Extensions (prolongations) of infinitesimal transformations

- in the first lecture, we showed how to extend (prolong) a local Lie group of transformations into the space of derivatives of any order
- here we are interested in the extension of infinitesimal transformations, let us consider 1-par. LGT

$$\tilde{x}^i = F^i(x,u,\varepsilon) = x^i + \varepsilon \xi^i(x,u) + \mathcal{O}(\varepsilon^2)$$
$$\tilde{u}^\mu = G^\mu(x,u,\varepsilon) = u^\mu + \varepsilon \eta^\mu(x,u) + \mathcal{O}(\varepsilon^2)$$

and find its extension, which we will write as

$$\tilde{u}^\mu_i = H^\mu_i(x,u,\partial u,\varepsilon) = u^\mu_i + \varepsilon \eta^\mu_i(x,u,\partial u) + \mathcal{O}(\varepsilon^2)$$
$$\vdots$$
$$\tilde{u}^\mu_{i_1\dots i_k} = H^\mu_{i_1\dots i_k}(x,u,\partial u,\dots \partial^k u,\varepsilon) = u^\mu_{i_1\dots i_k} + \varepsilon \eta^\mu_{i_1\dots i_k}(x,u,\partial u,\dots,\partial^k u) + \mathcal{O}(\varepsilon^2)$$

- we can use the previous result for $H^\mu_i, \dots, H^\mu_{i_1\dots i_k}$ and by keeping terms of the order ε, we get for the first derivatives

$$\begin{pmatrix} \tilde{u}^\mu_1 \\ \vdots \\ \tilde{u}^\mu_n \end{pmatrix} = \begin{pmatrix} H^\mu_1 \\ \vdots \\ H^\mu_n \end{pmatrix} = A^{-1} \begin{pmatrix} D_1 G^\mu \\ \vdots \\ D_n G^\mu \end{pmatrix} = \begin{pmatrix} D_1F^1 & \dots & D_1F^n \\ \vdots & \ddots & \vdots \\ D_nF^1 & \dots & D_nF^n \end{pmatrix}^{-1} \begin{pmatrix} D_1 G^\mu \\ \vdots \\ D_n G^\mu \end{pmatrix} \overset{\text{now we keep only terms up to order } \varepsilon}{=}$$

$$= \begin{pmatrix} 1+\varepsilon D_1\xi^1 & \varepsilon D_1 \xi^2 & \dots & \varepsilon D_1\xi^n \\ \varepsilon D_2\xi^1 & 1+\varepsilon D_2\xi^2 & & \vdots \\ \vdots & & \ddots & \\ \varepsilon D_n\xi^1 & \dots & & 1+\varepsilon D_n\xi^n \end{pmatrix}^{-1} \begin{pmatrix} u^\mu_1 + \varepsilon D_1\eta^\mu \\ \vdots \\ u^\mu_n + \varepsilon D_n \eta^\mu \end{pmatrix} =$$

here we use that for $A = 1+\varepsilon B$ we have $A^{-1} = 1-\varepsilon B$ to the order ε

$$= \begin{pmatrix} 1-\varepsilon D_1\xi & \dots & -\varepsilon D_1\xi^n \\ \vdots & \ddots & \vdots \\ -\varepsilon D_n\xi^1 & \dots & 1-\varepsilon D_n\xi^n \end{pmatrix} \begin{pmatrix} u^\mu_1 + \varepsilon D_1\eta^\mu \\ \vdots \\ u^\mu_n + \varepsilon D_n\eta^\mu \end{pmatrix} =$$

$$= \begin{pmatrix} u_1^\mu + \varepsilon\left[D_1\eta^\mu - \sum_{i=1}^{n}(D_1\xi^i)\,u_i^\mu\right] + \mathcal{O}(\varepsilon^2) \\ \vdots \\ u_n^\mu + \varepsilon\left[D_n\eta^\mu - \sum_{i=1}^{n}(D_n\xi^i)\,u_i^\mu\right] + \mathcal{O}(\varepsilon^2) \end{pmatrix} = \begin{pmatrix} u_1^\mu + \varepsilon\eta_1^\mu + \mathcal{O}(\varepsilon^2) \\ \vdots \\ u_n^\mu + \varepsilon\eta_n^\mu + \mathcal{O}(\varepsilon^2) \end{pmatrix}$$

or

$$\boxed{\eta_i^\mu = D_i\eta^\mu - \sum_{k=1}^{n}(D_i\xi^k)\,u_k^\mu} \quad \text{for } i=1,\dots,n$$

- we can continue in a similar way for higher derivatives, only instead of D_1G^μ we would use $D_1H_i^\mu$ etc.

