Solving ODE using point symmetries

- knowledge of point symmetries of a given ODE can help us to simplify or to solve this equation, in particular:

  1) one-param. LGT under which a given ODE of the first order is invariant transforms the ODE into a solvable form $\Rightarrow$ straightforward integration (if the symmetry is non-trivial, details later)

  2) if the ODE is of the second or higher order, such a symmetry can be used to reduce the order

  3) if there is an r-param. LGT under which the ODE is invariant, then it can be used to reduce the order by r if the LGT is solvable

- two basic methods:

  a) canonical (normal) variables - „natural" variables for a certain 1-par. LGT $\Rightarrow$ a simpler form of the ODE

  b) method of differential invariants - rewrite the ODE using expressions invariant under the 1-par. LGT

- in both cases we need to solve PDE of the first order of the type

$$Xr(x,y) = \xi(x,y)\frac{\partial r}{\partial x} + \eta(x,y)\frac{\partial r}{\partial y} = 0$$

or

$$Xs(x,y) = \xi(x,y)\frac{\partial s}{\partial x} + \eta(x,y)\frac{\partial s}{\partial y} = 1$$

such equations can be often solved using the method of characteristics

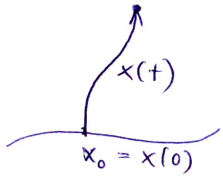
(details can be found in L.C. Evans: PDEs, chapter 3) even for more general case

- let us consider the PDE

$$\sum_{i=1}^{n} b^i(x) \frac{\partial u(x)}{\partial x^i} = c(x,u) \qquad (*)$$

for an unknown scalar function $u = u(x^1, \dots, x^n)$

- we search a solution along a characteristic starting at a certain point x_0 where we have the given initial value $u(x_0) = u_0$



- it can be proved that the solution along the characteristic given by

$$\frac{dx^i(t)}{dt} = b^i(x(t)) \quad \text{for } i = 1, \dots, n \,, \quad x(0) = x_0$$

satisfies the equation

$$\frac{du(t)}{dt} = c(x(t), u(t)) \quad , \quad u(0) = u(x_0) = u_0$$

the equations of characteristic can be also written as

$$\frac{dx^1}{b^1(x)} = \frac{dx^2}{b^2(x)} = \dots = \frac{dx^n}{b^n(x)}$$

which is a system of $(n-1)$ equations. The solutions of this system depend on $(n-1)$ integration constants

- It can be shown, that if we write these constants r_j as functions of x, they will be $(n-1)$ independent solutions of $(*)$ with $c(x,u) = 0$, i.e.

$$\sum_{i=1}^{n} b^i(x) \frac{\partial r_j(x)}{\partial x^i} = 0 \quad \text{for } j = 1, \dots, n-1$$

(in this case we have $\frac{du(t)}{dt} = 0 \Rightarrow u(t) = \text{const} = u_0$ along the characteristic)

- if $c(x,u) \neq 0$, then if $s(x)$ is a solution of $(*)$ then also $s(x) + \sum_{j=1}^{n-1} \alpha^j r_j(x)$ is a solution

- for $c(x,u) = 1$ we can write

$$dt = \frac{ds}{1} = \frac{dx^i}{b^i(x)} \quad \text{for a certain } i \text{ and integrate}$$