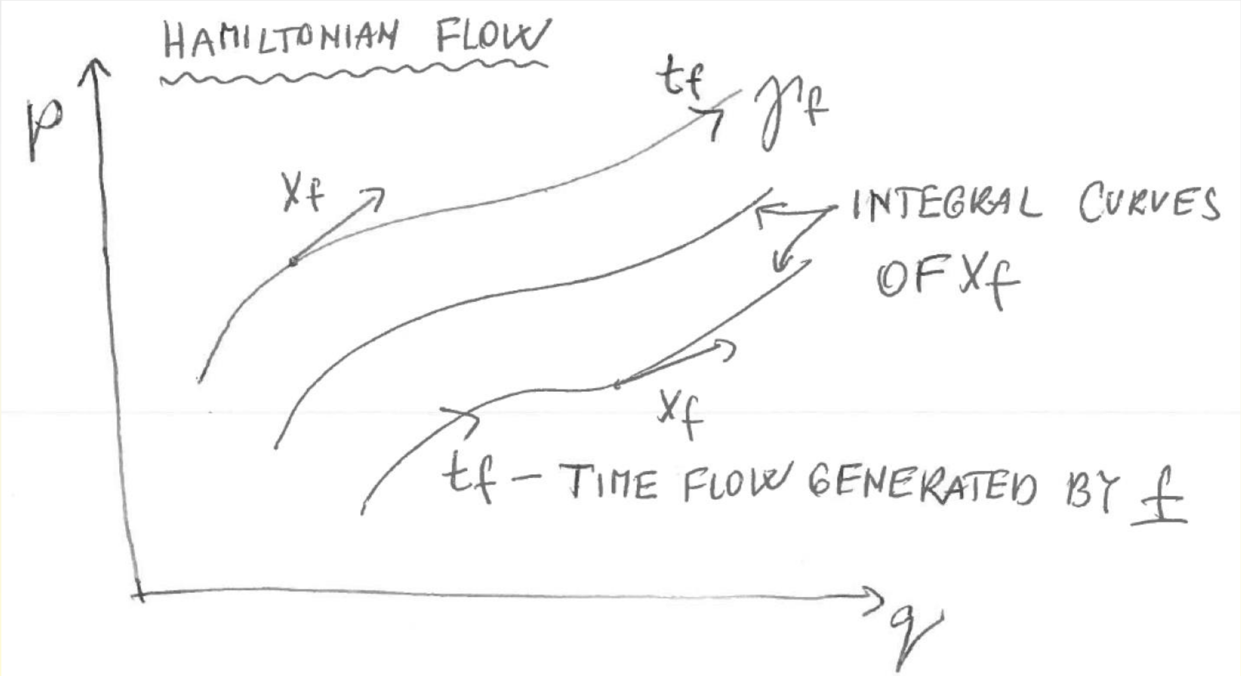
$$\vec{A} \cdot \vec{L} = 0 \qquad A^2 = m^2k^2 + 2mEL^2$$

…**hidden (dynamical) symmetry**

Symplectic 2-form

- **Symplectic manifold**: $$\omega = \frac{1}{2}\omega_{AB} d\xi^A \wedge d\xi^B$$ (non-degenerate, closed 2-form)
- **Hamiltonian vector field** generated by function f: $$X_f^A = \Omega^{AB}\partial_B f$$
- **Poisson bracket**

$$\{f, g\} = df \cdot \Omega \cdot dg = -\omega(X_f, X_g)$$



$$X_f = \{\cdot, f\} = \frac{d}{dt_f}\bigg|$$

Darboux theorem

- There exist Darboux coordinates: $\xi^A = (q^i, p_j)$ s.t

$$\Omega^{\alpha\beta} = \begin{array}{cc} & \begin{array}{cc} q^1 & p_1 \end{array} \\ \begin{array}{c} q^1 \\ p_1 \end{array} & \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & & \emptyset \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & & \ddots & \\ \emptyset & & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix} \end{array} \Leftrightarrow \omega_{\alpha\beta} = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & \emptyset \\ \emptyset & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix}$$

$$\omega = dp_i \wedge dq^i$$

- Poisson brackets then take the standard form:

