Canonical variables

- first, let us consider a general change of variables in the ODE from (x,y) to (r,s) (we know how to rewrite the ODE to new variables from the first lecture)

  if this ODE is invariant under a 1-par. LGT generated by

$$X^{(x,y)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

  then the ODE in new variables (r,s) will be invariant under a 1-par. LGT expressed in these new variables, now generated by $X^{(r,s)} = \alpha(r,s)\frac{\partial}{\partial r} + \beta(r,s)\frac{\partial}{\partial s}$

  - how to obtain α, β from the original ξ and η?

    let us apply $X^{(r,s)}$ to a general function $F(r,s)$ which can be considered as a function of (x,y) because $r=r(x,y)$ and $s=s(x,y)$, so we can also apply $X^{(x,y)}$ to it:

$$X^{(r,s)}F(r,s) = \alpha(r,s)\frac{\partial F}{\partial r} + \beta(r,s)\frac{\partial F}{\partial s}$$

and

$$X^{(x,y)}F(r(x,y),s(x,y)) = \xi(x,y)\left(\frac{\partial F}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial x}\right) + $$
$$+ \eta(x,y)\left(\frac{\partial F}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial y}\right) =$$
$$= \left[X^{(x,y)}r(x,y)\right]\frac{\partial F}{\partial r} + \left[X^{(x,y)}s(x,y)\right]\frac{\partial F}{\partial s}$$

by comparison we get

$$\alpha(r(x,y),s(x,y)) = X^{(x,y)}r(x,y)$$
$$\beta(r(x,y),s(x,y)) = X^{(x,y)}s(x,y)$$

or

$$\alpha(r,s) = X^{(x,y)}r(x,y)\Big|_{\substack{x=x(r,s)\\ y=y(r,s)}}, \quad \beta(r,s) = X^{(x,y)}s(x,y)\Big|_{\substack{x=x(r,s)\\ y=y(r,s)}}$$

Example: we know that $\frac{dy}{dx} = \frac{x+y}{x-y}$ is invariant under rotations generated by

$$\tilde{x} = x\cos\varphi - y\sin\varphi \qquad X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$
$$\tilde{y} = x\sin\varphi + y\cos\varphi$$

Let us transform the equation and X into the polar coordinates:

$$x = r\cos\varphi = F \qquad\qquad r = \sqrt{x^2+y^2} \quad \text{dependent var.}$$
$$\Longleftrightarrow$$
$$y = r\sin\varphi = G \qquad\qquad \varphi = \mathrm{arctg}\left(\frac{y}{x}\right) \quad \text{indep. var.}$$

for the derivative we get

$$\frac{dy}{dx} = \frac{D_\varphi G}{D_\varphi F} = \frac{\frac{dr}{d\varphi}\sin\varphi + r\cos\varphi}{\frac{dr}{d\varphi}\cos\varphi - r\sin\varphi} = \frac{x+y}{x-y} = \frac{r(\cos\varphi+\sin\varphi)}{r(\cos\varphi-\sin\varphi)}$$

or $$\frac{dr}{d\varphi}\left(\cancel{\sin\varphi\cos\varphi}\overbrace{-\sin^2\varphi - \cos^2\varphi}^{-1} - \cancel{\sin\varphi\cos\varphi}\right) =$$
$$= -r\left(\cancel{\sin\varphi\cos\varphi} + \underbrace{\sin^2\varphi + \cos^2\varphi}_{1} - \cancel{\cos\varphi\sin\varphi}\right).$$

$$\Rightarrow \quad \frac{dr}{d\varphi} = r$$

this new equation is clearly invariant under translations in φ generated by $\frac{\partial}{\partial\varphi}$ which is actually X expressed in (r,φ):

$$X^{(r,\varphi)} = (Xr)\Big|_{\substack{x=r\cos\varphi\\ y=r\sin\varphi}}\frac{\partial}{\partial r} + (X\varphi)\Big|_{\substack{x=r\cos\varphi\\ y=r\sin\varphi}}\frac{\partial}{\partial\varphi}$$

where $$X\left(\sqrt{x^2+y^2}\right) = X\big(f(\underbrace{x^2+y^2}_{z})\big) = -y\frac{\partial f}{\partial z}\underbrace{2x}_{\frac{\partial z}{\partial x}} + x\frac{\partial f}{\partial z}\underbrace{2y}_{\frac{\partial z}{\partial y}} = 0$$

$$X\left(\mathrm{arctg}\left(\frac{y}{x}\right)\right) = \frac{-y\left(-\frac{y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} + \frac{x\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = 1$$

$$\Rightarrow \quad X^{(r,\varphi)} = \frac{\partial}{\partial\varphi}$$

we will see that (r,φ) are actually the canonical variables for $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and we will get

$$\frac{d\varphi}{dr} = \frac{1}{r} \Rightarrow \ln r = \varphi + c \Rightarrow \frac{1}{2}\ln(x^2+y^2) = \mathrm{arctg}\frac{y}{x} + c$$

