Using point symmetries to solve PDEs

- there is no general method how to solve PDEs
$$R^\sigma(x, u, \partial u, \dots, \partial^k u) = 0 \ , \ \sigma = 1, \dots, N$$
but symmetries can help us to find particular solutions

- ODEs have solutions depending on k constants and they can be obtained in principle by k integrations
  on the other hand, PDEs have in general infinite number of independent solutions and a specific solution is given by boundary or initial conditions

- we will show here how to obtain some of them using point symmetries via three basic methods:

1) <u>construction of a new (often non-trivial) solution from another solution (usually trivial) using a finite point symmetry</u>

- we actually saw this in the first lecture, in general we use transformation
$$\tilde{u}(\tilde{x}, \tilde{t}) = G^n\left(F(\tilde{x}, \tilde{u}), \varepsilon^{-1}\right), \Theta\left(F(\tilde{x}, \tilde{u}, \varepsilon^{-1})\right), \varepsilon)$$
if we have a solution $u = \Theta(x)$ of a PDE $R^\sigma(x, u, \partial u, \dots \partial^k u) = 0$ invariant under $\tilde{x} = F(x, u)$, $\tilde{u} = G(x, u)$

<u>Example:</u> heat equation $u_t = u_{xx}$

the trivial solution $u(x,t) = A = \text{const}$ is transformed under
$$\tilde{x} = x + 2\varepsilon t \ , \ \tilde{t} = t$$
$$\tilde{u} = u e^{-\varepsilon x - \varepsilon^2 t}$$
(a point symmetry of the heat equation)

into
$$u(x,t) = A e^{-\varepsilon x + \varepsilon^2 t} \quad \text{for an arbitrary } \varepsilon$$

2) using infinitesimal operators of point symmetries to find new solutions of linear PDEs

- if we have a linear PDE or a system of lin. PDE

$$H^\sigma_{(x)} u = G^\sigma(x) \quad , \quad \sigma = 1, \dots, N$$

where H^σ is a linear differential operator

$$H^\sigma(x) u = \sum_{\alpha=1}^{m} b^\sigma_\alpha(x) u^\alpha + \sum_{\alpha, j} b^\sigma_{\alpha j}(x) \frac{\partial u^\alpha}{\partial x^j} + \dots$$

it is always invariant under point symmetries generated by $\quad X_\infty = \sum_{\alpha=1}^{m} \eta^\alpha(x) \frac{\partial}{\partial u^\alpha}$

where $\eta^\alpha(x)$ is an arbitrary solution of the homogeneous equation

$$H^\sigma \eta(x) = 0$$

because (notice that $\xi = 0$)

$$X^{(k)}_\infty = X_\infty + \sum_{\alpha, j} (D_j \eta^\alpha) \frac{\partial}{\partial u^\alpha_j} + \sum_{\alpha, j_1, j_2} (D_{j_1} D_{j_2} \eta^\alpha) \frac{\partial}{\partial u^\alpha_{j_1 j_2}} + \dots$$

and thus

$$X^{(k)}_\infty (H^\sigma u - G^\sigma)\Big|_{H^\sigma u = G^\sigma} = \sum_\alpha b^\sigma_\alpha(x) \eta^\alpha(x) + \sum_{\alpha, j} b^\sigma_{\alpha j}(x) \frac{\partial \eta^\alpha}{\partial x^j} + \dots - \underbrace{X^{(k)} G^\sigma(x)}_{\shortparallel \atop 0} =$$

$$= H^\sigma(x) \eta(x) = 0$$

- if there are other point symmetries generated by $X_1, \dots, X_r$ which are independent of X_∞

then for ~~two~~ arbitrary inf. operators

$$X = \sum_{j=1}^{r} a_j X_j + \sum_\alpha f^\alpha(x) \frac{\partial}{\partial u^\alpha} \quad \text{and} \quad Y = \sum_{j=1}^{r} b_j X_j + \sum_\alpha g^\alpha(x) \frac{\partial}{\partial u^\alpha}$$

is their commutator also an infin. operator given by

$$Z = [X, Y] = \sum_{j=1}^{r} c_j X_j + \sum_\alpha h^\alpha(x) \frac{\partial}{\partial u^\alpha}$$

where $h^\alpha(x)$ is a solution of $H^\sigma h = 0$

- hence we can add it to any solution $u(x)$ of $H^\sigma u = G^\sigma$ and get a new solution $u + h$

- for homogeneous equations we get directly a new solution of $H^\sigma u = 0$.