  Finally, we would get the recursive relations

$$\boxed{\eta^\mu_{i_1\dots i_k} = D_{i_k}\eta^\mu_{i_1\dots i_{k-1}} - \sum_{j=1}^{n}(D_{i_k}\xi^j)\,u^\mu_{i_1\dots i_{k-1}j}}$$

  for all derivatives up to order k (or higher)

- corresponding extension of the infinitesimal operator X into the space of derivatives up to order k is

$$X^{(k)} = \underbrace{\sum_{i=1}^{n}\xi^i\frac{\partial}{\partial x^i} + \sum_{\mu=1}^{m}\eta^\mu\frac{\partial}{\partial u^\mu}}_{X} + \sum_{i,\mu}\eta_i^\mu\frac{\partial}{\partial u_i^\mu} + \dots + \sum_{\substack{i_1\dots i_k\\ \mu}}\eta^\mu_{i_1\dots i_k}\frac{\partial}{\partial u^\mu_{i_1\dots i_k}}$$

- in the case of ordinary differential equations with x being independent and y being dependent variables we will use

$$\tilde{x} = F(x,y,\varepsilon) = x + \varepsilon\xi(x,y) + \mathcal{O}(\varepsilon^2)$$

$$\tilde{y} = G(x,y,\varepsilon) = y + \varepsilon\eta(x,y) + \mathcal{O}(\varepsilon^2)$$

$$\frac{d\tilde{y}}{d\tilde{x}} \equiv \tilde{y}_1 = H_1(x,y,y_1,\varepsilon) = y_1 + \varepsilon\eta^{(1)}(x,y,y_1) + \mathcal{O}(\varepsilon^2)$$

$$\vdots$$

$$\frac{d^k\tilde{y}}{d\tilde{x}^k} \equiv \tilde{y}_k = H_k(x,y,y_1,\dots,y_k,\varepsilon) = y_k + \varepsilon\eta^{(k)}(x,y,y_1,\dots,y_k) + \mathcal{O}(\varepsilon^2)$$

where now

$$\boxed{\eta^{(1)} = D_x\eta - y_1D_x\xi,\ \dots,\ \eta^{(k)} = D_x\eta^{(k-1)} - y_kD_x\xi}$$

• Examples:

1) <u>translation</u> in the plane (x,y) by a vector (a,b)

$$\left.\begin{aligned} \tilde{x} &= x + \varepsilon a \Rightarrow \xi(x,y) = a \\ \tilde{y} &= x + \varepsilon b \Rightarrow \eta(x,y) = b \end{aligned}\right\} \Rightarrow \eta^{(k)} = 0 \quad \text{for } k \geq 1$$

thus $X^{(k)} = X$ (when we translate a function, its derivatives do not change)

2) <u>scaling of dependent variable</u>

$$\left.\begin{aligned} \tilde{x} &= x \Rightarrow \xi(x,y) = 0 \\ \tilde{y} &= e^{\varepsilon} y \Rightarrow \eta(x,y) = y \end{aligned}\right\} = \eta^{(k)} = y_k \quad \text{for } k \geq 1$$

thus $X^{(k)} = X + \sum_{i=1}^{k} y_i \frac{\partial}{\partial y_i}$ (derivatives are scaled by the same factor)

3) rotation in the plane (x,y) by an angle ε

$$\left.\begin{aligned} \tilde{x} &= x \cos\varepsilon - y \sin\varepsilon \Rightarrow \xi(x,y) = -y \\ \tilde{y} &= x \sin\varepsilon + y \cos\varepsilon \Rightarrow \eta(x,y) = x \end{aligned}\right\} \Rightarrow \begin{aligned} \eta^{(1)} &= 1 + y_1^2 \\ \eta^{(2)} &= 3 y_1 y_2 \end{aligned}$$

$$\eta^{(3)} = D_x \eta^{(2)} - y_3 D_x \xi = 3 y_2^2 + 4 y_1 y_3$$

4) Galilean transformation under which the heat equation $u_{xx} = u_t$ is invariant:

$$\left.\begin{aligned} \tilde{x} &= x + 2t\varepsilon \Rightarrow \xi^x(x,t,u) = 2t \\ \tilde{t} &= t \Rightarrow \xi^t(x,t,u) = 0 \\ \tilde{u} &= u e^{-x\varepsilon - t\varepsilon^2} \Rightarrow \eta(x,t,u) = -ux \end{aligned}\right\} \Rightarrow$$

we get

$$\eta_x = D_x \eta - \overbrace{(D_x \xi^x)}^{0} u_x - \cancel{(D_x \xi^t) u_t} = -u - x u_x$$

$$\eta_t = D_t \eta - (D_t \xi^x) u_x = -x u_t - 2 u_x$$

and

$$\eta_{xx} = D_x \eta_x - \overbrace{(D_x \xi^x)}^{0} u_{xx} = -2 u_x - x u_{xx}$$

$$\eta_{xt} = D_t \eta_x - (D_t \xi^x) u_{xx} = -u_t - x u_{xt} - 2 u_{xx} = \eta_{tx}$$

$$\eta_{tt} = D_t \eta_t - (D_t \xi^x) u_{tx} = -x u_{tt} - 4 u_{tx}$$

etc.