(a solution written as an implicit equation for $y = y(x)$)

- if we would have more variables, e.g.

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \qquad \text{(scaling in space)}$$

we can change variables in the exactly same way, e.g. if

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \varphi &= \operatorname{arctg}\left(\frac{y}{x}\right) \\ \theta &= \operatorname{arctg}\frac{\sqrt{x^2+y^2}}{z} \\ r &= \sqrt{x^2+y^2+z^2} \end{aligned}$$

we would get in spherical coordinates

$$X^{(r,\theta,\varphi)} = \underbrace{(X\varphi)\Big|_{\substack{x=\dots\\ y=\dots\\ z=\dots}}}_{0}\frac{\partial}{\partial\varphi} + \underbrace{(X\theta)\Big|_{\substack{x=\dots\\ y=\dots\\ z=\dots}}}_{0}\frac{\partial}{\partial\theta} + \underbrace{(Xr)\Big|_{\substack{x=\dots\\ y=\dots\\ z=\dots}}}_{r}\frac{\partial}{\partial r} = r\frac{\partial}{\partial r}$$

• definition of canonical variables

- for a given infinitesimal generator

$$X^{(x,y)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

the canonical variables (or coordinates) are functions $r(x,y)$ and $s(x,y)$ satisfying equations

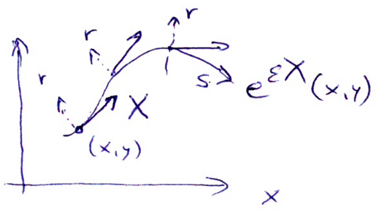
$$\boxed{X^{(x,y)}r(x,y) = 0 \quad \text{and} \quad X^{(x,y)}s(x,y) = 1}$$

In these coordinates the generator $X^{(x,y)}$ has a form

$$X^{(r,s)} = \frac{\partial}{\partial s}$$

i.e. it is a generator of translations (along s)

they are such coordinates, that s runs along the integral curves of the vector field X and r is a coordinate „complementary" to s



Example: canonical variables for rotations in (x,y)-plane generated by $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

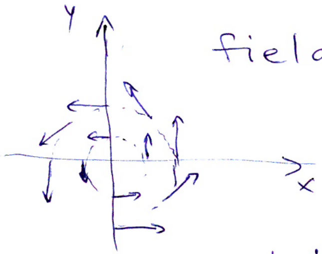
1) $r(x,y)$ satisfies
$$X r(x,y) = 0 = -y\frac{\partial r}{\partial x} + x\frac{\partial r}{\partial y}$$

using the method of characteristics we get equations

$$(1) \qquad \frac{dx(t)}{dt} = -y(t), \quad \frac{dy(t)}{dt} = x(t) \quad \text{and} \quad \frac{dz(t)}{dt} = 0 \quad \text{for } z(t) = r(x(t), y(t))$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$ and $z(0) = g(x_0, y_0)$
is a value of r in (x_0, y_0)

when looking for canonical variables, the characteristic can begin in an arbitrary point and also the initial value $g(x_0,y_0)$ can be arbitrary. Therefore we choose some simple initial conditions according to the field X, e.g. along a semiaxis $y_0 = 0$, $x_0 > 0$
and $g(x_0, y_0) = g(x_0)$ for now arbitrary



solution of (1) with these conditions is
(both $x(t)$ and $y(t)$ satisfy the equation of linear harmonic osc.)

