# Direct method of construction of conservation laws

- it is not necessary to have variational formulation fof the system $\Rightarrow$ it can be used for any system of PDEs

- we look for the local conservation law in its characteristic form
$$\sum_{\sigma=1}^{N} \Lambda_\sigma R^\sigma = \mathrm{Div}\, \Phi = 0 \quad \text{for solutions of } R^\sigma = 0,\ \sigma = 1, \dots, N$$

- if we find $\Lambda_\sigma$ for which $\sum_{\sigma=1}^{N} \Lambda_\sigma R^\sigma$ can be written as divergence of $\Phi$ then we have a conservation law

- equations for multipliers $\Lambda_\sigma$

  - we know that two lagrangians $\tilde{L}$ and $L$ which differ by a divergence
  $$\tilde{L} = L + \mathrm{Div}\, F$$
  give the same Euler-Lagrange equations because
  $$E_\alpha(\mathrm{Div}\, F) = 0 \quad \text{for arbitrary } F(x, u, \partial u \dots)$$

  - we also mentioned that the inverse is also true i.e. if $E_\alpha(G(x, u, \dots)) = 0$ then $G = \mathrm{Div}\, F(x, u, \dots)$ for some $F$ (see Olver: theorem 4.7)

  - therefore applying $E_\alpha$ to $\sum_{\sigma=1}^{N} \Lambda_\sigma R^\sigma = \mathrm{Div}\, \Phi$ gives
  $$\boxed{E_\alpha\left(\sum_{\sigma=1}^{N} \Lambda_\sigma R^\sigma\right) = 0} \quad \text{for } \alpha = 1, \dots, m \text{ and arbitrary } u(x)$$

  - for general $\Lambda_\sigma(x, u, \partial u \dots)$ we have a system of PDEs but in general of high order which can be difficult to solve

- usually we limit the dependence of $\Lambda_\sigma$ to a few variables or we assume some functional form (e.g. polynomial) to make things easier

- if we find $\Lambda_\sigma(x, u, \partial u ..)$ satisfying $E_\alpha(\Lambda_\sigma R^\sigma) = 0$ then we can find fluxes $\Phi^i$ directly by rewriting $\sum_{\sigma=1}^{N} \Lambda_\sigma R^\sigma$ to the form $\mathrm{Div}\, \Phi$, but there are also integral formulas, see Bluman, Anco, Cheviakov chapter 1.3.7.

- Notes: (details can be found also in the book by Bluman et al.)

1) in general, not all local conservation laws can be obtained in this way, only for systems of PDEs which are in certain suitable form; for example for

$$R^\sigma = u^{\alpha_\sigma}_{J_\sigma} - G^\sigma(x, u, \partial u, .. \partial^k u) = 0$$

where $G^\sigma$ is independent of $u^{\alpha_\sigma}_{J_\sigma}$ and these are independent

2) in general, several $\Lambda_\sigma$ can give the equivalent local conservation laws, but again there are systems for which there is a one-to-one correspondence between classes of equivalence

$$\mathrm{Div}\, \Phi \quad \longleftrightarrow \quad \Lambda_\sigma(x, u, ..)$$

if they are in the so-called Cauchy-Kovalevskaya form with respect to the independent variable $x^j$

$$\frac{\partial^{s_\sigma} u^\sigma}{\partial (x^j)^{s_\sigma}} - G^\sigma(x, u, .., \partial^k u) = 0, \quad \begin{array}{l} 1 \leq s_\sigma \leq k \\ \sigma = 1, \dots, N = m \end{array}$$

where $s_\sigma$ is the highest derivative of $u^\sigma$ with respect to $x^j$ in $R^\sigma$

$m$ equations for $m$ dependent variables

- sometimes it is possible to rewrite a certain system of PDEs into the C-K form using a suitable point transformation, e.g.

$$u_{tx} = 0 \xrightarrow[X = t+x]{T = t-x} u_{TT} = u_{XX}$$

but not always

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Example: let us consider the heat equation

$$u_{xx} = u_t$$

and, for simplicity, let $\Lambda = c_1 + c_2 x + c_3 t + c_4 u$

- using $E(L) = \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} - D_t \frac{\partial L}{\partial u_t} + D_x D_x \frac{\partial L}{\partial u_{xx}} + \dots$

we get $E(\Lambda R) = E\left((c_1 + c_2 x + c_3 t + c_4 u)(u_{xx} - u_t)\right) =$

$$= c_4 (u_{xx} - u_t) + D_t\left[c_1 + c_2 x + c_3 t + c_4 u\right] + D_x D_x \left[\cancel{c_1 + c_2 x + c_3 t} + c_4 u\right] =$$

$$= c_4 (u_{xx} - u_t) + c_4 u_t + c_4 u_{xx} + c_3 = c_3 + 2 c_4 u_{xx} = 0$$

hence $c_3 = 0$, $c_4 = 0$ and $c_1$ and $c_2$ arbitrary

- we got two independent multipliers and conservation laws

$$\Lambda_1 = 1 \Rightarrow D_x(u_x) + D_t(-u) = 0$$

$$\Lambda_2 = x \Rightarrow x u_{xx} - x u_t = D_x(x u_x - u) + D_t(-xu) = 0$$

- if we would assume $\Lambda = \Lambda(x,t,u)$ we would get

$$\frac{\partial \Lambda}{\partial u}(u_{xx} - u_t) + D_t\left[\Lambda(x,t,u)\right] + D_x D_x\left[\Lambda(x,t,u)\right] =$$

