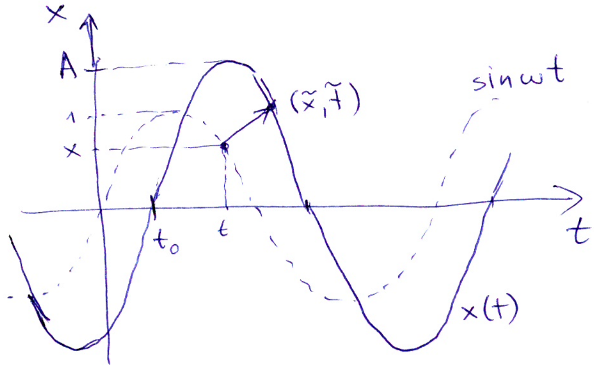
Symmetries of Equations of Mathematical Physics and Conservation Laws

- basic information: utf.mff.cuni.cz/~~vyuka~~/NTMF064(/en)
- literature: I recommend books by Bluman et al.
  or by Olver (especially Applications of LG to DE, 1993)
  see Library Genesis (libgen.rs)
  or utf.mff.cuni.cz/~houfek/esources04/list.htm

- motivation
  - classical linear harmonic oscillator

$$(1) \qquad m\ddot{x} = -kx \,, \qquad \ddot{x} = \frac{d^2x}{dt^2}$$

or $$\ddot{x} + \omega^2 x = 0 \,, \qquad \omega = \sqrt{\frac{k}{m}}$$



general solution

$$(2) \qquad x(t) = A \sin \omega (t - t_0)$$

where A and t_0 are constants.

- this solution can be obtained from a basic solution $\sin \omega t$ using transformations

$$(3) \qquad \begin{aligned} t &\mapsto \tilde{t} = t + t_0 && \text{(translation in time)} \\ x &\mapsto \tilde{x} = Ax && \text{(scaling in space)} \end{aligned}$$

[In general, such transformations convert arbitrary functions $x = f(t)$ into functions $x = Af(t - t_0)$, see later.]

- because transformations (3) convert a solution of (1) into another solution, we talk about a point symmetry of the equation (1) and it is a transformation which does not change this equation

$$\left(\ddot{x} = \frac{1}{A}\ddot{\tilde{x}} = -\omega^2 \frac{\tilde{x}}{A} \Rightarrow \ddot{\tilde{x}} + \omega^2 \tilde{x} = 0 \right)$$

- it is a two-parametric Lie group of transformations
  1) a composition of two such transformations gives again a transformation of the same kind
  2) for parameters $A = 1$, $t_0 = 0$, it is an identity
  3) there is an inverse transformation to each transformation (3), with parameters $\frac{1}{A}$ and $-t_0$
- it is actually only a subgroup of a much larger group of point symmetries of the equation (1)
  (there is an 8-parametric Lie group of point transf. under which (1) is invariant!)

- goals of lectures
  1) to learn how to find algorithmically these continuous symmetries of differential equations which, in general, form Lie groups of transformations
     (we will not deal with discrete symmetries such as reflections, time inversion etc.)
     but we will obtain them in a local (infinitesimal) form, not as global transformations
     e.g. $\tilde{x} = x + \varepsilon \eta(x,t) + O(\varepsilon^2)$ where ε is a small parameter
     $\tilde{t} = t + \varepsilon \xi(x,t) + O(\varepsilon^2)$
  2) how to use point symmetries to solve or to simplify ODE and PDE
  3) how to find an equation invariant under a certain (given) Lie group of transformations
  4) how to find a corresponding conservation laws