$$x(t) = x_0 \cos t \qquad z(t) = g(x_0)$$
$$y(t) = x_0 \sin t$$

by choosing a certain point (x,y), there is a corresponding x_0 and t given by
$$x_0 = \sqrt{x^2+y^2}$$
$$t = \operatorname{arctg}\frac{y}{x}$$

and the solution of $Xr = 0$ can be finally written as

$$r(x,y) = r(x(t), y(t)) = z(t) = g(x_0) = g\left(\sqrt{x^2+y^2}\right)$$

an arbitrary function

and as a standard choice of r we take

$$r = \sqrt{x^2+y^2}$$

2) $s(x,y)$: satisfies

$$X s(x,y) = -y \frac{\partial s}{\partial x} + x \frac{\partial s}{\partial y} = 1$$

the characteristic equations now are

$$\frac{dx(t)}{dt} = -y(t), \quad \frac{dy(t)}{dt} = x(t) \quad \text{and} \quad \frac{dz(t)}{dt} = 1$$

for $z(t) = s(x(t), y(t))$

with initial conditions

$$x(0) = x_0, \; y(0) = y_0 = 0 \text{ and } z(0) = g(x_0)$$

again an arbitrary function

the solution is

$$x(t) = x_0 \cos t \quad \text{and} \quad z(t) = g(x_0) + t$$
$$y(t) = x_0 \sin t$$

or

$$s(x,y) = z(t) = g(x_0) + \operatorname{arctg} \frac{y}{x} = \underbrace{g(\sqrt{x^2+y^2})}_{\substack{\text{general}\\ \text{solution of}\\ Xr=0}} + \underbrace{\operatorname{arctg} \frac{y}{x}}_{\substack{\text{particular}\\ \text{solution of}\\ Xs=1}}$$

the usual choice of the second canonical variable is

$$s(x,y) = \operatorname{arctg} \frac{y}{x} = \varphi$$

This means that the canonical variables for rotations in a plane are, of course, the polar coordinates (r, φ)

We woud get the same result by solving

$$(dt =) \frac{dx}{-y} = \frac{dy}{x} \Rightarrow x\,dx = -y\,dy \Rightarrow x^2 = -y^2 + c$$

where c is an arbitrary integration constant, which can be written e.g. as $c = r^2$ and thus $r = \sqrt{x^2+y^2}$ is a solution of $Xr = 0$

For $s(x,y)$ we have

$$(dt =) \frac{ds}{1} = \frac{dy}{x}$$ where x is taken along the characteristic as $x = \sqrt{r^2 - y^2}$

by integration we get

$$ds = \frac{dy}{\sqrt{r^2-y^2}} \Rightarrow s = \operatorname{arctg} \frac{y}{\sqrt{r^2-y^2}} + g(x_0) = \varphi + \varphi_0$$

Again we choose $\varphi_0 = 0$, thus $s = \varphi$.

- canonical variables and solution of ODE of the first order

- if we have a general ODE of the first order

$$y_1 = \frac{dy}{dx} = f(x,y) \qquad (*)$$

which is invariant under a 1-par. LGT generated by

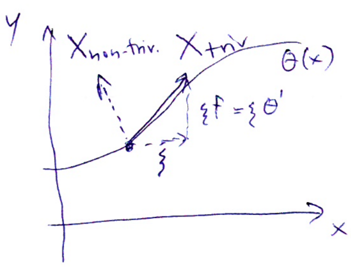
$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

we can use this point symmetry to solve $(*)$

if 1) it is non-trivial point symmetry, that is

if $\eta(x,y) \neq \xi(x,y)\, f(x,y)$

(or more generally $\eta(x,\theta(x)) \neq \xi(x,\theta(x))\,\theta'(x)$ where $\theta(x)$ is a solution of $(*)$)



As we can easily prove if $\eta = \xi f$ then X will generate a symmetry for an arbitrary ξ:

the inf. criterion gives $\left(\eta^{(1)}(x,y,y_1) = \xi D_x f + (D_x \xi)(f - y_1)\right)$

$$X^{(1)}(y_1 - f(x,y))\Big|_{y_1=f} = \eta^{(1)} - \xi\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y}\Big|_{\substack{y_1=f \\ (\eta=\xi f)}} = 0$$

- geometrically this means that X is in the direction of a solution $y=\theta(x)$ going through a point (x,y)
(such a 1-par LGT transforms a solution into the same solution)

2) it is easier to find canonical coordinates than to actually solve the equation $(*)$

we can either to guess what the canonical coordinates are, or we can solve $Xr=0$ and $Xs=1$ using

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{ds}{1}$$

Notice that for a trivial symmetry $\eta = \xi f$ we have

$$\frac{dx}{\xi} = \frac{dy}{\xi f} \Rightarrow \frac{dy}{dx} = f(x,y)$$ which is the same as $(*)$