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} .$$

Canonical equations

- **Time evolution:** generated by the **Hamiltonian** H

$$X_H = \{\cdot, H\} = \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j}$$

its **integral curves** $\gamma(t) = (p_j(t), q^j(t))$

$$X_H = \frac{d}{dt} = \frac{dq^j}{dt}\frac{\partial}{\partial q^j} + \frac{dp_j}{dt}\frac{\partial}{\partial p_j}$$

which yields **Hamilton's canonical equations**

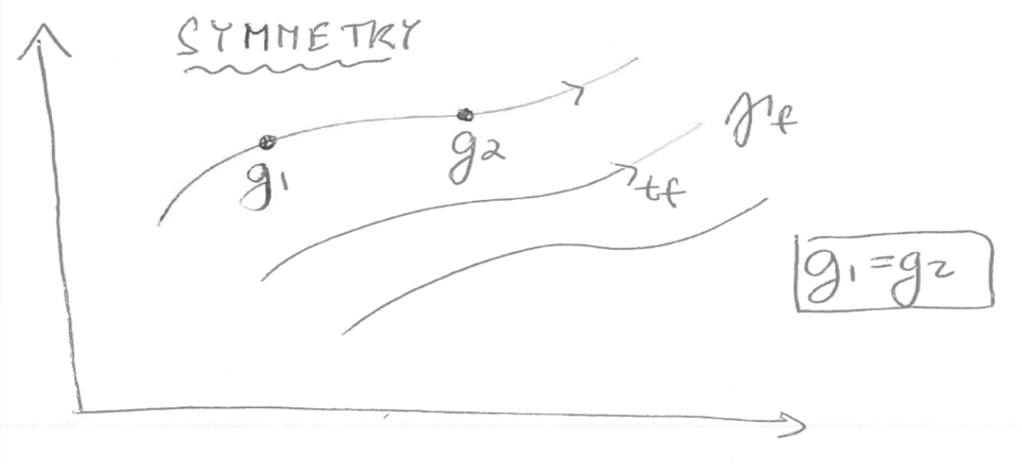
$$\boxed{\dot{q}^j = \frac{\partial H}{\partial p_j}\,, \qquad \dot{p}_j = -\frac{\partial H}{\partial q^j}\,.}$$

- **more generally:** $\dot{f} = \frac{df}{dt} = X_H \cdot df = \{f, H\}$

Symmetry in phase space

- **Symmetry** of a phase space function g:

