- characteristics of conservation laws

- if the conservation law

$$\mathrm{Div}\, \Phi(x,u,\partial u,\ldots \partial^{l} u) = 0$$

holds for solutions $u=f(x)$ of a system of PDEs

$$R^{\sigma}(x,u,\partial u,\ldots,\partial^{k} u) = 0 \quad , \quad \sigma = 1,\ldots,N$$

then we can write it as

$$\mathrm{Div}\, \Phi = \sum_{\mu,J} Q^{J}_{\mu}(x,u,\partial u \ldots)\, D_J R^{\mu} = 0 \qquad (*)$$

because $u=f(x)$ is also a solution of $D_J R^{\sigma} = 0$, $\sigma = 1,\ldots,N$ for all $J=(j_1,\ldots,j_p)$, $p \geq 0$.

- using Leibniz rule over and over as in

$$D_{j_p}(Q^{J}_{\mu} D_{j_1\cdots j_{p-1}} R^{\mu}) = (D_{j_p} Q^{J}_{\mu}) D_{j_1\cdots j_{p-1}} R^{\mu} + Q^{J}_{\mu} D_J R^{\mu}$$

we can rewrite $(*)$ as

$$\mathrm{Div}\, \Phi = \mathrm{Div}\, \Psi + \sum_{\mu} Q_{\mu} R^{\mu}$$

where $$Q_{\mu} = \sum_{J} (-D)_J\, Q^{J}_{\mu}$$

and $\mathrm{Div}\, \Psi = 0$ is a trivial conservation law of the first kind (Ψ contains R^{σ} or its derivatives).

- thus there is an equivalent conservation law

$$0 = \mathrm{Div}\, \Phi' = \mathrm{Div}\,(\Phi - \Psi) = \sum_{\mu} Q_{\mu} R^{\mu}$$

which is in the characteristic form and the N-tuple $Q = (Q_1,\ldots,Q_{\mu})$ is called the characteristic of the given conservation law.

- in general for $N \geq 2$, the characteristic Q is not unique but it can be shown that if $Q\cdot R = Q'\cdot R$ for two different Q and Q' then for non-degenerate (of maximal rank) systems $R^{\sigma}=0$, these characteristics satisfy

$$Q - Q' = 0 \quad \text{for } u=f(x) \text{ (solutions of } R=0)$$

and we call them equivalent characteristics.

- characteristic form of the infinitesimal generators

- in general, the point symmetries are generated by the inf. operators (here extended into the space of derivatives)

$$X^{(k)} = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\alpha=1}^{m} \sum_{\substack{J \\ 1\le \# J \le k}} \eta^\alpha_J \frac{\partial}{\partial u^\alpha_J}$$

where
$$\eta^\alpha_{\underbrace{j_1\cdots j_p}_{J}} = D_{j_p} \eta^\alpha_{j_1\cdots j_{p-1}} - \sum_{i=1}^{n} (D_{j_p} \xi^i)\, u^\alpha_{j_1\cdots j_{p-1} i}$$

which can be rewritten (again using Leibniz rule as
$D_{j_1}(\xi^i u^\alpha_i) = (D_{j_1}\xi^i)\, u^\alpha_i + \xi^i u^\alpha_{j_1 i}$)

into the form
$$\eta^\alpha_J = D_J \underbrace{\left(\eta^\alpha - \sum_{i=1}^{n} \xi^i u^\alpha_i\right)}_{\hat\eta^\alpha} + \sum_{i=1}^{n} \xi^i u^\alpha_{J,i}$$

the m-tuple $\hat\eta = (\hat\eta^1, \dots \hat\eta^m)$ is called the characteristic of the point symmetry generated by $X = \sum_i \xi^i \frac{\partial}{\partial x^i} + \sum_\alpha \eta^\alpha \frac{\partial}{\partial u^\alpha}$.