- it can be shown in general, that ODEs of the first order have always a non-trivial point symmetry, but it is difficult to find them (more difficult than to solve $(*)$ usually)

Example: let us have $y_1 = \frac{dy}{dx} = f\left(\frac{y}{x}\right)$ with f arbitrary

- we can „see" that this equation is invariant under scaling $\tilde{x} = \alpha x$, $\tilde{y} = \alpha y$ with the generator $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

  (check this using the inf. criterion)

- canonical coordinates are given by

  this is our choice of integration const.

  $$X r(x,y) = 0 \Rightarrow \frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x = \ln y - \ln r \Rightarrow r = \frac{y}{x}$$

  $$X s(x,y) = 1 \Rightarrow \frac{ds}{1} = \frac{dy}{y} \Rightarrow s = \ln y$$

  also our choice, we could also use $\frac{ds}{1} = \frac{dx}{x}$ and set $s = \ln x$, leading also to the same solution (try it :-)

- to change variables in $y_1 = f\left(\frac{y}{x}\right)$ we express

  $$y = e^s \quad \text{and} \quad x = \frac{e^s}{r}$$

  and find

  $$\frac{dy}{dx} = \frac{D_r(e^s)}{D_r\left(\frac{e^s}{r}\right)} = \frac{\frac{ds}{dr}e^s}{\frac{ds}{dr}\frac{e^s}{r} - \frac{e^s}{r^2}} = \frac{r^2\frac{ds}{dr}}{r\frac{ds}{dr} - 1} = f\left(\frac{y}{x}\right) = f(r)$$

  or

  $$\frac{ds}{dr} = \frac{f(r)}{f(r)\,r - r^2} = G(r)$$

  as it should be because we should get an equation invariant under translations in s generated by $X = \frac{\partial}{\partial s}$

- the solution of the original equation $y_1 = f\left(\frac{y}{x}\right)$ is given implicitly as

  $$s(x,y) = \int^{r(x,y)} G(r')\,dr' + C$$

- for $f\left(\frac{y}{x}\right) = \frac{y}{x}$ we would get $G(r) = \frac{r}{r^2 - r^2}$ and in this case we cannot use this procedure, because X is actully a trivial point symmetry of $y_1 = \frac{y}{x}$ as $\frac{\eta}{\xi} = \frac{y}{x} = f$, but in this case we can integrate $y_1 = \frac{y}{x}$ directly

- if $f\left(\frac{y}{x}\right) \neq \frac{y}{x}$, it is the non-trivial symmetry and we get a solution; for example

for $f\left(\frac{y}{x}\right) = \frac{x}{y}$ we have $f(r) = \frac{1}{r}$ and $G(r) = \frac{1}{r(1-r^2)}$

and $\ln y = s = \ln r - \frac{1}{2}\ln(r^2-1) + \ln C$

$$\Rightarrow \quad y = \pm\sqrt{x^2+c^2}$$

or for $f\left(\frac{y}{x}\right) = \frac{x}{x-y} = \frac{1}{1-\frac{y}{x}}$ we find

$G(r) = \frac{1}{r-r^2+r^3}$ $\Rightarrow$ by integrating using partial fractions

$$\ln y = s = -\frac{1}{2}\ln(r^2-r+1) + \ln r + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2r-1}{\sqrt{3}} + C$$

which is an implicit equation for $y = y(x)$
(compare with the solution obtained by Mathematica using DSolve)

<u>Example</u>: linear homogeneous equation

$$\frac{dy}{dx} + p(x)\,y = y_1 + p(x)\,y = 0$$

this equation can be solved directly by integration

$$\frac{y_1}{y} = -p(x) \quad \Rightarrow \quad \int^x \frac{y_1}{y}\,dx' = \int^x -p(x')\,dx' \quad \Rightarrow$$

$$\Rightarrow \quad \ln y = -\int^x p(x')\,dx' + \ln c \quad \Rightarrow \quad y = c\,e^{-\int^x p(x')\,dx'}$$

the possibility of integration actually follows from the scaling symmetry generated by $X = y\frac{\partial}{\partial y}$ $\Rightarrow$ $\hat{x} = x$, $\hat{y} = e^{\varepsilon}y$ which is clearly a symmetry of any linear homogeneous equations (of any order)

by finding the canonical variables

$Xr = 0 \Rightarrow r = x$ (trivial)

