Invariance of a system of differential equations under point transformations

- We say that r-par. local LGT acting on the space of independent and dependent variables (x, u) via

$$\tilde{x} = F(x, u, \varepsilon) \quad , \quad \tilde{u} = G(x, u, \varepsilon) \qquad (*)$$

is a group of (point) symmetries of the system

$$R^\sigma(x, u, \partial u, \dots, \partial^k u) = 0 \quad , \quad \sigma = 1, \dots, N$$

if this system is not changed when written in new coordinates $(\tilde{x}, \tilde{u})$ after extension of $(*)$ into the space of derivatives, i.e.

$$R^\sigma(\tilde{x}, \tilde{u}, \partial\tilde{u}, \dots \partial^k \tilde{u}) = 0 \quad , \quad \sigma = 1, \dots, N$$

(the functions R^σ are the same!)

- if there exists a solution of $R^\sigma = 0$, then this solution is transformed into another solution of this system, or to a class of new solutions dependent on parameters ε
- in other words, the surface in the space $(x, u, \partial u, \dots \partial^k u)$ given by equations $R^\sigma = 0$, $\sigma = 1, \dots, N$, on which all solutions lie, is not changed under transformations $(*)$
- we also say that the system $R^\sigma = 0$, $\sigma = 1, \dots, N$ is invariant under these transformations

- Infinitesimal criterion of invariance for the system of PDR

1) necessary condition of invariance

If an r-par. LGT $\tilde{x} = F(x,u,\varepsilon)$, $\tilde{u} = G(x,u,\varepsilon)$ is a group of symmetries of a certain system of PDR

$$R^\sigma(x,u,\partial u, \dots \partial^k u) = 0 \quad , \sigma = 1, \dots, N$$

then the extension

$$X^{(k)} = X + \sum_{i,\alpha} \eta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \dots \sum_{\substack{i_1 \dots i_k \\ \alpha}} \eta_{i_1 \dots i_k}^\alpha \frac{\partial}{\partial u_{i_1 \dots i_k}^\alpha}$$

of any infinitesimal operator

$$X = \sum_{i=1}^n \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^m \eta^\alpha(x,u) \frac{\partial}{\partial u^\alpha}$$

of this group must satisfy the infinitesimal criterion

$$\boxed{X^{(k)} R^\sigma(x,u,\partial u, \dots \partial^k u)\Big|_{R^\sigma = 0} = 0}$$ (evaluated on the surface given by $R^\sigma = 0$)

Particularly it must be valid for the basis X_α.

Proof: For a group of symmetries, we have

$$R^\sigma(\tilde{x}, \tilde{u}, \dots \partial^k \tilde{u}) = 0 \quad \text{for arbitrary } \varepsilon$$

If we take a derivative of these equations with respect to the parameter ε_α, we get

$$0 = \frac{\partial R^\sigma}{\partial \varepsilon_\alpha}\Big|_{\varepsilon = e} = \sum_{j=1}^n \underbrace{\frac{\partial R^\sigma}{\partial \tilde{x}^j}\Big|_{\varepsilon=e}}_{\frac{\partial R^\sigma}{\partial x^j}} \underbrace{\frac{\partial \tilde{x}^j}{\partial \varepsilon_\alpha}\Big|_{\varepsilon=e}}_{\xi^{\alpha j}(x,u)} + \sum_M \frac{\partial R^\sigma}{\partial \tilde{u}^M}\Big|_{\varepsilon=e} \frac{\partial \tilde{u}^M}{\partial \varepsilon_\alpha}\Big|_{\varepsilon=e} +$$

$$+ \dots + \sum_{M, j_1, \dots j_k} \underbrace{\frac{\partial R^\sigma}{\partial \tilde{u}^M_{j_1 \dots j_k}}\Big|_{\varepsilon=e}}_{\frac{\partial R^\sigma}{\partial u^M_{j_1 \dots j_k}}} \underbrace{\frac{\partial \tilde{u}^M_{j_1 \dots j_k}}{\partial \varepsilon_\alpha}\Big|_{\varepsilon=e}}_{\eta^{\alpha M}_{j_1 \dots j_k}} =$$

$$= X_\alpha^{(k)} R^\sigma(x,u,\partial u, \dots \partial^k u)$$ evaluated on $R^\sigma = 0$ because only there it is valid $R^\sigma(\tilde{x}, \tilde{u}, \dots \partial^k \tilde{u}) = 0$

Thanks to linearity, it has to be valid for any $X = \sum \sigma_\alpha X_\alpha$.

• Examples:

1) linear harmonic oscillator

$$\ddot{x} + \omega^2 x = 0 \quad = R(t, x, \dot{x}, \ddot{x})$$

We know that this equation is invariant under the 2-par LGT:

$$\tilde{t} = t + \beta \quad \Rightarrow \quad X_1 = \frac{\partial}{\partial t} \quad \text{(generator of translations)}$$

$$\tilde{x} = \alpha x \quad \Rightarrow \quad X_2 = x \frac{\partial}{\partial x} \quad \text{(generator of scaling)}$$

