Variational symmetry

- summary of the variational calculus

- as usual, we have a space (x,u) of independent $x=(x^1,\dots,x^n)$ and dependent $u=(u^1,\dots,u^m)$ variables

- let $\Omega \subset X$ (space of all independent variables) be an open subset with a smooth boundary $\partial\Omega$

then the variational problem is to find the extrema of the linear functional (integral of the Lagrangian L)

$$\mathcal{L}[u] = \int_\Omega L(x,u,\partial u,\dots,\partial^\ell u)\, d^n x$$

in a certain class of functions $u=f(x)$ defined on Ω (this class depends on the order of derivatives in L and boudary conditions)

- variational derivative of the functional $\mathcal{L}[u]$ is the unique m-tuple

$$\delta\mathcal{L}[u] = (\delta_1\mathcal{L}, \delta_2\mathcal{L}, \dots, \delta_m\mathcal{L})$$

given by
$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{L}[f+\varepsilon\eta] = \int_\Omega \delta\mathcal{L}[f(x)]\cdot\eta(x)\, d^n x$$

where $f(x)$ is a smooth function defined on Ω.

and $\eta(x) = (\eta^1(x),\dots,\eta^m(x))$ is a smooth function with compact support in Ω (thus $f(x)$ and $f(x)+\varepsilon\eta(x)$ satisfy the same boundary conditions)

and $\delta_\alpha\mathcal{L}[u]$ is the variational derivative of $\mathcal{L}$ with respect to u^α, which can be determined using the Euler operator

$$\delta_\alpha\mathcal{L} = E_\alpha(L) = \sum_J (-D)_J \frac{\partial L(x,u..)}{\partial u^\alpha_J}$$

where $(-D)_J = (-1)^k D_J = (-D_{j_1})(-D_{j_2})\dots(-D_{j_k})$

for a multiindex $J=(j_1,\dots,j_k)$ and D_j is as usually the total derivative operator

$$D_j = \frac{\partial}{\partial x^j} + \sum_{\alpha} u^{\alpha}_j \frac{\partial}{\partial u^{\alpha}} + \dots + \sum_{\alpha, i_1, \dots i_p} u^{\alpha}$$

- if $u=f(x)$ is an extremal of $\mathcal{L}[u]$, then

$$\delta\mathcal{L}[f]=0 \quad , \text{i.e.} \quad \delta_\alpha \mathcal{L} = 0 \quad \text{for all } \alpha$$

and if $f(x)$ is smooth enough then it must be the solution of the Euler-Lagrange equations

$$E_\alpha(L) = 0 \quad \text{for } \alpha = 1, \dots, m$$

Example: Dirichlet's principle for Laplace's equation

for the functional

$$\mathcal{L}[u] = \int_{\Omega} \frac{1}{2} |\nabla u|^2 d^n x \quad \text{for some } \Omega \subset X = \mathbb{R}^n$$

we set

$$\delta\mathcal{L}[u] = 0 = E(L) = \sum_{i=1}^{n} (-D_i) \frac{\partial L}{\partial u_i} = \sum_{i=1}^{n} -D_i u_i = -\Delta u$$

the derivative with respect to u does not contribute

- null and equivalent Lagrangians

if the Euler-Lagrange equations are identically zero (i.e. for arbitrary u, not only for the solutions) we call such a Lagrangian null

Example: for $\mathcal{L}[u] = \int_a^b u u_x \, dx$ we set $\delta\mathcal{L} = u_x - D_x(u) = 0$

(note that we can write $\mathcal{L}[u] = \int_a^b D_x\left(\frac{u^2}{2}\right) dx = \frac{u^2}{2}\Big|_{x=a}^{b}$ and thus we have the same value of $\mathcal{L}$ for any function with correct bound. cond.)

· in general, if $L = \mathrm{Div}\, P$ for a certain

$P = (P^1, \dots, P^n)$, where

$$\mathrm{Div}\, P = D_1 P^1 + \dots + D_n P^n \quad \text{(total divergence)}$$

then $E(L) = 0$ and L is a null Lagrangian

because $\mathcal{L}[u] = \int_{\Omega} \mathrm{Div}\, P \, d^n x = \int_{\partial\Omega} P \cdot ds = \mathrm{const}$

Note: the inverse of this statement is also true,

that is if for a function $L(x, u, \dots)$ and for an arb. $u = f(x)$

$$E(L) = 0$$

then L must be of the form

$$L = \mathrm{Div}\, P(x, u, \dots)$$

(the proof can be found in Olver, see theorem 4.7)

- this result will be used later to find the equation for multipliers giving conservation laws

- as a consequence if we have two Lagrangians L and $\tilde{L}$ differing only by a divergence, i.e.

$$L = \tilde{L} + \mathrm{Div}\, P$$

then the Euler-Lagrangian equations are the same

$$E(L) = E(\tilde{L})$$

and these Lagrangians describe basically the same physical system.

