Point symmetries of the classical central-force problem

- let us consider the Newton's equations of motion for spherically symmetric potential $V(r)$, that is the system of ODEs

$$m\frac{d^2x}{dt^2} = m\ddot{x} = -\frac{\partial V}{\partial x} = -V'(r)\frac{x}{r}$$
$$m\ddot{y} = -V'(r)\frac{y}{r}$$
$$m\ddot{z} = -V'(r)\frac{z}{r}$$

- to find all point symmetries of this system, we ~~assume~~ take first the infinitesimal generator in a form

$$X = \xi(t,x,y,z)\frac{\partial}{\partial t} + \eta^x(t,x,y,z)\frac{\partial}{\partial x} + \eta^y(t,x,y,z)\frac{\partial}{\partial y} + \eta^z(t,x,y,z)\frac{\partial}{\partial z}$$

and we apply its second extension $X^{(2)}$ on each of these equations as in

$$(*) \qquad X^{(2)}\left(m\ddot{x} + V'(r)\frac{x}{r}\right)\Bigg|_{m\ddot{\vec{r}} = -V'(r)\frac{\vec{r}}{r}} = 0 \qquad \left(\begin{array}{l}\text{similarly}\\ \text{for } y \text{ and } z\end{array}\right)$$

- Using Mathematica with the ansatz

$$\xi = \Xi, \quad \eta^x = \alpha, \quad \eta^y = \beta \quad \text{and} \quad \eta^z = \gamma$$

we get conditions (we will use only simple ones, there are many others)
(from coefficients of the polynomial in $\dot{x}, \dot{y}, \dot{z}$)

$$\frac{\partial^2\alpha}{\partial y^2} = 0 = \frac{\partial^2\alpha}{\partial z^2} = \frac{\partial^2\alpha}{\partial y\partial z}$$

and similar conditions for β and γ with x,z and x,y insted of y,z

and

$$\frac{\partial^2\Xi}{\partial x^2} = 0 = \frac{\partial^2\Xi}{\partial y^2} = \frac{\partial^2\Xi}{\partial z^2}$$
$$\frac{\partial^2\Xi}{\partial x\partial y} = 0 = \frac{\partial^2\Xi}{\partial y\partial z} = \frac{\partial^2\Xi}{\partial x\partial z}$$

we can see that $\Xi(t,x,y,z)$ must be linear in x, y, and z

and $\alpha(t,x,y,z)$ is linear in y and z,
$\beta(t,x,y,z)$ is linear in x and z,
$\gamma(t,x,y,z)$ is linear in x and y.

- the second ansatz is

$$\xi(t,x,y,z) = \delta_x(t)\,x + \delta_y(t)\,y + \delta_z(t)\,z + \delta_0(t)$$
$$\eta^x(t,x,y,z) = \alpha_y(t,x)\,y + \alpha_z(t,x)\,z + \alpha_0(t,x)$$
$$\eta^y(t,x,y,z) = \beta_x(t,y)\,x + \beta_z(t,y)\,z + \beta_0(t,y)$$
$$\eta^z(t,x,y,z) = \gamma_x(t,z)\,x + \gamma_y(t,z)\,y + \gamma_0(t,z)$$

- using the infinitesimal criterion (*) again we get, for example, conditions

$$\frac{\partial^2 \alpha_0(t,x)}{\partial x^2} = 2\,\frac{d\delta_x(t)}{dt}$$ and similarly for $\frac{\partial^2\beta_0}{\partial y^2}$ and $\frac{\partial^2\gamma_0}{\partial z^2}$

$$\Rightarrow \quad \alpha_0(t,x) = \delta_x'(t)\,x^2 + \alpha_0^1(t)\,x + \alpha_0^0(t)$$
$$\beta_0(t,y) = \delta_y'(t)\,y^2 + \beta_0^1(t)\,y + \beta_0^0(t)$$
$$\gamma_0(t,z) = \delta_z'(t)\,z^2 + \gamma_0^1(t)\,y + \gamma_0^0(t)$$

furthermore we have

$$\frac{\partial \alpha_y(t,x)}{\partial x} = \frac{d\delta_y(t)}{dt} \quad\Rightarrow\quad \alpha_y(t,x) = \delta_y'(t)\,x + \alpha_y^0(t)$$
$$\frac{\partial \alpha_z(t,x)}{\partial x} = \frac{d\delta_z(t)}{dt} \quad\Rightarrow\quad \alpha_z(t,x) = \delta_z'(t)\,x + \alpha_z^0(t)$$

etc. for $\beta_x(t,y)$ and $\beta_z(t,y)$ which are linear in y, and for $\gamma_x(t,z)$ and $\gamma_y(t,z)$ which are linear in z.

- now we have the third ansatz in which there are only unknown functions of time, thus the infinitesimal criterion gives a polynomial not only in $\dot{x},\dot{y},\dot{z}$, but also in x,y,z (except there is the potential $V(r)$ ~~and~~ or more precisely its derivatives) which must be zero for any $x, y,$ and z.

