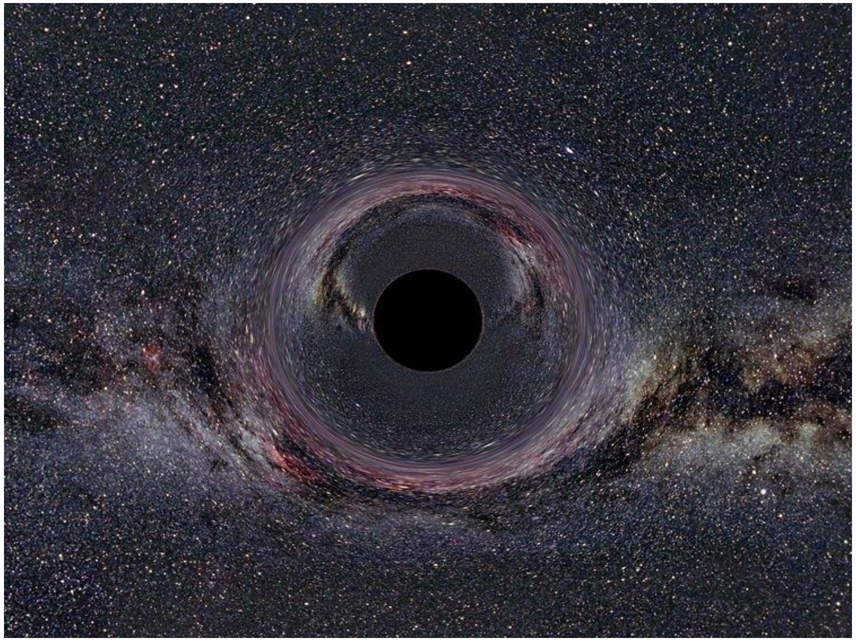
Lecture 3: Spinning particles & SUSY in the sky

David Kubizňák
(Charles University)



Selected Topics in General Relativity

Plan for lecture 3

a) More on KY tensors

b) Spinning particles in rotating black hole spacetimes

c) Classical spinning particle & SUSY in the sky

I) More on Killing-Yano tensors

Killing-Yano tensors: Parallel transport

$$w^\mu = f^{\mu\nu} p_\nu$$

Impose that:

$$p^\mu \nabla_\mu w^\nu = 0 \qquad \boldsymbol{w} \cdot \boldsymbol{p} = 0$$

⟺ **(Exercise)**

$$\nabla_{(\kappa} f_{\mu)\nu} = 0 \qquad f_{\mu\nu} = f_{[\mu\nu]}$$

…Killing-Yano equation

K. Yano, Ann. Math. 55, 328 (1952). [Kashiwada 1968 & Tachibana 1969]
R. Penrose, Ann. N.Y. Acad. Sci. , 125 (1973).

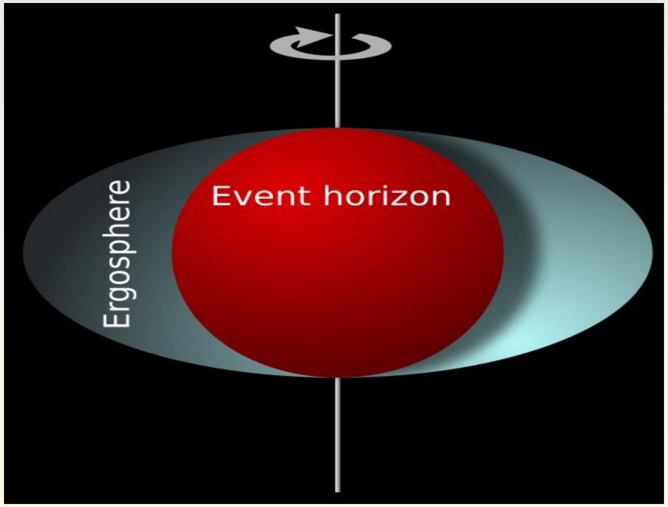
Killing-Yano tensors

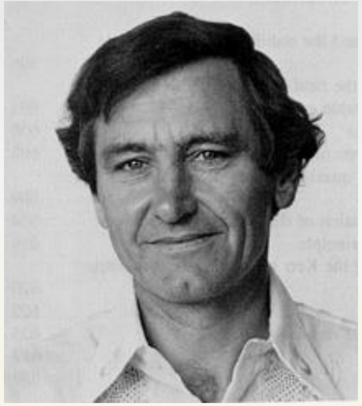
- They are **more "fundamental"** – they are a "square root" of Killing tensors

$$K_{\mu\nu} = f_{\mu\alpha} f_{\nu}{}^{\alpha}$$

(**Exercise**)

- Exist in the sky – rotating black hole described by **Kerr geometry** admits rank-2 KY tensor





Roy Patrick Kerr

Unique vacuum solution of Einstein equations describing a rotating black hole in 4d – discovered by Kerr in 1963

What about an algebra of Killing-Yano tensors?

- For (symmetric) Killing tensors – **SN brackets** induced by Poisson brackets of corresponding integrals for geodesic motion
-
- For KY tensors **not known**!
- One needs to include antisymmetric (Grassmann) observables & consider tensors of “mixed symmetries”

Research project!!!

II) Spinning particles in rotating black hole spacetimes

SPINNING PARTICLE IN CURVED ROTATING BH BACKGROUND

ARXIV: 1110.0495

D=4

QUANTUM

DIRAC EQ

$(D+m)\psi=0$

- SEPARABLE

$\psi = R_A(r) S_B(\theta) e^{i(m\varphi-\omega t)}$

$R_\pm(r)$, $S_\pm(\theta)$ } only decoupled 2nd-order ODE's

- "ENOUGH INTEGRALS OF MOTION ≈ SYMMETRY OPERATORS"

$\{D, K_t, K_\varphi, M_h\}$

COMPLETE SET OF MUT. COMM. OPS

NON-TRIVIAL ↓ NO SPIN

KG EQUATION

SEPARABLE —WKB→ GEODESIC EQ

IN BETWEEN

PI quantization (SUSY → DIRAC EQ); nontrivial? (DIRAC EQ → SUSY)

SUSY Classical spinning p.

INTEGRABLE (?)

"classical Ham system"

$\{x^a, \theta^a\}$

↑ GRASSMANN

$$\frac{D^2x^a}{d\tau^2} = R^a{}_{bcd}\frac{dx^b}{d\tau}\theta^c\theta^d \quad \text{"BOSONIC"}$$

$$\frac{d\theta^a}{d\tau}=0 \quad \text{"FERMIONIC"}$$

$\theta^a\theta^b = S^{ab}$ ⇢ (to PAPAPETROU EQ)

$\theta^a = 0$ → GEODESIC EQ

CLASSICAL

PAPAPETROU EQ. (⇠⇢ SUSY)

CHAOTIC MOTION (SCHW spin-orb. int.)

$\{u^a, p^a, S^{ab}\}$

↓ GAUGE FIXING (NOT UNIQUE)

$$\frac{Du^a}{d\tau} = R^a{}_{bcd}u^b S^{cd}$$

$$\frac{dS^{ab}}{d\tau}=0$$

$S^{ab}=0$ → GEODESIC EQ

GEODESIC EQ: COMPLETELY INTEGRABLE

III) Classical spinning particle & SUSY in the sky

A little more about spinning particle

"Classical Hamiltonian system" $\{x^a\,, \theta^a\}$

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta + \frac{i}{2} \eta_{ab} \theta^a \frac{D\theta^b}{d\tau}$$

$$\frac{D^2 x^\alpha}{d\tau^2} = \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = \frac{i}{2} R^\alpha{}_{\beta ab} \theta^a \theta^b \dot{x}^\beta \,,$$
$$\frac{D\theta^a}{D\tau} = \dot{\theta}^a + \omega_\beta{}^a{}_b \dot{x}^\beta \theta^b = 0\,.$$