Let us check that the infinitesimal criterion is valid:

$$X_1^{(2)} (\ddot{x} + \omega^2 x)\Big|_{R=0} = \frac{\partial}{\partial t} (\ddot{x} + \omega^2 x)\Big|_{R=0} = 0 \ \checkmark$$

and

$$X_2^{(2)} (\ddot{x} + \omega^2 x)\Big|_{R=0} = \left(x \frac{\partial}{\partial x} + \dot{x} \frac{\partial}{\partial \dot{x}} + \ddot{x} \frac{\partial}{\partial \ddot{x}} \right) (\ddot{x} + \omega^2 x)\Big|_{R=0} =$$

$$= (\ddot{x} + \omega^2 x)_{R=0} = 0 \ \checkmark$$

2) heat equation $u_{xx} = u_t$ is invariant under

$$\tilde{x} = x + 2t\varepsilon, \quad \tilde{t} = t, \quad \tilde{u} = u e^{-x\varepsilon - t\varepsilon^2}$$

generated by $X = \underbrace{2t}_{\xi^x} \frac{\partial}{\partial x} \underbrace{- ux}_{\eta} \frac{\partial}{\partial u}$

$\xi^t = 0$

therefore it should be

$$X^{(2)} (u_{xx} - u_t)\Big|_{u_{xx} = u_t} = 0$$

we found $X^{(2)}$ before:

$$X^{(2)} = X + \eta_x \frac{\partial}{\partial u_x} + \eta_t \frac{\partial}{\partial u_t} + \eta_{xx} \frac{\partial}{\partial u_{xx}} + \eta_{xt} \frac{\partial}{\partial u_{xt}} + \eta_{tt} \frac{\partial}{\partial u_{tt}}$$

we need only

$$\eta_t = -x u_t - 2u_x \quad \text{and} \quad \eta_{xx} = -2u_x - x u_{xx}$$

we get

$$X^{(2)} (u_{xx} - u_t)_{u_{xx} = u_t} = (\eta_{xx} - \eta_t)\Big|_{u_{xx} = u_t} = (-x u_{xx} + x u_t)\Big|_{u_{xx} = u_t} = 0$$

as it should be.

2) sufficient condition of invariance

- we can actually use the infinitesimal criterion also in the other way; as a tool to show that a certain LGT is a group of symmetries of the given system of PDEs, but we need one additional condition

Let us have a system of PDEs

$$R^\sigma(x, u, \partial u, \dots, \partial^k u) = 0 \quad , \sigma = 1, \dots, N$$

which is <u>of maximal rank</u>, i.e. the Jacobian matrix

$$J = \left.\begin{pmatrix} \frac{\partial R^1}{\partial x^1}, \dots, \frac{\partial R^1}{\partial u^1}, \dots \frac{\partial R^1}{\partial u^\alpha_{i_1 \dots i_k}}, \dots \\ \frac{\partial R^2}{\partial x^1} \\ \vdots \\ \frac{\partial R^N}{\partial x^1}, \dots, \frac{\partial R^N}{\partial u^1}, \dots \frac{\partial R^N}{\partial u^\alpha_{i_1 \dots i_k}}, \dots \end{pmatrix}\right|_{R^\sigma = 0}$$

has rank N.

again it has to be evaluated on the surface of solutions

If for all infinitesimal operators X of an r-par. LGT and its extensions $X^{(k)}$ the infinitesimal criterion is satisfied, i.e.

$$X^{(k)} R^\sigma(x, u, \partial u, \dots \partial^k u)\Big|_{R^\sigma = 0} = 0$$

then this group is the group of symmetries of $R^\sigma = 0$.

- the condition of maximal rank of J is necessary, otherwise we could have a situations when the system is not appropriately written and the infinitesimal criterion would be satisfied for any X, e.g.

if $\quad R = (\ddot{x} + \omega^2 x)^2 = 0$

then $\quad X^{(2)}\left[(\ddot{x} + \omega^2 x)^2\right]_{\ddot{x} + \omega^2 x = 0} = 2(\ddot{x} + \omega x) X^{(2)}(\ddot{x} + \omega x)\Big|_{\ddot{x} + \omega x = 0} = 0$

in this case we would have

$$J = \left(\frac{\partial R}{\partial t}, \frac{\partial R}{\partial x}, \frac{\partial R}{\partial \dot{x}}, \frac{\partial R}{\partial \ddot{x}}\right)\Big|_{R=0} = (0, 0, 0, 0)$$

but for $R = \ddot{x} + \omega x = 0$ we get

$$J = (0, \omega, 0, 1)$$ which is of rank 1.

- the proof (see the book by P. Olver for details) is based on transition to the so-called normal (canonical) variables in which X is simply a generator of translations $X = \frac{\partial}{\partial s}$ and ~~from~~ the infinitesimal criterion we get $XR\big|_{R=0} = \frac{\partial R}{\partial s} = 0$

and thus R has to be independent (explicitly) on s.
This can be done for all one-parametric subgroups of the LGT and then extended to the whole group.
(we use the normal (canonical) variables later for solving differential equations using their symmetries)

• Applications of the infinitesimal criteorion

1) direct verification whether a certain point transformation generated by X belongs to the group of symmetries of a given system

2) it gives conditions for $\xi^i(x,u)$ and $\eta^\mu(x,u)$ (actually a system of PDE of the first order linear in the derivatives, which can be sometimes solved in a straightforward way)

   from a general $X = \sum_i \xi^i \frac{\partial}{\partial x^i} + \sum_\mu \eta^\mu \frac{\partial}{\partial u^\mu}$

   when applied to the given system $R^\sigma = 0$

   $\Rightarrow$ searching for symmetries algorithmically

3) it gives also conditions for the system $R^\sigma = 0$ if we look for equations invariant under the given group of transformations