$$= \frac{\partial \Lambda}{\partial u}(u_{xx} \cancel{- u_t}) + \frac{\partial \Lambda}{\partial t} + \cancel{\frac{\partial \Lambda}{\partial u} u_t} + D_x\left[\frac{\partial \Lambda}{\partial x} + \frac{\partial \Lambda}{\partial u} u_x\right] =$$

$$= \frac{\partial \Lambda}{\partial t} + \frac{\partial^2 \Lambda}{\partial x^2} + 2 \frac{\partial^2 \Lambda}{\partial u \partial x} u_x + \frac{\partial^2 \Lambda}{\partial u^2} u_x^2 + 2 \frac{\partial \Lambda}{\partial u} u_{xx} = 0$$

for arbitrary $u_x, u_{xx}$

thus $\frac{\partial \Lambda}{\partial u} = 0 \Rightarrow \Lambda \neq \Lambda(u)$

and $\Lambda(x,t)$ can be any solution of $\Lambda_t + \Lambda_{xx} = 0$

$\Rightarrow$ infinite number of multipliers

- to find more general $\Lambda = \Lambda(x,t,u,u_x,u_t \dots)$ would be more complicated but let us check that $u_x$ and $u_t$ are not multipliers giving some conservation laws although the heat equation is invariant under translations in time generated by $X_1 = \frac{\partial}{\partial t} \iff \hat{X}_1 = -u_t \frac{\partial}{\partial u}$ and translation in space generated by $X_2 = \frac{\partial}{\partial x} \iff \hat{X}_2 = -u_x \frac{\partial}{\partial u}$

$$E[u_t(u_{xx}-u_t)] = -\cancel{D_t(u_{xx})} + D_t(2u_t) + \cancel{D_x D_x u_t} = 2u_{tt} \neq 0$$

$$E[u_x(u_{xx}-u_t)] = -D_x(\cancel{u_{xx}}-u_t) + D_t(u_x) + \cancel{D_x D_x(u_x)} = 2u_{tx} \neq 0$$

- on the other hand, for the wave equation

$$u_{xx} - u_{tt} = 0$$

we would get

$$E[u_t(u_{xx}-u_{tt})] = -D_t(\cancel{u_{xx}}-u_{tt}) + D_x D_x u_t + \cancel{D_t D_t(-u_t)} = 0$$

$$E[u_x(u_{xx}-u_{tt})] = -D_x(\cancel{u_{xx}}-\cancel{u_{tt}}) + \cancel{D_x D_x u_x} + D_t D_t(-u_x) = 0$$

and thus $\Lambda = u_t$ and $\Lambda = u_x$ are multipliers giving local conservation laws (of energy and momentum) as they should because here we have variational formulation and Noether's theorem can be used to find LCL

- when is it possible to use Noether's theorem?

or when the given system of PDEs is the Euler-Lagrange equations of some variational problem?

- if there is a corresponding variational problem to the system of PDR $R^\sigma = 0$, $\sigma = 1, \dots, N$ then according to Noether's theorem the multipliers $\Lambda_\sigma$ are given by the characteristic of the symmetry $\Lambda_\sigma = \hat{\eta}^\sigma$

- to formulate the main result we need the notion of

- linearizing operator (Fréchet derivative)

- motivation: linearizing operator for nonlinear equations

  - let $u(x)$ be a solution of the system $R^\sigma(x, u, \partial u \ldots) = 0$, $\sigma = 1, \ldots, N$

  - then in a certain neighborhood of $x_0$ we can search for $v(x)$ instead of $u(x)$ such that

$$u(x) = u(x_0) + \varepsilon v(x)$$
$$u_1(x) = u_1(x_0) + \varepsilon v_1(x) \quad \text{etc.}$$

where $\varepsilon$ is „small", using the Taylor series of $R^\sigma$ in $\varepsilon$

we get

$$R^\sigma[u(x_0) + \varepsilon v(x)] = R^\sigma[u(x_0)] + \varepsilon\left(\sum_{\alpha, J} \frac{\partial R^\sigma}{\partial u^\alpha_J} D_J v^\alpha\right) + \sigma(\varepsilon^2) = 0$$

or the „correction" $v(x)$ must satisfy a system of linear

equations $$\underbrace{(D_R[u])}^\sigma_\alpha v^\alpha = \sum_{0 \le \# J \le k} \frac{\partial R^\sigma}{\partial u^\alpha_J} D_J v^\alpha = 0$$

this is the Fréchet derivative of $R$ at $u$

or it is a linearizing operator for the system $R$

( it is actually $N \times m$ operators, see indices $\sigma, \alpha$)

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Note: in general, it is a bounded linear operator $D_f(x) : V \to W$

from the Banach space $V$ into the Banach space $W$

if there exists the limit

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - D_f(x) h \|_W}{\| h \|_V} = 0$$

for a certain operator $D_f(x)$.

