Point symmetries of the heat equation $u_{xx} = u_t$

- using Mathematica we search for the most general
$$X = \xi^x(x,t,u)\frac{\partial}{\partial x} + \xi^t(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}$$
which generates a symmetry for $u_{xx} = u_t$
- the infinitesimal criterion now gives
$$X^{(2)}(u_{xx} - u_t)\Big|_{u_{xx}=u_t} = (\eta_{xx} - \eta_t)\Big|_{u_{xx}=u_t} = 0$$
- the result will be a polynomial in all derivatives which remain after substitution $u_t = u_{xx}$, i.e. in u_x, u_{xt} and u_{xx} (u_{tt} is not in η_{xx})
- as in the previous case, all coefficients of this polynomial must be zero, because it has to be zero for all values of u_x, u_{xt} and u_{xx}
- we get for example conditions:
$$\underbrace{\frac{\partial \xi^x}{\partial u} = 0 = \frac{\partial \xi^t}{\partial u} = \frac{\partial \xi^t}{\partial x} = \frac{\partial^2 \eta}{\partial u^2}}_{} \quad \text{and} \quad 2\frac{\partial \xi^x}{\partial x} = \frac{\partial \xi^t}{\partial t} \quad (*)$$
from these we have restrictions like ξ^x is independent of u etc.
- by using only these conditions we get the second ansatz:
$$\xi^t(x,t,u) = \tau(t) \quad \text{(independent of } x \text{ and } u)$$
$$\xi^x(x,t,u) = \tfrac{1}{2}\tau'(t)\,x + \chi(t) \quad \text{(follows from } (*))$$
$$\eta(x,t,u) = \alpha(x,t)\,u + \beta(x,t) \quad \text{(linear in } u)$$
- from the inf. criterion we get new conditions, e.g.
$$\frac{\partial \alpha}{\partial x} = -\frac{1}{4}\tau''(t)\,x - \frac{1}{2}\chi'(t)$$
and we see that α is at most quadratic in x

- 3rd ansatz is the same as the 2nd with

$$\alpha(x,t) = -\frac{1}{8}\tau''(t)\,x^2 - \frac{1}{2}\chi'(t)\,x + \gamma(t)$$

and finally we get conditions

$$\chi''(t) = 0\,, \quad \tau'''(t) = 0 \quad \text{and} \quad 4\gamma'(t) = -\tau''(t)$$

From which we can see that

χ is linear in t,
τ is quadratic in t,
and γ is linear in t

- the result is

$$\xi^x(x,t,u) = c_1 + c_4 x + 2c_5 t + 4c_6 x t$$
$$\xi^t(x,t,u) = c_2 + 2c_4 t + 4c_6 t^2$$
$$\eta(x,t,u) = (c_3 - c_5 x - 2c_6 t - c_6 x^2)\,u + \beta(x,t)$$

where we have 6 independent parameters and $\beta(x,t)$ is an arbitrary solution of the heat equation because we have a condition $\beta_{xx} = \beta_t$.
(this is actually a consequence of linearity of the heat equation, as we will ~~show~~ later for general linear equations)

- we have 6 independent inf. generators (which provide non-trivial symmetries which can be used e.g. to find solutions of the heat equation) plus infinite-dimensional space of generators given by solutions of this equation

$$X_\infty = \beta(x,t)\frac{\partial}{\partial u} \quad \Rightarrow \quad \begin{aligned} \tilde{x} &= x \\ \tilde{t} &= t \\ \tilde{u} &= u + \varepsilon\beta(x,t) \end{aligned}$$

it is actually the superposition principle following from linearity

- 6 non-trivial point symmetries of the heat equation

$$X_1 = \frac{\partial}{\partial x} \Rightarrow \begin{array}{l} \tilde{x} = x + \varepsilon \\ \tilde{t} = t \\ \tilde{u} = u \end{array}$$ translation in x

$$X_2 = \frac{\partial}{\partial t} \Rightarrow \begin{array}{l} \tilde{x} = x \\ \tilde{t} = t + \varepsilon \\ \tilde{u} = u \end{array}$$ translation in t

$$X_3 = u\frac{\partial}{\partial u} \Rightarrow \begin{array}{l} \tilde{x} = x \\ \tilde{t} = t \\ \tilde{u} = e^{\varepsilon} u \end{array}$$ scaling in u