$$\mathcal{L}_{X_f} g = X_f(g) = X_f \cdot dg = \frac{dg}{dt_f} = \{g, f\} = 0$$



- **Spec**: f=H, then g is preserved during the time evolution

$$\{g, H\} = 0$$

… g is an **integral of motion**

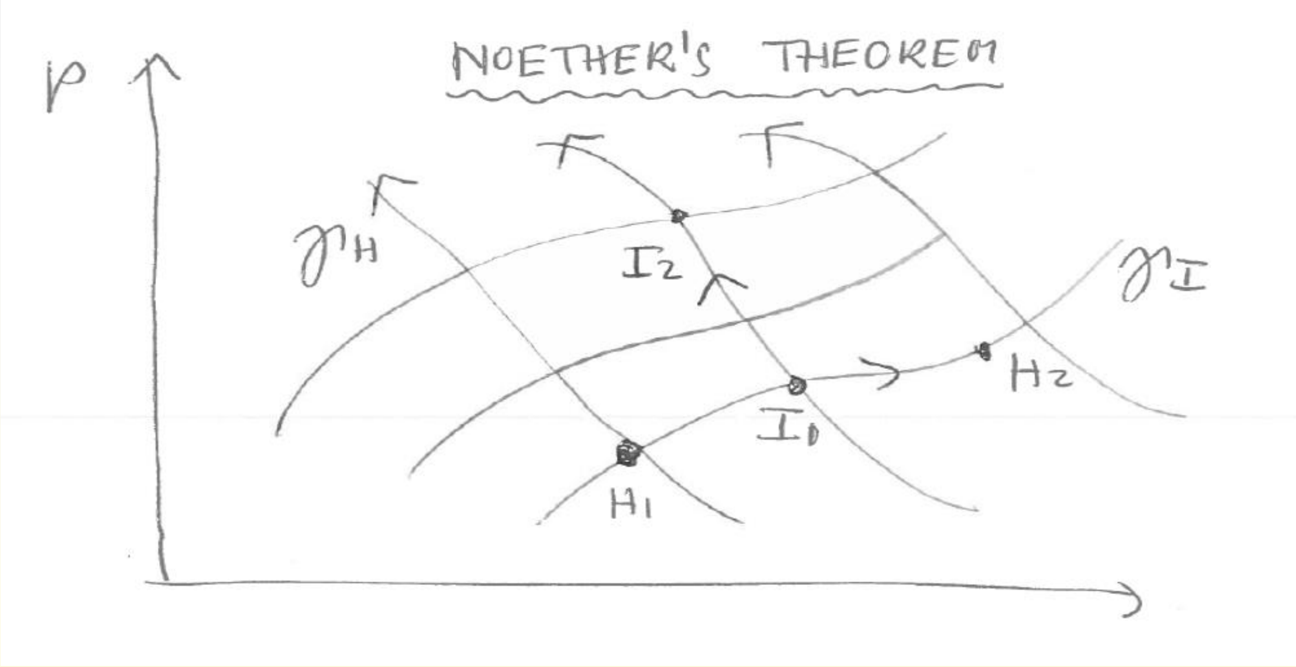
Noether's theorem

Noether theorem (Version 4: Hamiltonian version). Let there be a symmetry of the Hamiltonian generated by the Hamiltonian vector field X_I. Then I is an integral of motion.

proof:

$$\{H, I\} = \frac{dH}{dt_I} = 0 \quad \Rightarrow \quad \{I, H\} = \frac{dI}{dt} = 0$$

- Geometrical meaning is non-trivial:



Gauge freedom of Hamiltonian dynamics

- Encoded in **Canonical transformations**:

$$Q^j = Q^j(q^i, p_i)\,, \qquad P_j = P_j(q^i, p_i)$$

such that

$$\omega = dp_i \wedge dq^i = dP_i \wedge dQ^i$$

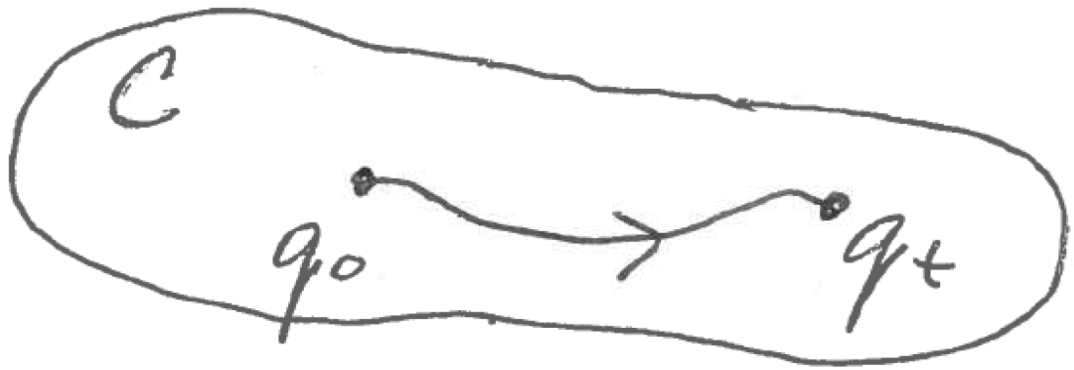
(form of Hamilton’s equations remain the same)

One **cannot distinguish** p’s from q’s!!!