Hamiltonian formulation:

- $$H = \frac{1}{2} g^{\alpha\beta} \pi_\alpha \pi_\beta \,, \quad \pi_\alpha = p_\alpha - \frac{i}{2} \theta^a \theta^b \omega_{\alpha ab} = g_{\alpha\beta} \dot{x}^\beta$$

Covariant (think of EM) [π_α]

canonical [p_α]

- Poisson bracket

$$\{F, G\} = \frac{\partial F}{\partial x^\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial x^\alpha} + i(-1)^{a_F} \frac{\partial F}{\partial \theta^a} \frac{\partial G}{\partial \theta_a}$$
$$= D_\alpha F \frac{\partial G}{\partial \Pi_\alpha} - \frac{\partial F}{\partial \Pi_\alpha} D_\alpha G + \frac{i}{2} \theta^a \theta^b R_{ab\alpha\beta} \frac{\partial F}{\partial \Pi_\alpha} \frac{\partial G}{\partial \Pi_\beta}$$
$$+ i(-1)^{a_F} \frac{\partial F}{\partial \theta^a} \frac{\partial G}{\partial \theta_a} \,, \tag{2.8}$$

A little more about spinning particle

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta + \frac{i}{2} \eta_{ab} \theta^a \frac{D\theta^b}{d\tau}$$

Generic supercharge

$$Q = \theta^a e_a^\alpha \pi_\alpha$$

$$\{H, Q\} = 0$$

$$\{Q, Q\} = -2iH$$

SUSY

Physical (gauge) conditions

$$Q = 0\,, \quad H = -\frac{1}{2}$$

Nongeneric superinvariants: SUSY in the sky

Gibbons, Rietdijk, van Holten, Nucl. Phys. B404 (1993) 42; hep-th/9303112.

$$\{Q, Q_\omega\} = 0$$

Automatically an integral of motion $\{H, Q_\omega\} = 0$

$$Q_\omega = \theta^{a_1} \dots \theta^{a_{p-1}} \omega^\alpha{}_{a_1 \dots a_{p-1}} \Pi_\alpha - \frac{i}{(p+1)^2} \theta^{a_1} \dots \theta^{a_{p+1}} \tilde{\omega}_{a_1 \dots a_{p+1}}$$

$$\nabla_\gamma \omega_{\alpha_1 \dots \alpha_p} = \nabla_{[\gamma} \omega_{\alpha_1 \dots \alpha_p]} = \frac{1}{p+1} (d\omega)_{\gamma \alpha_1 \dots \alpha_p}$$

Nongeneric superinvariants: SUSY in the sky

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$$\{Q, Q_\omega\} = 0$$

Automatically an integral of motion $\{H, Q_\omega\} = 0$

Linear in momenta superinvariants

- $Q_k = k^\alpha \pi_\alpha - \frac{i}{2}(dk)_{ab}\theta^a\theta^b$ $\nabla_{(a}k_{b)} = 0$
- $Q_f = \theta^a f_a{}^\alpha \pi_\alpha - \frac{i}{3!}(df)_{abc}\theta^a\theta^b\theta^c$

$\nabla_a f_{bc} = \nabla_{[a} f_{bc]}$ Killing-Yano 2-form

- $\{Q_f, Q_f\} = -2iK_f$

$$K_f \begin{cases} = H & \text{extended SUSY} \\ \neq H & \text{`non-generic supercharge'} \end{cases}$$

SUSY in the sky: Kerr geometry

Set of commuting operators: $\{H, Q_{\partial_t}, Q_{\partial_\varphi}, Q_f\}$

"bosonic" ($H, Q_{\partial_t}, Q_{\partial_\varphi}$) "fermionic" ($Q_f$)
(no classical analogue)

$$K_f = \{Q_f, Q_f\} = K^{\alpha\beta}\pi_\alpha\pi_\beta + .\ \theta\theta + .\ \theta\theta\theta\theta$$

terms

Bosonic set of commuting operators :

$$\{H, Q_{\partial_t}, Q_{\partial_\varphi}, K_f\}$$

- SUSY in the sky
- can take a limit $\theta \to 0$ and recover Carter's result

Problem: "integrates" only bosonic equations. What about fermionic?