- if we admit that the infinitesimals η^α may be dependent on the derivatives u^α_J (as we will do later for generalized symmetries) then we can define the infinitesimal operator

$$\hat X^{(k)} = \sum_{\alpha=1}^{m} \hat\eta^\alpha(x, u, \partial u) \frac{\partial}{\partial u^\alpha} + \sum_{\alpha=1}^{m} \sum_{\substack{J \\ 1\le \# J\le k}} (D_J \hat\eta^\alpha) \frac{\partial}{\partial u^\alpha_J}$$

called the inf. op. in the characteristic form, which is related to $X^{(k)}$ via

$$X^{(k)} = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j} + \sum_{\alpha, J} \eta^\alpha_J \frac{\partial}{\partial u^\alpha_J} = \sum_j \xi^j \frac{\partial}{\partial x^j} + \underbrace{\sum_{\alpha, J} \left(D_J \hat\eta^\alpha + \sum_i \xi^i u^\alpha_{J,i}\right) \frac{\partial}{\partial u^\alpha_J}}_{\hat X^{(k)}} =$$

$$= \hat X^{(k)} + \sum_j \xi^j \frac{\partial}{\partial x^j} + \sum_{\alpha, i} \xi^i u^\alpha_i \frac{\partial}{\partial u^\alpha} + \sum_{i,j,\alpha} \xi^i u^\alpha_{ji} \frac{\partial}{\partial u^\alpha_j} + \dots =$$

$$\boxed{X^{(k)} = \hat X^{(k)} + \sum_{j=1}^{n} \xi^j D_j}$$

from which it follows that if $X^{(k)}$ generates a symmetry of $R^\sigma = 0$, then also $\hat X^{(k)}$ generates a symmetry and viceversa.

Noether's theorem

- this theorem (proved in 1915 by Emmy Noether already for generalized symmetries, see later) shows the relation between symmetries of the differential equations (which can be derived from some variational principle) and their conservation laws
- here we formulate it for the special case of point symmetries and later we generalize it for more general cases

- let one-parametric LGT generated by
$$X = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \eta^\alpha \frac{\partial}{\partial u^\alpha}$$
be the group of point symmetry of the variational problem for
$$\mathcal{L}[u] = \int_\Omega L(x, u, \partial u, \dots \partial^\ell u)\, d^n x$$
and thus also a point symmetry of the corresponding Euler-Lagrange equations
$$E_\alpha(L) = 0,$$
then the characteristic
$$\hat{\eta}^\alpha = \eta^\alpha - \sum_{i=1}^{n} \xi^i u_i^\alpha$$
of the inf. generator X is also the characteristic of the conservation law of the Euler-Lagrange equations, that is, there exists $\Phi(x, u, \partial u \dots) = (\Phi^1, \dots, \Phi^n)$ such that
$$\mathrm{Div}\, \Phi = \hat{\eta} \cdot E(L) = \sum_{\alpha=1}^{m} \hat{\eta}^\alpha \cdot E_\alpha(L) = 0$$
and the fluxes Φ^i are given by (for $L = L(x, u, \partial u)$)
$$\Phi^i = -\sum_{\alpha=1}^{m} \hat{\eta}^\alpha \frac{\partial L}{\partial u_i^\alpha} - L\xi^i = \sum_{\alpha=1}^{m}\sum_{j=1}^{n} \xi^j u_j^\alpha \frac{\partial L}{\partial u_i^\alpha} - \sum_{\alpha=1}^{m} \eta^\alpha \frac{\partial L}{\partial u_i^\alpha} - L\xi^i$$
but it is straightforward to find Φ^i also for Lagrangians dependent on higher derivatives (see the proof)