$Xs = 1 \Rightarrow s = \ln y$ $\left(\text{solution of } y\frac{\partial s}{\partial y} = 1\right)$

we get

$$\frac{ds}{dr} = \frac{D_x s}{D_x r} = \frac{y_1}{y} = -p(r)$$

and

$$\ln y = s = -\int^{r=x} p(r')\,dr' + \ln c$$

<u>Example:</u> linear inhomogeneous equation

$$y_1 + p(x)\,y = g(x) \qquad (*)$$

each lin. inhom. equation of any order is invariant under

$$\tilde{x} = x \qquad \tilde{y} = y + \varepsilon \phi(x)$$

where $\phi(x)$ is an arbitrary solution of homogeneous equation (here $\phi'(x) + p(x)\phi(x) = 0$)

generated by

$$X = \phi(x)\frac{\partial}{\partial y}$$

thus, if we know $\phi(x)$ we can use it to find a particular solution of $(*)$:

the canonical variables now are

$$Xr = 0 \implies r = x$$

$$Xs = 1 \implies s = \frac{y}{\phi(x)}$$

(our simplest choice, r could be any function of x and we could add to s also any function of x)

and we have

$$\frac{ds}{dr} = \frac{D_x s}{\underbrace{D_x r}_{=1}} = -\frac{\phi'(x)\,y}{\phi^2(x)} + \frac{y_1}{\phi(x)} = \frac{1}{\phi(x)}\left(y_1 + p(x)\,y\right) = \frac{g(x)}{\phi(x)} = \frac{g(r)}{\phi(r)}$$

$$\frac{\phi'(x)}{\phi(x)} = -p(x)$$

thus we can integrate

$$s = \int^r \frac{g(r')}{\phi(r')}\,dr' + C$$

and in the original coordinates

$$s = \frac{y}{\phi(x)} = \int^x \frac{g(r')}{\phi(r')}\,dr' + C \implies y = \underbrace{\phi(x)\int^x \frac{g(r')}{\phi(r')}\,dr'}_{\text{a particular solution}} + \underbrace{C\phi(x)}_{\text{any solution of hom. eq.}}$$

particularly for

$$y' + y = x \qquad \text{we have} \qquad \phi(x) = e^{-\int^x 1\,dx'} = e^{-x}$$

and we get

$$y = e^{-x}\int^x \frac{r}{e^{-r}}\,dr + Ce^{-x} = e^{-x}\left[e^x(x-1) + C\right] = x - 1 + Ce^{-x}$$

- reduction of order of ODEs using canonical variables

- if we have an ODE of higher order

$$\frac{d^k y}{dx^k} \equiv y_k = f(x, y, y_1, \dots y_{k-1}) \qquad (*)$$

which is invariant under a point symmetry generated by $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$

which is non-trivial, i.e. $\eta \neq \xi\, \theta'(x)$, where $\phi(x)$ is a solution of $(*)$,

then by changing variables to canonical (r,s)

we finally get

$$\frac{d^k s}{dr^k} = G\left(r, \frac{ds}{dr}, \dots, \frac{d^{k-1} s}{dr^{k-1}}\right)$$

because it must be invariant under $X^{(r,s)} = \frac{\partial}{\partial s}$

and therefore, independent explicitly on s

- by substituting $z = \frac{ds}{dr}$ the equation will have the order reduced by 1

$$\frac{d^{k-1} z}{dr^{k-1}} = G\left(r, z, \frac{dz}{dr}, \dots, \frac{d^{k-2} z}{dr^{k-2}}\right)$$

and from the solution $z = \phi(x)$ of this equation we can obtain the solution of $(*)$ given implicitly by

$$s(x,y) = \int^{r(x,y)} \phi(r')\, dr' + C$$

Example: linear homogeneous ODE of the second order

$$y_2 + p(x)\, y_1 + q(x)\, y = 0$$

again invariant under scaling in y

$$\tilde{x} = x$$
$$\tilde{y} = e^{\varepsilon} y \qquad \text{generated by } X = y\frac{\partial}{\partial y}$$

with canonical variables $r = x$, $s = \ln y$

- from the inverse formulas $x = r,\ y = e^s$

we get
$$\frac{dy}{dx} = \frac{D_r y(r,s)}{D_r x(r,s)} = e^s \frac{ds}{dr}$$
$$\frac{d^2y}{dx^2} = \frac{D_r\left(e^s \frac{ds}{dr}\right)}{D_r x} = e^s\left[\frac{d^2s}{dr^2} + \left(\frac{ds}{dr}\right)^2\right]$$

and thus
$$y_2 + p(x)\,y_1 + q(x)\,y = e^s\left[\frac{d^2s}{dr^2} + \left(\frac{ds}{dr}\right)^2 + p(r)\frac{ds}{dr} + q(r)\right] = 0$$