• again using Mathematica we get first

$$\frac{d\alpha_y^o(t)}{dt} = 0$$ and the same condition for

$\alpha_z^o(t)$, $\beta_x^o(t)$, $\beta_z^o(t)$, $\gamma_x^o(t)$, and $\gamma_y^o(t)$

that is none of these is a function of time

and then $\alpha_y^o = -\beta_x^o$, $\alpha_z^o = -\gamma_x^o$ and $\beta_z^o = -\gamma_y^o$

there are no other conditions for these parameters

and we get expected generators of rotations

$$X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \quad (\text{for } \gamma_y^o = 1 \text{ and other parameters } 0)$$

$$X_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \quad (\text{for } \alpha_z^o = 1)$$

$$X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad (\text{for } \beta_x^o = 1)$$

These are of course a consequence of $V(r)$ being spherically symmetric.

• from the third ansatz we get further

$$\alpha_o^o(t)\left[V'(r) - r\,V''(r)\right] = 0$$ and the same for β_o^o and γ_o^o

if the potential $V(r)$ satisfies the equation

$$V'(r) = rV''(r) \Rightarrow V(r) = a_1 r^2 + a_2$$

where a_1 and a_2 are arbitrary

then the system has special point symmetries because α_o^o, β_o^o, and γ_o^o may be non-zero (generating translations)

This is the case of a free particle and also the case of a linear harmonic oscillartor.

In both cases we have the system of three independent one-dimensional equations and there are actually many more symmetries (remember point symmetries of $\frac{d^2y}{dx^2} = 0$)

- in the following we will treat only the cases for which $V'(r) \neq rV''(r)$

then, of course, $\alpha_0^0 = \beta_0^0 = \gamma_0^0 = 0$ and we also get

$$\delta_x = \delta_y = \delta_z = 0$$

from conditions of the form

$$\delta_x(t)\, V'(r) + mr\, \delta_x''(t) = 0 \quad \text{etc.}$$

which can give non-trivial $\delta_x(t)$ only for $V(r) = a_1 r^2 + a_2$

- finally, from conditions

$$\frac{d^2 \alpha_0^1(t)}{dt^2} = 0 = \frac{d^2 \beta_0^1(t)}{dt^2} = \frac{d^2 \gamma_0^1(t)}{dt^2} \Rightarrow \text{linearity in } t$$

and $$2\,\frac{d\alpha_0^1(t)}{dt} = \frac{d^2 \delta_0(t)}{dt^2} \Rightarrow \delta_0 \text{ is quadratic in } t$$

- the fourth ansatz is (the most general for $V'(r) \neq rV''(r)$ or $V(r) \neq a_1 r^2 + a_2$)

$$\xi(t,x,y,z) = \delta_0^2 t^2 + \delta_0^1 t + \delta_0^0$$

$$\eta^x(t,x,y,z) = c_1 y + c_2 z + (\delta_0^2 t + \alpha_0)\, x$$

$$\eta^y(t,x,y,z) = -c_1 x + c_3 z + (\delta_0^2 t + \beta_0)\, y$$

$$\eta^z(t,x,y,z) = -c_2 x - c_3 y + (\delta_0^2 t + \gamma_0)\, z$$

- the last conditions following from this ansatz are

$$\delta_0^2 \left[3V'(r) + rV''(r) \right] = 0$$

or $\delta_0^2 = 0$ unless $V(r) = \dfrac{b_1}{r^2} + b_2$

and $(2\delta_0^1 - \alpha_0)\, V'(r) + \alpha_0\, r V''(r) = 0$ (the same for β_0 and γ_0)

or $\alpha_0 = \delta_0^1 = 0 = \beta_0 = \gamma_0$ unless $V(r) = B_1 r^N + B_2$

when $(2\delta_0^1 - \alpha_0) N + \alpha_0 N(N-1) = 0$

and thus $$\alpha_0 = \beta_0 = \gamma_0 = \frac{2\delta_0^1}{2-N} \quad \text{for } N \neq 2$$

(special scaling of space and time)

- for $N=2$ we get $\delta_0^1 = 0$ and $\alpha_0, \beta_0, \gamma_0$ arbitrary

  in the other words, we can scale linear harmonic oscillator in space arbitrary and the period is still the same

- Summary:

  1) for a general potential $V(r)$
     $\Rightarrow$ 4-parametric LGT of rotations in space generated by X_1, X_2, and X_3 and time translation generated by
     $$X_4 = \frac{\partial}{\partial t}$$

  2) for $V(r) = B_1 r^N + B_2$ there is another point symmetry, scaling of time and space
     $$X_5 = t\frac{\partial}{\partial t} + \frac{2}{2-N}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$
     from this symmetry for $N=-1$ we get the third Kepler's law
     $$\left.\begin{aligned}\tilde{t} &= \alpha t\\ \tilde{r} &= \alpha^{2/3} r\end{aligned}\right\} \Rightarrow \frac{t^2}{r^3} = \text{konst}$$

  3) for $N=-2$ there is even another symmetry generated by
     $$X_6 = t^2\frac{\partial}{\partial t} + t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$