- variational symmetry

  - local group of point transformations G defined on $M \subset \Omega_0 \times U$ is the group of variational symmetry of the functional

$$\mathcal{L}[u] = \int_{\Omega_0} L(x, u, \partial u, \dots \partial^{\ell} u)\, d^n x$$

if

$$\int_{\tilde{\Omega}} L(\tilde{x}, \tilde{f}(\tilde{x}), \dots, \tilde{f}^{(\ell)}(\tilde{x}))\, d^n\tilde{x} = \int_{\Omega} L(x, f(x), \dots f^{(\ell)}(x))\, d^n x$$

for arbitrary $\Omega \subset \Omega_0$, $u = f(x)$ smooth on Ω and its transformation image $\tilde{u} = \tilde{f}(\tilde{x})$ under the point transformation

$$\tilde{x} = F(x, u, \varepsilon), \quad \tilde{u} = G(x, u, \varepsilon) \qquad (*)$$

defined on $\tilde{\Omega} \subset \Omega_0$.

- infinitesimal criterion of variational symmetry

  - considering the point transformation $(*)$ as a change of coordinates in the multiple integral, we get

$$\int_{\tilde{\Omega}} L(\tilde{x}, \tilde{u}(\tilde{x}), \dots)\, d^n\tilde{x} = \int_{\Omega} L(\tilde{x}(x, u(x), \varepsilon), \tilde{u}(x, u, \varepsilon), \dots) \det J_\varepsilon(x, u(x) \dots)\, d^n x =$$

$$= \int_{\Omega} L(x, u(x), \dots)\, d^n x$$

where $(J_\varepsilon)_{ij} = D_i F^j(x, u(x), \varepsilon)$,

and this must be true for arbitrary $\Omega \subset \Omega_0$ and $u = f(x)$

and thus for the infinitesimal transformation we obtain

$$L(x + \varepsilon \xi_* + \mathcal{O}(\varepsilon^2), u + \varepsilon \eta + \mathcal{O}(\varepsilon^2), \dots) \begin{vmatrix} \frac{d(x^1 + \varepsilon \xi^1(x,u) + \mathcal{O}(\varepsilon^2))}{dx^1} & \frac{d(x^1 + \varepsilon \xi^1 + \dots)}{dx^2} & \dots \\ \vdots & \ddots & \\ \underbrace{\frac{d(x^n + \varepsilon \xi^n(x,u) + \mathcal{O}(\varepsilon^2))}{dx^1}}_{\varepsilon D_1 \xi^n + \mathcal{O}(\varepsilon^2)} & \dots & \underbrace{\frac{d(x^n + \varepsilon \xi^n \dots)}{dx^n}}_{1 + \varepsilon D_n \xi^n + \mathcal{O}(\varepsilon^2)} \end{vmatrix} =$$

$$\frac{dL}{d\varepsilon} = \xi^i \frac{\partial L}{\partial x^i} + \eta^\alpha \frac{\partial L}{\partial u^\alpha} + \dots$$

$$= \left(L(x, u, \dots) + \varepsilon X^{(\ell)} L + \mathcal{O}(\varepsilon^2)\right)\left(1 + \varepsilon \operatorname{Div} \xi + \mathcal{O}(\varepsilon^2)\right) = L(x, u, \dots)$$

and from here

$$\boxed{X^{(\ell)} L + L \operatorname{Div} \xi = 0}$$

for an arbitrary u

- it can be shown that if X generates a point symmetry of $\mathcal{L}[u]$ then it also generates a point symmetry of the corresponding Euler-Lagrange equations but it is not true the other way (see later e.g. scaling symmetry of the wave equation)

Conservation laws

- you probably know many examples from physics lectures like conservation of energy, momentum, angular momentum but there are also conservation laws like the continuity equation for the Schrödinger equation which is an example of a local conservation law

  - let us consider a system of PDR

$$R^\sigma(x, u, \partial u, \dots \partial^k u) = 0 \quad , \sigma = 1, \dots, N$$

where as usual we have indep. variables $x = (x^1, \dots x^n)$ and dep. variables $u = (u^1, \dots u^m)$

if the total divergence of a certain quantity Φ is zero:

$$\operatorname{Div} \Phi = \sum_{i=1}^{n} D_i \Phi^i = D_1 \Phi^1 + \dots + D_n \Phi^n = 0$$

for solutions of $R^\sigma = 0$

then it is called the local conservation law

  - the functions $\Phi^i(x, u, \partial u \dots)$, $i = 1, \dots, n$ are usually called fluxes or densities depending of physical meaning.
  - the highest derivative in Φ^i is called the order of the local conservation law