Gauge freedom of Hamiltonian dynamics

- Spec: $Phase\ space = \text{cotangent bundle } T^*C$

$\underline{Configuration\ space}\ C$:



- C is a manifold M

$$(q^1, \ldots, q^n)$$

- Typically equipped with the **metric**

- There exists a **canonical projection**:

$$\boxed{\pi \ : \ T^*(\mathcal{M}) \ \to \ \mathcal{M}}$$

$$\pi \ : \ (q,p) \to q$$

breaks the symmetry!

- Induces a map on vectors

$$\tilde{X}_I = \pi^*(X_I) = \pi^*\Big(\frac{\partial I}{\partial p_i}\frac{\partial}{\partial q_i} - \underbrace{\frac{\partial I}{\partial q^i}\frac{\partial}{\partial p_i}}_{\text{kills these directions}}\Big) = \frac{\partial I}{\partial p_i}\frac{\partial}{\partial q_i}$$

Hidden (dynamical) symmetries

Noether theorem (Version 4: Hamiltonian version). Let there be a symmetry of the Hamiltonian generated by the Hamiltonian vector field X_I. Then I is an integral of motion.

Can distinguish isometries from dynamical symmetries:

$$\pi^*(X_I) = \begin{cases} \text{vector field on } C & \underline{\textit{isometry}} \\ \text{not well defined on } C & \underline{\textit{dynamical (hidden) symmetry}} \end{cases}$$

The latter is a genuine symmetry of the phase space and does not have well defined geometric meaning on C.

- **Trivial example** of **hidden symmetry**: energy

$$E = H = \frac{p^2}{2m} \Rightarrow X_E = \{\cdot, E\} = \frac{p}{m}\frac{\partial}{\partial q} = \pi^*(X_E)$$

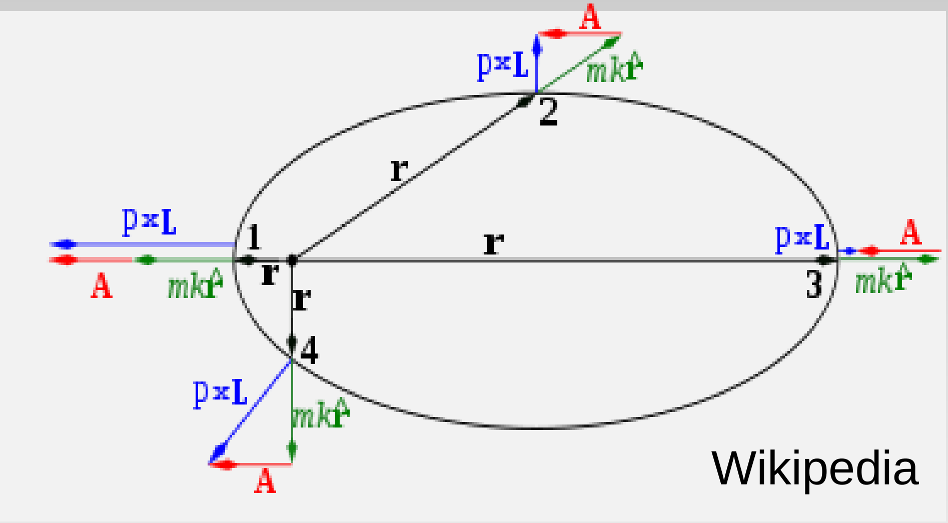
… no transformation of C in a given instant of time indicates that E should be conserved (time invariance of H)