- proof of the Noether's theorem for point symmetries

- we start with the infinitesimal criterion for variational symmetry and use $X^{(\ell)} = \hat{X}^{(\ell)} + \sum_{j=1}^{n} \xi^j D_j$

$$X^{(\ell)} L + L \operatorname{Div} \xi = 0$$

$$\hat{X}^{(\ell)} L + \sum_{j=1}^{n} \left[\xi^j D_j L + L D_j \xi^j \right] = \hat{X}^{(\ell)} L + \operatorname{Div}(L\xi) = 0 \quad (*)$$

the term $\hat{X}^{(\ell)} L$ canbe rearranged using the Leibniz rule:

$$\hat{X}^{(\ell)} L = \sum_{\alpha, J} (D_J \hat{\eta}^\alpha) \frac{\partial L}{\partial u^\alpha_J} =$$

$$= \sum_{\alpha=1}^{m} \left\{ \hat{\eta}^\alpha \frac{\partial L}{\partial u^\alpha} + \sum_{i=1}^{n} \left[D_i \left(\hat{\eta}^\alpha \frac{\partial L}{\partial u^\alpha_i} \right) - \hat{\eta}^\alpha D_i \left(\frac{\partial L}{\partial u^\alpha_i} \right) \right] + \right.$$

$$\left. + \sum_{i_1, i_2 = 1}^{n} \left[D_{i_2} \left((D_{i_1} \hat{\eta}^\alpha) \frac{\partial L}{\partial u^\alpha_{i_1 i_2}} \right) - D_{i_1} \left(\hat{\eta}^\alpha D_{i_2} \frac{\partial L}{\partial u^\alpha_{i_1 i_2}} \right) + \hat{\eta}^\alpha D_{i_1} D_{i_2} \frac{\partial L}{\partial u^\alpha_{i_1 i_2}} \right] + \dots \right\} =$$

$$= \sum_{\alpha=1}^{m} \hat{\eta}^\alpha \sum_J (-D)_J \frac{\partial L}{\partial u^\alpha_J} + \operatorname{Div} A = \sum_{\alpha=1}^{m} \hat{\eta}^\alpha \cdot E_\alpha(L) + \sum_{i=1}^{n} D_i A^i$$

wher $$A^i = \sum_{\alpha=1}^{m} \left\{ \left[\hat{\eta}^\alpha \frac{\partial L}{\partial u^\alpha_i} + \sum_{j=1}^{n} \left[(D_j \hat{\eta}^\alpha) \frac{\partial L}{\partial u^\alpha_{ji}} - \hat{\eta}^\alpha D_j \frac{\partial L}{\partial u^\alpha_{ji}} \right] + \dots \right\}$$

- substituting into $(*)$ we get

$$\sum_{\alpha=1}^{m} \hat{\eta}^\alpha \cdot E_\alpha(L) + \operatorname{Div}(A + L\xi) = 0$$

or $$\operatorname{Div} \Phi = -\operatorname{Div}(A + L\xi) = \sum_{\alpha=1}^{m} \hat{\eta}^\alpha E_\alpha(L) = 0$$

for solutions of the Euler-Lagr. eq.

we got the conservation law of the E-L equations corresponding to the point symmetry generated by X.

Examples:

1) Let us consider a system of N particles in classical mechanics described by the Lagrangian

$$L = \sum_{i=1}^{N} \left[\frac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right) \right] - V(x) = T - V$$

the Euler-Lagrange equations are standard Newton's equations of motion

$$m_i \ddot{x}_i = -\frac{\partial V}{\partial x_i}, \quad m \ddot{y}_i = -\frac{\partial V}{\partial y_i}, \quad m \ddot{z}_i = -\frac{\partial V}{\partial z_i} \quad \text{for all } i = 1, \dots, N$$

- all these equations are invariant under translations in time generated by $X_t = \frac{\partial}{\partial t} \quad (\xi = 1, \ \eta^{x_i} = \eta^{y_i} = \eta^{z_i} = 0)$

which is clearly also the variational symmetry because $X_t^{(1)} = X_t$ and

$$\underset{\substack{\uparrow \\ \text{independent of time}}}{X_t^{(1)} L} + L \overbrace{\mathrm{Div}\, \xi}^{0} = 0$$

the corresponding conservation law $\mathrm{Div}\, \Phi = \frac{d\Phi}{dt} = 0$ is for

$$\Phi = -L + \sum_{\alpha=1}^{3N} u_t^\alpha \frac{\partial L}{\partial u_t^\alpha} = -L + \underbrace{\sum_{i=1}^{N} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right)}_{2T} = T + V = E$$

$\xi = 1$, $u^1 = x_1, u^2 = y_1, \dots$

that is the conservation of energy.