- if $x^n = t$ is time variable then we usually write

$$D_t \varrho + \operatorname{div} \Phi = 0$$

where div is space divergence

$$\operatorname{div} \Phi = D_1 \Phi^1 + \dots + D_{n-1} \Phi^{n-1}$$

and ϱ is density and $\Phi^1, \dots, \phi^{n-1}$ are fluxes in space

- by integration over the whole region Ω where the physical system is contained, we get

$$\underbrace{D_t \int_\Omega \varrho \, d^{n-1}x}_{\text{conserved quantity ("charge")}} = -\int_\Omega \operatorname{div} \Phi \, d^{n-1}x = \oint_{\partial\Omega} (\phi \cdot \vec{n}) d^{n-2}x \underset{\uparrow}{=} 0$$

if ϕ^i is zero on the boundary

Example: Euler's equations of motion of the adiabatic processes in gas

ϱ is density, v is velocity, p is pressure

(*) $\quad D_t \varrho + D_j (\varrho v^j) = 0 \quad$ (conservation law of mass)

(**) $\quad \varrho (D_t + v^j D_j) v^i + D_i p = 0$

(***) $\quad \varrho (D_t + v^j D_j) p + \gamma \varrho p D_j v^j = 0 \quad$ (γ is adiabatic exp.)

conservation law of „momentum"

$$v^i(*) + (**) = D_t(\varrho v^i) + D_j(\varrho v^i v^j + p \delta^{ij}) = 0, \quad i = 1,2,3$$

of „energy"

$$\sum_{i=1}^{3} \left[\frac{(v^i)^2}{2} (*) + \varrho v^i (**) + \frac{1}{\varrho(\gamma-1)} (***) \right] =$$

$$= D_t(E) + D_j [v^j (E+p)] = 0$$

where $\quad E = \frac{1}{2} \varrho v^2 + \frac{p}{\gamma - 1} \quad$ (density of energy)

and

$$\underbrace{D_t \int_\Omega E d^3x}_{\text{energy}} = \underbrace{-\int_{\partial\Omega} (E+p)(\vec{v} \cdot \vec{n}) \, dS}_{\text{flux of energy through boundary}}$$

Example: Korteweg-de Vries equation (for long shelf waves)

$$u_t + u u_x + u_{xxx} = 0 \qquad (*)$$

it is actually in the form of conservation law

$1\cdot(*)$: $\quad D_t(u) + D_x\left(\frac{1}{2}u^2 + u_{xx}\right) = 0$

but there are infinite number of conservation laws of higher and higher order, e.g.

$u\cdot(*)$: $\quad D_t\left(\frac{1}{2}u^2\right) + D_x\left(\frac{1}{3}u^3 + u u_{xx} - \frac{1}{2}u_x^2\right) = 0$

- as we can see, each conservation law was obtained by multiplying the equations of motion by some „integrating" factors or multipliers and summing, or as a „linear combination" of them

$$\mathrm{Div}\,\Phi = \Lambda_\sigma R^\sigma = 0 \quad \text{for solutions}$$

we will return to this way of determining of the conservation laws later

- trivial and equivalent conservation laws

- local conservation law of a certain system of PDEs $R^\sigma = 0$, $\sigma = 1,\dots,N$ is called trivial if

$$\Phi^i(x,u,\dots) = M^i(x,u,\dots) + H^i(x,u,\dots)$$

where $M^i(x,u,\dots) = 0$ for all solutions $u=f(x)$ of $R^\sigma = 0$ (trivial LCL of the first kind)

and $H^i(x,u,\dots)$ satisfies $D_i H^i \equiv 0$ for any u (not only for solutions) (trivial LCL of the second kind)

Example: $\quad \mathrm{div}\,(\mathrm{rot}\,\vec{f}) = \nabla\cdot(\nabla\times\vec{f}) = 0$ for arb. $\vec{f}(x)$ in $\mathbb{R}^3$

Example: let us consider the system

$$v_x = u, \quad v_t = K(u)\, u_x$$

then e.g. $D_t[u(u - v_x)] + D_x[v_t - K(u)u_x] = 0$ (first kind)

$$D_t(u_{xx}) - D_x(u_{tx}) = 0$$ (second kind)

are trivial local conservation laws.

- two local conservation laws

$$D_i \Phi^i = 0 \quad \text{and} \quad D_i \Psi^i = 0$$

are equivalent if

$$D_i(\Phi^i - \Psi^i) = 0$$

is a trivial local conservation law.

(thus there are equivalence classes of conservation laws, one such a class consists of all LCLs equivalent to a certain non-trivial LCL)

- a set of conservation laws $\{D_i \Phi^i_j = 0\}_{j=1}^{L}$ is formed by linearly dependent LCLs if there exist $\{a^{(j)}\}_{j=1}^{L}$ non-zero such that

$$\sum_{i,j=1}^{L} D_i(a^{(j)} \Phi^i_j) = 0$$

is a trivial local conservation law.

- our goal is to find (possibly in a systematic way) all non-trivial, linearly independent LCLs
  - for systems which can be formulated variationally we can do that using the Noether theorem, but only if we consider also generalized symmetries (see later)
  - for general systems there is a „direct method" by Bluman et al which is searching for multipliers Λ_σ (see also later)