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- using the Fréchet derivative we can rewrite the infinitesimal criterion for generalized symmetries in the characteristic form

$$\hat{X}^{(\infty)} R^\sigma[u] \Big|_{D_J R^\sigma = 0} = 0$$

as $$\boxed{\left(D_R[u]\right)^{\sigma}_{\alpha}\hat{\eta}^{\alpha} = \sum_J (D_J\eta^{\alpha})\frac{\partial R^{\sigma}}{\partial u^{\alpha}_J} = 0}$$ evaluated on $D_J R^{\sigma} = 0$ (for solutions)

Example: for linear equations (operators) we get the same operator as the Fréchet derivative:

$$R = U_t - U_{xx} = 0 \quad \Rightarrow D_R V = D_t V - D_{xx} V = 0$$

single operator

but for nonlinear equations we get linear operator,

e.g. the Burgers' equation $R = U_t - U_{xx} - U_x^2 = 0$

gives after linearization

$$D_R V = D_t V - D_{xx} V - 2u_x D_x V = V_t - V_{xx} - 2u_x V_x = 0$$

here we see that it is linearization of the original eq. at certain „point" $u$

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- adjoint operator to $(D_R[u])^{\sigma}_{\alpha}$ is the operator ($m \times N$ operators)

$$\left(D^{*}_{R}[u]\right)^{\sigma}_{\alpha} W_{\sigma} = \sum_{0 \le \#J \le k} (-1)^{\#J} D_J\left(\frac{\partial R^{\sigma}[u]}{\partial u^{\alpha}_J} W_{\sigma}\right) =$$

$$= \frac{\partial R^{\sigma}}{\partial u^{\alpha}} W_{\sigma} - D_i\left(\frac{\partial R^{\sigma}}{\partial u^{\alpha}_i} W_{\sigma}\right) + D_i D_j\left(\frac{\partial R^{\sigma}}{\partial u^{\alpha}_{ij}} W_{\sigma}\right) - \dots$$

defined in such a way to satisfy

$$\int_{\Omega} W_{\sigma} (D_R)^{\sigma}_{\alpha} V^{\alpha} d^n x = \int_{\Omega} V^{\alpha} (D^{*}_{R})^{\sigma}_{\alpha} W_{\sigma} d^n x$$

- we say that the linear operator $(D_R[u])^{\sigma}_{\alpha}$ is self-adjoint

if $\quad (D_R[u])^{\sigma}_{\alpha} = (D^{*}_{R}[u])^{\sigma}_{\alpha}$

notice that $D_R$ is $N \times m$ matrix ($\sigma$ for rows, $\alpha$ for columns)
but $D^{*}_{R}$ is $m \times N$ matrix ($\alpha$ for rows, $\sigma$ for columns)

therefore for $D_R$ to be self-adjoint it must be $N = m$
(as for the Euler-Lagrange equations)

Example: For $R = u_{tt} - u_{xx} = 0$ we have

$$D_R[u] = D_{tt} - D_{xx} \quad \text{and} \quad D_R^*[u] = D_{tt} - D_{xx}$$

but for $R = u_t - u_{xx} = 0$ we have

$$D_R[u] = D_t - D_{xx} \quad \text{but} \quad D_R^*[u] = -D_t - D_{xx}$$

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- it can be shown (quite generally, but see conditions in Olver theorem 5.92)

that the given system of PDEs

$$R^\sigma(x, u, \partial u, .. \partial^k u) = 0 \qquad \sigma = 1, ..., N$$

in the form as it is written, are the Euler-Lagrange equations of a certain variational problem if and only if the linearizing operator $D_R[u]$ is self-adjoint,

and moreover $$L[u] = \int_0^1 \sum_\alpha u^\alpha R^\alpha[\lambda u]\, d\lambda$$

Note: using this formula we can get in general diffent Lagrangian then we have started, but these Lagrangians will be equivalent

- using the Fréchet derivative we can also rewrite the conditions for the multipliers $\Lambda_\sigma$:

$$0 = E_\alpha(\Lambda_\sigma R^\sigma) = \underbrace{(D_\Lambda^*)_\sigma^\alpha R^\sigma}_{\overset{\shortparallel}{0} \text{ for solutions}} + (D_R^*)_\alpha^\sigma \Lambda_\sigma$$

thus necessary (but not sufficient) conditions for $\Lambda_\sigma$ to be multipliers for LCL are

$$(D_R^*)_\alpha^\sigma \Lambda_\sigma = 0 \quad \text{for solutions } u(x) \text{ of } R^\sigma = 0$$

which gives the same conditions as

$$(D_R[u])_\alpha^\sigma \hat{\eta}^\alpha = 0 \quad \text{for solutions } u(x) \text{ of } R^\sigma = 0$$

for <u>self-adjoint operators $D_R$</u>

- because of the term $(D_R^*)_\alpha^\sigma \Lambda_\sigma$ there is actually less multipliers and also variational symmetries then symmetries of $R^\sigma = 0$