Summary of lecture 3

1) Spinning particles described by: i) **Dirac equation** (QM) ii) **Papapetrou equations** (classically), and iii) semiclasically by **Grasmann variables** and spinning particle equations. We have the following connections:

a) Dirac limit: $\theta^a \to \gamma^a$ $\pi_\alpha \to -i\nabla_\alpha$ $Q \to D$ $H \to \Box$

- Grassmann algebra $\neq$ Clifford algebra

$$\{\theta^a, \theta^b\} = 0 \qquad \{\gamma^a, \gamma^b\} = 2\eta^{ab}$$

- operator ordering

b) Papapetrou's limit: $S^{ab} = -i\theta^a\theta^b$ (satisfies Lorentz algebra)

$$S^{ab}S^{cd} \neq S^{[ab}S^{cd]}$$ (Integrals OK to linear order)

2) **Killing-Yano tensors** (antisymmetric generalizations of Killing tensors) provide hidden symmetries for these spin descriptions (non-generic SUSY).

Spinning particle in curved rotating BH background

a) Quantum description: Dirac equation

$$(D + m)\,\psi = 0\,.$$

- **Separable!**

$$\psi = R(r)S(\theta)e^{i(m\varphi - \omega t)}$$

$$\left.\begin{matrix} R_{\pm}(r) \\ S_{\pm}(\theta) \end{matrix}\right\}$$ obey decoupled 2nd-order ODEs

- "Enough integrals of motion $\approx$ symmetry operators"

$$\{D, K_t, K_\varphi, M_h\}$$ complete set of mutually commuting operators

Spinning particle in curved rotating BH background

b) Classical GR description: Papapetrou's Eq.

$$\{u^a, p^a, S^{ab}\}$$

$\Downarrow$ gauge fixing (not unique)

$$\frac{Du^a}{d\tau} = R^a{}_{bcd} u^b S^{cd} \,,$$

$$\frac{DS^{ab}}{d\tau} = 0 \,.$$

Chaotic motion!

(see however, Skoupy & Witzany, 2411.16855)

Spinning particle in curved rotating BH background

c) SUSY semi-classical spinning particle

Integrable?

"Classical Hamiltonian system"

$$\{x^a\,,\theta^a\}$$

$$\frac{D^2x^a}{d\tau^2} = R^a{}_{bcd}\frac{dx^b}{d\tau}\theta^c\theta^d$$

$$\frac{D\theta^a}{d\tau} = 0\,.$$

$\Longleftrightarrow$

$$\frac{Du^a}{d\tau} = R^a{}_{bcd}u^bS^{cd}\,,$$

$$\frac{DS^{ab}}{d\tau} = 0\,.$$

$$\theta^a\theta^b \approx S^{ab}$$

Spinning particle in curved rotating BH background

Quantum

$$(D+m)\,\psi = 0$$

Separable!

complete set
of comm.ops

SUSY: spinning

$$\frac{D^2x^a}{d\tau^2} = R^a{}_{bcd}\frac{dx^b}{d\tau}\theta^c\theta^d$$
$$\frac{D\theta^a}{d\tau} = 0\,.$$

Integrable?!

Classical

$$\frac{Du^a}{d\tau} = R^a{}_{bcd}u^bS^{cd}$$
$$\frac{DS^{ab}}{d\tau} = 0\,.$$

Chaotic?!?

No spin
(nontriv)

$$\theta^a = 0$$

$$S^{ab} = 0$$

Klein-Gordon Eq.

Separable!

WKB

Geodesic Eq.

Carter: Completely integrable!