Point symmetries of ODE $\frac{d^2y}{dx^2}=0$ („free particle in 1D)

- we can use the infinitesimal criterion to find all point symmetries generated by a general inf. operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

- because we have the ODE of the second order, we need extension of X up to the second derivatives, i.e.

$$X^{(2)} = X + \eta^{(1)}(x,y,y_1)\frac{\partial}{\partial y_1} + \eta^{(2)}(x,y,y_1,y_2)\frac{\partial}{\partial y_2}$$

where

$$\eta^{(1)}(x,y,y_1) = D_x\eta - (D_x\xi)\,y_1 = \frac{\partial\eta}{\partial x} + y_1\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}y_1 - \frac{\partial\xi}{\partial y}y_1^2$$

and

$$\eta^{(2)}(x,y,y_1,y_2) = D_x\eta^{(1)} - (D_x\xi)\,y_2 = \frac{\partial^2\eta}{\partial x^2} + 2\frac{\partial^2\eta}{\partial x\partial y}y_1 + \frac{\partial^2\eta}{\partial y^2}y_1^2 - $$
$$- \frac{\partial^2\xi}{\partial x^2}y_1 - 2\frac{\partial^2\xi}{\partial x\partial y}y_1^2 - \frac{\partial\xi}{\partial x}y_2 - \frac{\partial^2\xi}{\partial y^2}y_1^3 - \frac{\partial\xi}{\partial y}2y_1y_2$$

- from the inf. criterion we get

$$X^{(2)}y_2\Big|_{y_2=0} = \eta^{(2)}(x,y,y_1,y_2)\Big|_{y_2=0} = 0$$

- <u>this condition must be satisfied for all x, y, and y_1</u> and because ξ and η are not functions of y_1, we get a polynomial in y_1 (all terms containing y_2 are zero)

$$\underbrace{-\frac{\partial^2\xi}{\partial y^2}}_{=0}y_1^3 + \underbrace{\left(\frac{\partial^2\eta}{\partial y^2} - 2\frac{\partial^2\xi}{\partial x\partial y}\right)}_{=0}y_1^2 + \underbrace{\left(2\frac{\partial^2\eta}{\partial x\partial y} - \frac{\partial^2\xi}{\partial x^2}\right)}_{=0}y_1 + \underbrace{\frac{\partial^2\eta}{\partial x^2}}_{0} = 0$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\frac{\partial^3\eta}{\partial y^2} - 2\frac{\partial^3\xi}{\partial x\partial y^2} = \frac{\partial^3\eta}{\partial y^3} = 0 \qquad 2\frac{\partial^3\eta}{\partial x^2\partial y} - \frac{\partial^3\xi}{\partial x^3} = -\frac{\partial^3\xi}{\partial x^3} = 0$$

- from these equations we see that
  $\xi(x,y)$ must be at most linear in y and quadratic in x
  and $\eta(x,y)$ must be at most linear in x and quadratic in y

that is

$$\xi(x,y) = c_1 + c_2 x + c_3 x^2 + c_4 y + c_5 xy + c_6 x^2 y$$
$$\eta(x,y) = d_1 + d_2 x + d_3 y + d_4 xy + d_5 y^2 + d_6 xy^2$$

- by substitution into

$$\frac{\partial^2 \eta}{\partial y^2} - 2\frac{\partial^2 \xi}{\partial x \partial y} = 0 \quad \text{we get} \quad 2(d_5 + d_6 x) - 2(c_5 + 2c_6 x) = 0$$

and into

$$2\frac{\partial^2 \eta}{\partial x \partial y} - \frac{\partial^2 \xi}{\partial x^2} = 0 \quad \text{we get} \quad 2(d_4 + 2d_6 y) - 2(c_3 + c_6 y) = 0$$

because x and y can be arbitrary we have

conditions

$$\begin{aligned} d_5 &= c_5, & \left.\begin{aligned} d_6 &= 2c_6 \\ 2d_6 &= c_6 \end{aligned}\right\} &\Rightarrow c_6 = d_6 = 0 \\ d_4 &= c_3, \end{aligned}$$

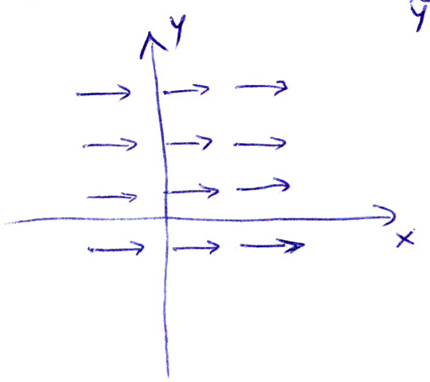
- the result is 8-parametric LGT generated by X

with
$$\xi(x,y) = c_1 + c_2 x + c_3 x^2 + c_4 y + c_5 xy$$
$$\eta(x,y) = d_1 + d_2 x + d_3 y + c_3 xy + c_5 y^2$$

- because $c_1, \dots, c_5, d_1, d_2, d_3$ can be arbitrary (there are no other conditions following from the inf. criterion) we have a basis of 8 linearly independent inf. generators; which we can get by setting of c's or d's equal to 1 and all others equal to 0:

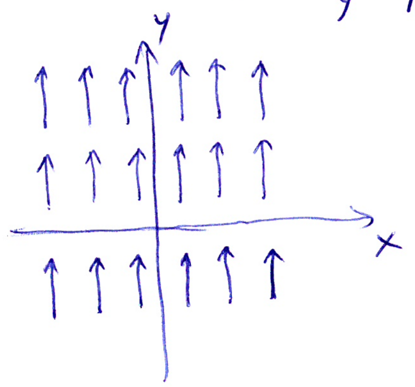
$X_1 = \frac{\partial}{\partial x}$ (for $c_1 = 1$)

translations in x: $\tilde{x} = x + \varepsilon$, $\tilde{y} = y$



$X_2 = \frac{\partial}{\partial y}$ (for $d_1 = 1$)

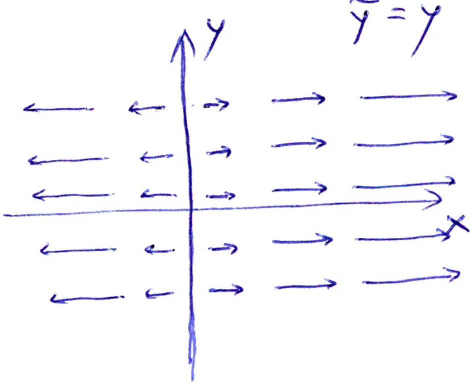
translation in y: $\tilde{x} = x$, $\tilde{y} = y + \varepsilon$



$$X_3 = x\frac{\partial}{\partial x} \quad (\text{for } c_2 = 1)$$

scaling in x: $\tilde{x} = e^{\varepsilon} x$

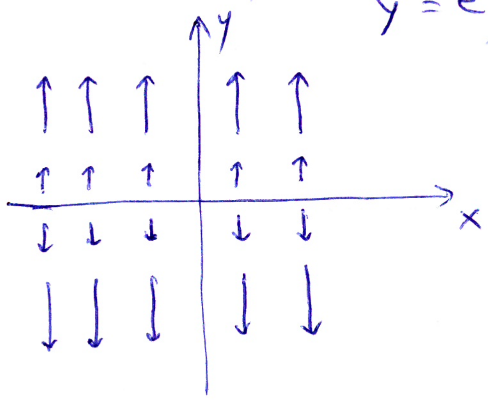
$\tilde{y} = y$



$$X_4 = y\frac{\partial}{\partial y} \quad (\text{for } d_3 = 1)$$

scaling in y: $\tilde{x} = x$

$\tilde{y} = e^{\varepsilon} y$

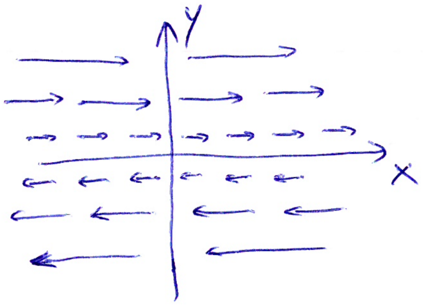


$$X_5 = y\frac{\partial}{\partial x} \quad (\text{for } c_4 = 1)$$

Galilean transformation in x:

$\tilde{x} = x + \varepsilon y$

$\tilde{y} = y$

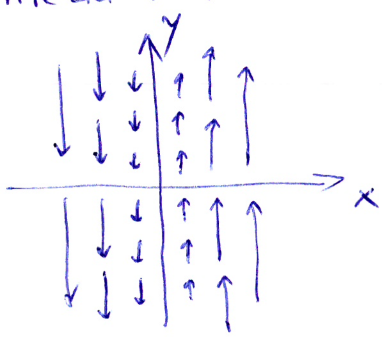


$$X_6 = x\frac{\partial}{\partial y} \quad (d_2 = 1)$$

Galilean transformation in y:

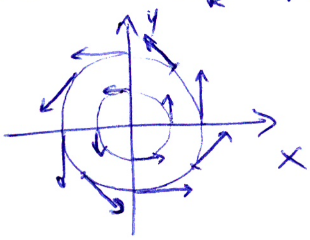
$\tilde{x} = x$

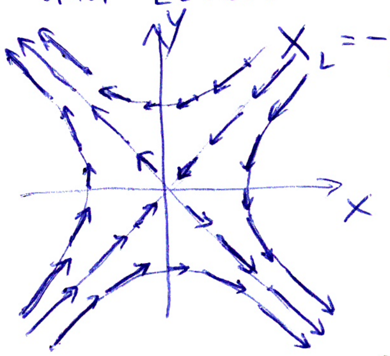
$\tilde{y} = y + \varepsilon x$



their linear combinations give

rotations $X_R = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and Lorentz transformations $X_L = -y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$



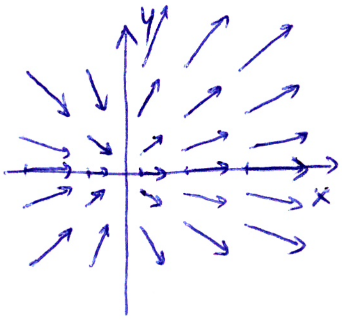


$$X_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \quad (\text{for } c_3 = d_4 = 1)$$

„projective" transformation in x:

$$\tilde{x} = \frac{x}{1 - \varepsilon x}$$

$$\tilde{y} = \frac{y}{1 - \varepsilon x}$$



$$X_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \quad (\text{for } d_5 = c_5 = 1)$$

„projective" transformation in y:

$$\tilde{x} = \frac{x}{1 - \varepsilon y}$$

$$\tilde{y} = \frac{y}{1 - \varepsilon y}$$