Example: let us consider again the heat equation $u_{xx} = u_t$

and take as $X = X_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$

and as $Y = g(x,t)\frac{\partial}{\partial u}$ where g is any solution of $u_{xx} = u_t$

we get
$$Z = [X, Y] = \left(2t\frac{\partial g}{\partial x} + xg\right)\frac{\partial}{\partial u} = h(x,t)\frac{\partial}{\partial u}$$

where $h(x,t)$ must be a solution of $u_{xx} = u_t$

For example if we take $g(x,t) = A = \text{const}$
we can generate a sequance of solutions
$$h_1(x,t) = Ax \Rightarrow h_2(x,t) = A(2t + x^2) \Rightarrow h_3 = A(6tx + x^3) \text{ etc.}$$

Example: let us consider the time-independent Schrödinger equation with the spherically symmetric potential $V(r)$
$$\left[-\frac{1}{2m}\Delta + V(r)\right]\psi(\vec{r}) = E\psi(\vec{r})$$
which is invariant under $SO(3)$ generated by
$$X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad X_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$
i.e. we have $L_x = -iX_1, \; L_y = -iX_2, \; L_z = -iX_3$

there is $2\ell+1$ independent solutions for energy $E_{n\ell}$
$$\psi_{n\ell m} = R_{n\ell}(r)\, Y_{\ell m}(\theta, \varphi)$$

a generator of symmetry is also $Y = \psi_{n\ell m}\frac{\partial}{\partial \psi}$
and thus from e.g. $[X_1, Y] = (X_1\psi_{n\ell m})\frac{\partial}{\partial \psi}$ or $[X_2, Y] = (X_2\psi_{n\ell m})\frac{\partial}{\partial \psi}$

we get also solutions

- remember the ladder operators $L_\pm = L_x \pm iL_y$ which also produce new solutions from other, e.g.
$$L_+Y_{1-1} \sim Y_{10}, \quad L_+Y_{10} \sim Y_{11}, \quad L_+Y_{11} \sim 0$$
this is also a (trivial) solution!

3) invariant solutions of PDEs

- for simplicity, let us consider a scalar PDE

$$R(x, u, \partial u, \dots \partial^k u) = 0$$

invariant under a point transformation generated by

$$X = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}$$

that is, according to the inf. criterion,

$$X^{(k)} R \Big|_{R=0} = 0$$

- a solution of $R=0$ is called <u>invariant</u> under X if a surface given by $u - f(x) = 0$ is invariant under X that is if

$$X(u - f(x))\Big|_{u=f} = 0$$

or

$$\sum_{i=1}^{n} \xi^i(x, f(x)) \frac{\partial f(x)}{\partial x^i} = \eta(x, f(x)) \qquad (*)$$

- the simplest case is when $R=0$ is not dependent explicitly on one of the indep. varidbles, let us say x^n, then $R=0$ is invariant under $X = \frac{\partial}{\partial x^n}$ and the condition $(*)$ is reduced to

$$\frac{\partial f}{\partial x^n} = 0$$

and thus we are looking for a solution independent of x^n by setting all derivatives with respect to x^n in $R=0$ to zero

<u>Example</u>: the heat equation $u_{xx} = u_t$ does not depend explicitly on x and t and therefore, there are solutions $u = f_1(x)$ and $u = f_2(t)$

for f_1 we set $\frac{\partial^2 f_1}{\partial x^2} = 0 \Rightarrow f_1(x) = Ax + B$
which is invariant under $\frac{\partial}{\partial t}$

and for f_2 we set $\frac{\partial f_2}{\partial t} = 0 \Rightarrow f_2(t) = A = \text{const}$
which is invariant under $\frac{\partial}{\partial x}$

- another simple case is if $\xi^i = 0$ for all i,

then the solution is given implicitly by

$$\eta(x, f(x)) = 0$$

For example if $X = u \frac{\partial}{\partial u}$ is a generator of scaling symmetry of $R = 0$, then $u = 0$ is a solution of $R = 0$

- in a general case, at least one of $\xi^i \neq 0$ and from (*)

we can express

$$\frac{\partial u}{\partial x^i} = \frac{M}{\xi^i} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\xi^j}{\xi^i} \frac{\partial u}{\partial x^j} \qquad (\dagger)$$

and substitute into the PDE $R = 0$ for all derivatives with respect to x^i

this technique is called ~~the~~ <u>method of direct substitution</u>

- we get a new PDE where x^i plays a role of a parameter,

* after finding its solution we have to substitute it

into (*) again and then we get an invariant solution

- if $n = 2$, then we reduce a PDE into a ODE, the solution

of which depends on constants of integration which

can be in general functions of x^i and thus

by substituting into (*) we find these functions

<u>Example</u>: again let us consider the heat equation $u_{xx} = u_t$

and let us find the invariant solution under

$$X = 2t \frac{\partial}{\partial x} - xu \frac{\partial}{\partial u} \qquad (\xi^t = 0,\ \xi^x = 2t)$$