- by using $z = \frac{ds}{dr} = \frac{y_1}{y}$ (the so-called Riccati transformation)

we get
$$\frac{dz}{dr} + z^2 + p(r)\,z + q(r) = 0$$

the equation of the Riccati type (reduced by one order, but nonlinear)

- particularly, for linear harmonic oscillator
$$y_2 + \omega^2 y = 0 \qquad (\cdot\ p(x) = 0\ ,\ q(x) = \omega^2)$$

we have
$$\frac{dz}{dr} + z^2 + \omega^2 = 0$$

which can be integrated (thanks to invariance under translations in $r = x$)

and the solution is
$$\frac{ds}{dr} = z(r) = -\omega\, \mathrm{tg}[\omega(r - c_1)]$$
$$\Rightarrow \ln y = s(r) = \ln \cos \omega(r - c_1) + \ln A$$
$$\Rightarrow y = A \cos \omega(x - c_1) \qquad \text{(as expected)}$$

· Notes: 1) It should not be surprising we could integrate out the equation $y_2 + \omega^2 y = 0$, because it is invariant under the 2-par. LGT with generators $X_1 = \frac{\partial}{\partial x}$, $X_2 = y\frac{\partial}{\partial y}$ which is solvable: $[X_1, X_2] = 0$ (see notes later for solvable Lie algebras)

2) To solve $y_2 + \omega^2 y = 0$ we could also begin with the symmetry $X_1 = \frac{\partial}{\partial x}$ (translations in x) with the canonical variables

$r = y$, $s = x$ (swapping of dependent and independent variables)

- we would get

$$\frac{dy}{dx} = \frac{D_r y}{D_r x} = \frac{1}{\frac{ds}{dr}}, \quad \frac{d^2y}{dx^2} = \frac{D_r\left(\frac{1}{\frac{ds}{dr}}\right)}{D_r x} = -\frac{\frac{d^2s}{dr^2}}{\left(\frac{ds}{dr}\right)^3}$$

and from $y_2 = -\omega^2 y$ we have

$$\frac{d^2s}{dr^2} = \omega^2 r \left(\frac{ds}{dr}\right)^3 \Rightarrow \frac{dz}{dr} = \omega^2 r z^3$$

which is invariant under $\tilde{z} = \frac{1}{\alpha} z$, $\tilde{r} = \alpha r$ $\left(\text{scales as } \frac{1}{y_1} \text{ for } \tilde{y} = \alpha y \text{ because } z = 1/y_1\right)$

and we can finally integrate again up to the solution of $y_2 + \omega^2 y = 0$:

$$\int \frac{dz}{z^3} = \int \omega^2 r\, dr + c \Rightarrow -\frac{1}{z^2} = \omega^2 r^2 - c_1^2$$

$$\Rightarrow \frac{ds}{dr} = z = \frac{1}{\sqrt{c_1^2 - \omega^2 r^2}} \Rightarrow x = s = \frac{1}{\omega} \operatorname{arctg} \frac{\omega \overset{=y}{r}}{\sqrt{c_1^2 - \omega^2 \underset{=y}{r^2}}} + c_2$$

$$\Rightarrow \frac{\omega y}{c_1} = \sin \omega (x - c_2) \Rightarrow y = A \sin \omega (x - x_0)$$

- <u>Finding ODEs of a given symmetry using canonical variables</u>

- we can always use directly the infinitesimal criterion to find $y_k = f(x, y, y_1, \dots, y_{k-1})$ invariant under the r-par. LGT generated by X_j, $j = 1, \dots, r$ from

$$X_j^{(k)} (y_k - f)\Big|_{y_k = f} = 0, \quad j = 1, \dots, r$$

we would get r conditions (PDEs of the first order linear in derivatives $\Rightarrow$ method of characteristic)

- but if we have a particular X_j, we can also find its canonical variables (r, s) in whic $X_j^{(r,s)} = \frac{\partial}{\partial s}$ and thus the most general equation invariant under the 1-par. LGT generated by X_j must be

$$\frac{d^k s}{dr^k} = G\left(r, \frac{ds}{dr}, \dots, \frac{d^{k-1}s}{dr^{k-1}}\right)$$

where G is an arbitrary function

Then we can transform this equation back to the original varibles (x, y)

