Example of finding a PDE of a given symmetry

- for simplicity we will look for a linear homogeneous partial differential equation of the second order for a scalar function $\psi(x^0, x^1, x^2, x^3)$ which is invariant under the Poincaré group generated by 10 infinitesimal operators

$$X_j = \varepsilon_{jkl} x^k \frac{\partial}{\partial x_l}, \quad j = 1,2,3 \quad \text{(space rotations)}$$

$$X_{3+j} = x^j \frac{\partial}{\partial x^0} + x^0 \frac{\partial}{\partial x^j}, \quad j=1,2,3 \quad \text{(Lorentz transformations)}$$

$$X_7 = \frac{\partial}{\partial x^0}, \quad X_{7+j} = \frac{\partial}{\partial x^j}, \quad j=1,2,3) \quad \text{(time and space translations)}$$

- even though we could look for a general equation

$$R\left(x^\mu, \psi, \frac{\partial \psi}{\partial x^\mu}, \frac{\partial^2 \psi}{\partial x^\mu \partial x^\nu}\right) = 0$$

we limit ourself (for simplicity) to a linear hom. equation

$$R = c\psi + a^\alpha \psi_{,\alpha} + b^{\mu\nu} \psi_{,\mu,\nu} = 0 \quad \text{where } \psi_{,\alpha} \equiv \frac{\partial \psi}{\partial x^\alpha} \text{ etc.}$$

Note that any explicit dependence on x^μ is forbidden due to invariance under all translations in x^μ.

Note also that $\psi_{,\mu\nu} = \psi_{,\nu\mu}$ but we will consider them separately to get more „symmetric" expressions

- the infinitesimal criterion must be satisfied for all 10 inf. operators. We have already used translations by omitting any dependence on x^μ. Now we apply the Lorentz transformations and then rotations, but we will see that rotations will be satisfied already, because d'Alambert operator is rotationally symmetric

- the second extensions of the generators of Lorentz transformations are $\left(\xi^{j0} = x^j ,\ \xi^{jk} = / \, x^0 \delta^{jk} \ \cancel{\text{for}} \right)$

$$X^{(2)}_{3+j} = X_{3+j} + \eta^j_\mu \frac{\partial}{\partial \psi_{,\mu}} + \eta^j_{\mu\nu} \frac{\partial}{\partial \psi_{,\mu\nu}}$$

where $\quad \eta^j_\mu = -(D_\mu \xi^\rho)\psi_{,\rho} = -\delta_{\mu j}\psi_{,0} - \delta_{\mu 0}\psi_{,j}$

and $\quad \eta^j_{\mu\nu} = D_\nu \eta^j_\mu - (D_\nu \xi^\rho)\psi_{,\mu\rho} =$

$$= -\delta_{\mu j}\psi_{,0\nu} - \delta_{\mu 0}\psi_{,j\nu} - \delta_{\nu j}\psi_{,\mu 0} - \delta_{\nu 0}\psi_{,\mu j}$$

- from the infinitesimal criterion we get

$$X^{(2)}_{3+j}\left(c\psi + a^\mu \psi_{,\mu} + b^{\mu\nu}\psi_{,\mu\nu}\right)\Big|_{R=0} = \left(a^\mu \eta^j_\mu + b^{\mu\nu}\eta^j_{\mu\nu}\right)\Big|_{R=0} = 0 \text{ for } j=1,2,3$$

we can express ψ from $R=0$, then we have no restriction

and thus

$$0 = \underbrace{-a^0\psi_{,j} - a^j\psi_{,0}}_{\Rightarrow\ a^0 = a^j = 0} + \underbrace{b^{j\nu}\psi_{,0\nu} + b^{0\nu}\psi_{,j\nu} + b^{\mu 0}\psi_{,\mu j} + b^{\mu j}\psi_{,\mu 0}}_{\Downarrow}$$

$$\psi_{,00}\left(b^{j0} + b^{0j}\right) = 0 \quad \Rightarrow \quad b^{0j} = b^{j0} = 0$$

(thanks to $b^{\mu\nu} = b^{\nu\mu}$)

and $\quad \psi_{,01}\left[b^{j1} + b^{1j} + \delta_{j1}\left(b^{00} + b^{00}\right)\right] = 0 \Rightarrow$

$$\Rightarrow \quad b^{12} = b^{13} = 0 \ , \quad b^{00} + b^{11} = 0$$

and similarly for $\psi_{,02}$ and $\psi_{,03}$ $\Rightarrow b^{23} = 0$

$$b^{00} + b^{22} = 0 \quad \text{and} \quad b^{00} + b^{33} = 0$$

- finally we reduced $R=0$ to

$$R = c\psi + b^{00}\left(\psi_{,00} - \psi_{,11} - \psi_{,22} - \psi_{,33}\right) = 0$$

in other words we get the Klein-Gordon equation

$$m^2\psi + \Box\psi = 0 \quad \text{for } m^2 = \frac{c}{b^{00}}$$

Check yourself that $\quad X^{(2)}_j\left(m^2\psi + \Box\psi\right)\Big|_{m^2\psi + \Box\psi = 0} = 0$