from $(\dagger)$ we get
$$\frac{\partial u}{\partial x} = \frac{M}{\xi^x} = -\frac{xu}{2t}$$

and
$$\frac{\partial^2 u}{\partial x^2} = -\frac{u}{2t} - \frac{x}{2t} \frac{\partial u}{\partial x} = -\frac{u}{2t} + \left(\frac{x}{2t}\right)^2 u$$

by substitution into $u_{xx} = u_t$ we get an ODE with x as param.

$$\frac{\partial u}{\partial t} = u\left(\frac{x^2}{4t^2} - \frac{1}{2t}\right) \implies \ln u = -\frac{x^2}{2t} - \frac{1}{2}\ln t + \underbrace{\ln G(x)}_{\text{integrating constant}}$$

or $$u(x,t) = \frac{G(x)}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

and by substitution into $\frac{\partial u}{\partial x} = -\frac{xu}{2t}$ we have an ODE for $G(x)$:

$$e^{-\frac{x^2}{4t}} \left[\frac{G'(x)}{\sqrt{t}} \cancel{- \frac{x}{2t} \frac{G(x)}{\sqrt{t}}} \right] = \cancel{-\frac{x}{2t} \frac{G(x)}{\sqrt{t}} e^{-\frac{x^2}{4t}}}$$

or $$G'(x) = 0 \quad \Rightarrow \quad G(x) = A$$

and we have the invariant solution
$$u(x,t) = \frac{A}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

as we can easily check
$$X\left(u - \frac{A}{\sqrt{t}} e^{-\frac{x^2}{4t}}\right)\Bigg|_{u=\dots} = -x \frac{A}{\sqrt{t}} e^{-\frac{x^2}{4t}} - 2t \frac{A}{\sqrt{t}} e^{-\frac{x^2}{4t}} \left(-\frac{2x}{4t}\right) = 0$$

- another possibility (instead of substituting of $\frac{\partial u}{\partial x^i}$ into $R=0$) is to change variables and use the canonical ones given in this case by

$$\left.\begin{aligned} X y^j &= 0 \quad \text{for } j \neq i \\ X y^i &= 1 \\ X v &= 0 \end{aligned}\right\} \Rightarrow X^{(y,v)} = \frac{\partial}{\partial y^i}$$

and we thus get a PDE (we consider $v = v(y)$) $\tilde{R}(y, v, \partial v, \dots \partial^k v) = 0$ with a solution $v = H(y^1, \dots y^{i-1}, y^{i+1}, \dots, y^n)$ independent of y^i

the solution of the original $R=0$ is then given implicitly by
$$v(x,u) = H(y^1(x,u), \dots \quad, y^n(x,u))$$

- in a special case when $y^1, \dots, y^{i-1}, y^{i+1}, \dots, y^n$ are independent of u then we can express u directly from $v = H(y^1, \dots, y^n)$ and by substitution into $R=0$ we get the equation for H without changing variables

in this case we get the <u>invariant form method</u>

Example: again we have $u_{xx} = u_t$ and $X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$

the canonical variables are

$$Xy^1 = 0 \quad \Rightarrow \quad y^1 = t$$

$$Xy^2 = 1 \quad \Rightarrow \quad \text{for example} \quad y^2 = \frac{x}{2t}$$

$$Xv = 0 \quad \Rightarrow \quad \frac{dx}{2t} = \frac{du}{-xu} \quad \Rightarrow \quad -2x\,dx = 4t\,\frac{du}{u}$$

$$\Rightarrow \quad v(x,t,u) = x^2 + 4t\ln u$$

because y^1 is independent of u we can express u

from $$x^2 + 4t\ln u = v(x,t,u) = H(y^1(x,t,u)) = H(t)$$

as $$u = e^{\frac{H(t) - x^2}{4t}}$$

and by substituting into $u_{xx} = u_t$ we get for $H(t)$

$$\frac{dH}{dt} = \frac{H}{t} - 2$$

with the solution (scaling symmetry, remember $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$)

$$H(t) = -2t\ln\frac{t}{A^2}$$

and $$u(x,t) = e^{\frac{H(t) - x^2}{4t}} = \frac{A}{\sqrt{t}}\,e^{-\frac{x^2}{4t}}$$

which is, of course, the same invariant solution as we got using the method of direct substitution