Example: let us check that $y_1 = f\left(\frac{y}{x}\right)$ is the most general ODE of the first order invariant under scaling generated by $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

a) direct use of the inf. criterion

$$X^{(1)} = X \quad \Rightarrow \quad X^{(1)}\left(y_1 - \tilde{f}(x, y)\right)\Big|_{y_1 = \tilde{f}} = -x\frac{\partial \tilde{f}}{\partial x} - y\frac{\partial \tilde{f}}{\partial y} = 0$$

gives equation for characteristic

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x = \ln y + c \Rightarrow$$

our choice of a constant which is a solution if expressed as a function of (x, y)

$$\Rightarrow \quad \tilde{f}(x, y) = f\left(\frac{y}{x}\right)$$

any function

b) using the canonical variables

we know that $r = \frac{y}{x}$, $s = \ln y$ are a suitable choice of the can. var. for X

the most general ODE then is

$$\frac{ds}{dr} = G(r) \quad \text{where } G \text{ is any function}$$

but

$$\frac{ds}{dr} = \frac{D_x s}{D_x r} = \frac{\frac{y_1}{y}}{-\frac{y}{x^2} + \frac{y_1}{x}} = \frac{y_1}{r y_1 - r^2} = G(r)$$

and thus

$$y_1 = \frac{r^2 G(r)}{r G(r) - 1} = f(r) = f\left(\frac{y}{x}\right)$$

notice that y_1 and $\frac{y}{x}$ are both invariants satisfying

$$X^{(1)} y_1 = 0, \quad X^{(1)}\left(\frac{y}{x}\right) = 0$$

Example: let us find the most general ODE of the second order $\ddot{x} = f(t, x, \dot{x})$

invariant under the Galilean transformation generated by $X = t\frac{\partial}{\partial x}$.

- as the canonical variables we can take

$$\left.\begin{array}{lll} Xr = 0 & \Rightarrow & r = t \\ Xs = 1 & \Rightarrow & s = \frac{x}{t} \end{array}\right\} \Rightarrow \frac{ds}{dr} = \frac{t\dot{x} - x}{t^2}$$

and
$$\frac{d^2 s}{dr^2} = D_t\left(\frac{t\dot{x}-x}{t^2}\right) = -\frac{2(t\dot{x}-x)}{t^3} + \frac{\ddot{x}}{t}$$

- the most general ODE then is

$$\frac{d^2 s}{dr^2} = G\left(r, \frac{ds}{dr}\right)$$

or
$$\ddot{x} = tG\left(t, \frac{t\dot{x}-x}{t^2}\right) + \frac{2(t\dot{x}-x)}{t^2} \Rightarrow$$

$$\Rightarrow \quad \ddot{x} = f(t, t\dot{x} - x)$$

(again $\ddot{x}$, t, and $t\dot{x} - x$ are all invariants: $X^{(2)}\ddot{x} = 0$, $X^{(2)}t = 0$, $X^{(2)}(t\dot{x}-x) = 0$)

- check that we would get the same result using the inf. criteriton with

$$X^{(2)} = t\frac{\partial}{\partial x} + \frac{\partial}{\partial \dot{x}}$$

Example: find $y_1 = f(x, y)$ invariant under $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

canonical coordinates

$$Xr = 0 \Rightarrow \frac{dx}{x} = \frac{dy}{1} \Rightarrow \ln x = y + \ln r \Rightarrow r = xe^{-y}$$

$$Xs = 1 \Rightarrow dy = ds \Rightarrow s = y$$

the most general equation is

$$\frac{ds}{dr} = G(r) \Rightarrow \frac{ds}{dr} = \frac{D_x s}{D_x r} = \frac{y_1}{e^{-y}(1 - xy_1)} = G(xe^{-y})$$

or
$$y_1\left[1 + xe^{-y}G(xe^{-y})\right] = \frac{xe^{-y}}{x}G(xe^{-y})$$

$$X^{(1)} = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - y_1\frac{\partial}{\partial y_1}$$

$$xy_1 = h(xe^{-y})$$

$$X^{(1)}(xy_1) = 0$$

once again xy_1 and xe^{-y} are invariants: $X^{(1)}(xe^{-y}) = 0$