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \Rightarrow \begin{array}{l} \tilde{x} = e^{\varepsilon} x \\ \tilde{t} = e^{2\varepsilon} t \\ \tilde{u} = u \end{array}$$ simultaneous scaling of x and t

$$X_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} \Rightarrow \begin{array}{l} \tilde{x} = x + 2\varepsilon t \\ \tilde{t} = t \\ \tilde{u} = u e^{-x\varepsilon - t\varepsilon^2} \end{array}$$ Galilean transformation

$$X_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (2t + x^2)u\frac{\partial}{\partial u}$$

$$\Rightarrow \quad \tilde{x} = \frac{x}{1-4t\varepsilon} \qquad \tilde{u} = u\sqrt{1-4\varepsilon t}\; e^{-\frac{\varepsilon x^2}{1-4t\varepsilon}}$$

$$\tilde{t} = \frac{t}{1-4t\varepsilon}$$

„projective" transformation in t

Point symmetries of 1D Schrödinger equation for free particle

- it is basically the same equation as the heat equation, only with a complex function ψ and komplex coefficients

- we search for symmetries of

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} \Rightarrow i \frac{\partial \psi}{\partial t} = -\lambda \frac{\partial^2 \psi}{\partial x^2} \quad \text{with } \lambda = \frac{\hbar}{2m}$$

generated by

$$X = \xi^x(x,t,\psi)\frac{\partial}{\partial x} + \xi^t(x,t,\psi)\frac{\partial}{\partial t} + \eta(x,t,\psi)\frac{\partial}{\partial \psi}$$

using the infinit. criterion

$$X^{(2)}\left(i\frac{\partial \psi}{\partial t} + \lambda \frac{\partial^2 \psi}{\partial x^2}\right)\Bigg|_{\frac{\partial \psi}{\partial t} = i\lambda \frac{\partial^2 \psi}{\partial x^2}} = 0$$

- in a similar way as for the heat equation we would get

$$\xi^x(x,t,\psi) = c_1 + c_3 x + c_5 t + c_6 t x$$

$$\xi^t(x,t,\psi) = c_2 + 2c_3 t + c_6 t^2$$

$$\eta(x,t,\psi) = \left(c_4 + \frac{ic_5}{2\lambda}x + \frac{ic_6}{4\lambda}x^2 - \frac{c_6}{2}t\right)\psi + \beta(x,t)$$

where $\beta(x,t)$ is an arbitrary solution of the Sch. Eq.

- again we have 6 independent generators + infinite number of generators $X_\infty = \beta(x,t)\frac{\partial}{\partial \psi}$

$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$ ... translations in x and t

$X_3 = c\psi \frac{\partial}{\partial \psi}$ ... scaling of ψ, it contains also a change of phase $\tilde{\psi} = e^{i\varepsilon}\psi$ for $c = i$

$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ .. scaling of x and t

$$X_5 = t\frac{\partial}{\partial x} + \frac{ix}{2\lambda}\psi\frac{\partial}{\partial\psi}$$ ... Galilean transformation with a change of phase of ψ

$\Rightarrow$ (for $\varepsilon = v$) $\tilde{x} = x + vt$, $\tilde{t} = t$, $\quad \tilde{\psi} = \psi\, e^{i(mvx + \frac{1}{2}mv^2 t)/\hbar}$

the solution $\theta(x,t) = A$ transforms into

$$\tilde{\psi} = A\, e^{\frac{i}{\hbar}[mv(\tilde{x} - v\tilde{t}) + \frac{1}{2}mv^2\tilde{t}]}$$

or (without tildas)

$$\psi(x,t) = A\, e^{\frac{i}{\hbar}(mvx - \frac{1}{2}mv^2 t)}$$ plane-wave solution

$$X_6 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + \left(\frac{ix^2}{4\lambda} - \frac{t}{2}\right)\psi\frac{\partial}{\partial\psi}$$ ... „projective" transform

$$\Rightarrow \quad \tilde{t} = \frac{t}{1-\varepsilon t}, \quad \tilde{x} = \frac{x}{1-\varepsilon t}, \quad \tilde{\psi} = \psi\sqrt{1-\varepsilon t}\; e^{\frac{im\varepsilon x^2}{2\hbar(1-\varepsilon t)}}$$

and the solution $\theta(x,t) = A$ transforms into a nontrivial solution (written without tildas)

$$\psi(x,t) = \frac{A}{\sqrt{1+\varepsilon t}}\, e^{\frac{im\varepsilon x^2}{2\hbar(1+\varepsilon t)}}$$ Gaussian wave packet for $\varepsilon = i$

- the same result we would get if we would write $\psi = \psi_R + i\psi_I$ and considered two real-valued functions and coupled equations

$$\frac{\partial\psi_I}{\partial t} = \lambda\frac{\partial^2\psi_R}{\partial x^2}$$
$$\frac{\partial\psi_R}{\partial t} = -\lambda\frac{\partial^2\psi_I}{\partial x^2}$$

only this time we would get scaling and rotation in the complex plane (ψ_R, ψ_I) seperately, i.e.

$$X_4 = \psi_R\frac{\partial}{\partial\psi_R} + \psi_I\frac{\partial}{\partial\psi_I}, \quad X_4' = -\psi_R\frac{\partial}{\partial\psi_I} + \psi_I\frac{\partial}{\partial\psi_R}$$

(this is a change of phase)