Hidden (dynamical) symmetries

- **Non-trivial example**: Laplace-Runge-Lenz vector

$$\vec{F} = -\frac{k}{r^2}\hat{r}$$

$$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}$$



- Corresponding vector flow: **(Exercise)**

$$X_{A^i} = \left(2x^i p^k - \delta^i_k x \cdot p - p^i x^k\right)\frac{\partial}{\partial x^k} - \left(\delta^i_k p^2 - p^i p_k - mk\delta^i_k \frac{1}{r} + mk\frac{x^i x^k}{r^3}\right)\frac{\partial}{\partial p^k}$$

- Canonical projection yields:

$$\pi^*(X_{A^i}) = \left(2x^i p^k - \delta^i_k x \cdot p - p^i x^k\right)\frac{\partial}{\partial x^k}$$

Hidden symmetry

II) Symmetries in GR

Symmmetries in GR

Particle motion
(geodesics)

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$

$$p^\mu \nabla_\mu p^\nu = 0$$

Linear in momentum conserved quantities:

$$\boxed{C_K = K^\mu p_\mu} \quad \Leftrightarrow \quad \boxed{\nabla_{(\mu} K_{\nu)} = 0}$$

…Killing vector equation

Proof:

$$\dot{C}_K = p^\nu \nabla_\nu (K^\mu p_\mu) = p^\nu p^\mu \nabla_{(\nu} K_{\mu)} + K^\mu \underbrace{p^\nu \nabla_\nu p_\mu}_{0} = 0$$

Hamiltonian vector field:

$$X_{C_K} = K^\mu \frac{\partial}{\partial x^\mu} - \frac{\partial K^\lambda}{\partial x^\mu} p_\lambda \frac{\partial}{\partial p_\mu}$$

$$\boxed{\pi_*(X_{C_K}) = K^\mu \frac{\partial}{\partial x^\mu} = K}$$

…isometry

$$\mathcal{L}_\xi g = 0$$

Higher-order conserved quantities

$$\boxed{C_K = K^{\mu_1 \ldots \mu_p} p_{\mu_1} \ldots p_{\mu_p}}$$

(Exercise)

$$\Longleftrightarrow \quad \boxed{\nabla_{(\mu} K_{\nu_1 \ldots \nu_p)} = 0}$$

…Killing tensor equation

Walker & Penrose, Comm. Math. Phys. 18 , 265 (1970). (Stackel 1895).

Hamiltonian vector field:

$$\pi_*(X_{C_K}) = p K^{\mu_1 \ldots \mu_{p-1} \nu} p_{\mu_1} \ldots p_{\mu_{p-1}} \frac{\partial}{\partial x^\nu}$$

…dynamical symmetry

Parallel transport

$$w^{\mu} = f^{\mu\nu} p_{\nu}$$

Impose that:

$$p^{\mu} \nabla_{\mu} w^{\nu} = 0 \qquad \boldsymbol{w} \cdot \boldsymbol{p} = 0$$

$\Longleftrightarrow$ **(Exercise)**

$$\boxed{\nabla_{(\kappa} f_{\mu)\nu} = 0} \qquad f_{\mu\nu} = f_{[\mu\nu]}$$

…Killing-Yano equation

K. Yano, Ann. Math. 55, 328 (1952). [Kashiwada 1968 & Tachibana 1969]
R. Penrose, Ann. N.Y. Acad. Sci. , 125 (1973).

Summary of lecture 1

1) **Hidden (dynamical) symmetries** are genuine symmetries of the phase space – they are “invisible” on the (preferred) configuration space.

2) In GR, hidden symmetries described by:
   - (symmetric) **Killing** tensors, and
   - (antisymmetric) **Killing-Yano** tensors.

Although we derived these as symmetries of the particle motion, they have far-reaching consequences for the properties of the spacetime and the dynamics of fields in it.

3) The latter are **more fundamental** – they are a “square root” of Killing tensors

$$\boxed{K_{\mu\nu} = f_{\mu\alpha} f_{\nu}{}^{\alpha}}$$

(**Exercise**)