- if there are no external forces then $-\sum_{i=1}^{N} \frac{\partial V}{\partial x_i} = 0$ (the net force is zero)

and we have invariance under translations of all particles at once in space, that is under translations generated by

$$X_1 = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \quad \left(\begin{matrix} \xi = 0 \\ \eta^\alpha = 1 \text{ for } \alpha = 1, 4 \dots \\ \eta^\alpha = 0 \text{ for } \alpha = 2, 3, 5 \dots \end{matrix} \right), \quad X_2 = \sum_{i=1}^{N} \frac{\partial}{\partial y_i}, \quad X_3 = \sum_{i=1}^{N} \frac{\partial}{\partial z_i}$$

they are also variational symmetries because e.g.

$$X_1^{(1)} L + \underbrace{L \, \mathrm{Div} \overset{0}{\xi}}_{0} = X_1 L = -\sum_{i=1}^{N} \frac{\partial V}{\partial x_i} = 0$$

the corresponding conservation laws are those for momentum

that is $\frac{d\Phi_1}{dt} = -\frac{dp_x}{dt} = 0$; $\frac{d\Phi_2}{dt} = -\frac{dp_y}{dt} = 0$, $\frac{d\Phi_3}{dt} = -\frac{dp_z}{dt} = 0$

where $\Phi_1 = -\sum_{\alpha=1}^{3N} \eta^\alpha \frac{\partial L}{\partial u^\alpha_t} = -\sum_{i=1}^{N} m_i \dot{x}_i = -p_x$ etc.

2) Let us consider a simple wave equation in one-dimensional space

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

which can be derived from the variational problem with the Lagrangian

$$L = \frac{1}{2}\left[\left(\frac{\partial \psi}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial \psi}{\partial t}\right)^2\right]$$

- the wave equation is invariant under LGT generated by

$X_1 = \frac{\partial}{\partial x} \Rightarrow X_1^{(1)} = \frac{\partial}{\partial x} \Rightarrow X_1^{(1)}L + L\,\mathrm{Div}\,\xi = 0$ ✓

$X_2 = \frac{\partial}{\partial t} \Rightarrow X_2^{(1)} = \frac{\partial}{\partial t} \Rightarrow X_2^{(1)}L + L\,\mathrm{Div}\,\xi = 0$ ✓

$X_3 = \frac{\partial}{\partial \psi} \Rightarrow X_3^{(1)} = \frac{\partial}{\partial \psi} \Rightarrow X_3^{(1)}L + L\,\mathrm{Div}\,\xi = 0$ ✓

$X_4 = \psi\frac{\partial}{\partial \psi} \Rightarrow X_4^{(1)} = \psi\frac{\partial}{\partial \psi} + \psi_{,x}\frac{\partial}{\partial \psi_{,x}} + \psi_{,t}\frac{\partial}{\partial \psi_{,t}} \Rightarrow X_4^{(1)}L = 2L \neq 0$

$X_5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \Rightarrow X_5^{(1)} = X_5 - \psi_{,x}\frac{\partial}{\partial \psi_{,x}} - \psi_{,t}\frac{\partial}{\partial \psi_{,t}} \Rightarrow X_5^{(1)}L + L\,\mathrm{Div}\,\xi =$

$= -2L + 2L = 0$ ✓

thus except X_4 these are all variational symmetries for L and the corresponding conservation laws $\mathrm{Div}\,\Phi = 0$ are:

- for X_1 and X_2 we get (in general $\Phi^x = (\xi^x\psi_{,x} + \xi^t\psi_{,t} - \eta)\frac{\partial L}{\partial \psi_{,x}} - L\xi^x$
$\Phi^t = (\xi^x\psi_{,x} + \xi^t\psi_{,t} - \eta)\frac{\partial L}{\partial \psi_{,t}} - L\xi^t$)

for X_1
$$\begin{cases} \Phi_1^x = \psi_{,x}\frac{\partial L}{\partial \psi_{,x}} - L\xi_1^x = \frac{1}{2}\left(\psi_{,x}^2 + \frac{1}{c^2}\psi_{,t}^2\right) \\ \Phi_1^t = \psi_{,x}\frac{\partial L}{\partial \psi_{,t}} = -\frac{1}{c^2}\psi_{,x}\psi_{,t} \end{cases}$$

$$\Phi_2^x = \psi_{,t}\frac{\partial L}{\partial \psi_{,x}} = \psi_{,x}\psi_{,t}$$

$$\Phi_2^t = \psi_{,t}\frac{\partial L}{\partial \psi_{,t}} - L\xi_2^t = -\frac{1}{2}\left(\psi_{,x}^2 + \frac{1}{c^2}\psi_{,t}^2\right)$$

you can check that for both Φ_i we get $\mathrm{Div}\,\Phi_i = 0$ for solutions of $\psi_{,xx} = \frac{1}{c^2}\psi_{,tt}$

we can write them in a usual form $T_{\mu,\nu}^{\nu} = 0$ where $T_\mu^{\nu} = \begin{pmatrix} \Phi_1^x & \Phi_1^t \\ \Phi_2^x & \Phi_2^t \end{pmatrix}$

- for X_3 we set the original wave equation

$$\Phi_3^x = -\frac{\partial L}{\partial \psi_{,x}} = -\psi_{,x} \quad , \quad \Phi_3^t = -\frac{\partial L}{\partial \psi_{,t}} = \frac{1}{c^2}\psi_{,t}$$

thus $$\mathrm{Div}\,\Phi_3 = -D_x\psi_{,x} + \frac{1}{c^2}D_t\psi_{,t} = -\psi_{,xx} + \frac{1}{c^2}\psi_{,tt} = 0$$

- for X_5 we set

$$\Phi_5^x = \frac{1}{2}x\left(\psi_{,x}^2 + \frac{1}{c^2}\psi_{,t}^2\right) + t\psi_{,t}\psi_{,x}$$

$$\Phi_5^t = -\frac{x}{c^2}\psi_{,t}\psi_{,x} - \frac{1}{2}t\left(\psi_{,x}^2 + \frac{1}{c^2}\psi_{,t}^2\right)$$

<u>Straightforward generalization of the Noether's theorem</u>

- if for a certain one-parametric LGT generated by X holds $$X^{(1)}L + L\,\mathrm{Div}\,\xi = \mathrm{Div}\,B \quad \text{(instead of zero)}$$ then it is also true that X generates a point symmetry for the corresponding Euler-Lagrange equations
- such a symmetry is called Noether's symmetry instead variational symmetry and the corresponding conservation law is $\mathrm{Div}\,\Phi = 0$ for $\Phi = B - A - L\xi$

<u>Example:</u> again we consider a system of N particles without external forces with

$$L = \sum_{i=1}^{N} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - V(x)$$

for Galilean transformation generated by

$$X = \sum_{i=1}^{t} t\frac{\partial}{\partial x_i} \quad \Rightarrow \quad \begin{aligned}\tilde{t} &= t\\ \tilde{x}_i &= x_i + \varepsilon t\end{aligned}$$

(similarly for the Gal. transf. in y and z)

we have $$X^{(1)} = X + \sum_i \frac{\partial}{\partial \dot{x}_i}$$

and $$X^{(1)}L + L\underbrace{\mathrm{Div}\,\xi}_{0} = \sum_{i=1}^{N}\left(m_i\dot{x}_i - t\frac{\partial V}{\partial x_i}\right) = \sum_{i=1}^{N} m_i\dot{x}_i = \frac{d}{dt}\sum_{i=1}^{N} m_i x_i = \mathrm{Div}\,B$$

thus the quantity which is conserved $\left(\frac{d\Phi}{dt} = 0\right)$ is

$$\Phi = B - A = \sum_{i=1}^{N} m_i x_i - t\sum_{i=1}^{N} m_i\dot{x}_i = \text{const} \approx$$ the initial position of the center of mass in the x direction