Lorentz transformation and one-dimensional wave equation

Problems to solve

1. Determine the infinitesimal generator

$$X = \xi^x(x,t)\frac{\partial}{\partial x} + \xi^t(x,t)\frac{\partial}{\partial t} \tag{1}$$

of the Lorentz transformation

$$\tilde{x} = \gamma(x - vt)\,, \quad \tilde{t} = \gamma(t - vx/c^2)\,, \quad \gamma = \sqrt{1 - v^2/c^2}\,. \tag{2}$$

2. By solving the following system of ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\varepsilon} &= \xi^x(\tilde{x},\tilde{t})\,, & \tilde{x}(0) &= x\,, \\ \frac{\mathrm{d}\tilde{t}}{\mathrm{d}\varepsilon} &= \xi^t(\tilde{x},\tilde{t})\,, & \tilde{t}(0) &= t \end{aligned} \tag{3}$$

express the transformation (2) using a parameter ε for which the group binary operation will be $\phi(\varepsilon_1, \varepsilon_2) = \varepsilon_1 + \varepsilon_2$. What is the relation between this parameter ε and the velocity v?

3. Check that the wave equation

$$\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi(x,t)}{\partial t^2} = 0 \tag{4}$$

is invariant under the Lorentz transformation (2). Guess (or find) other three basic symmetries of this equation.

4. Which of the symmetries found in the previous problem are also variational symmetries? The Lagrangian corresponding to the wave equation (4) is

$$L[\psi] = \frac{1}{2}\left[\left(\frac{\partial\psi(x,t)}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2\right]. \tag{5}$$

Choose one of these symmetries and find the corresponding conservation law.

5. Determine the particular solution of the equation (4), which is the invariant solution under Lorentz transformation.

Solution

1. The infinitesimal Lorentz transformation can be written as

$$\tilde{x}(x,t;v) = x + v\xi^x(x,t) + O(v^2)\,,$$
$$\tilde{t}(x,t;v) = t + v\xi^t(x,t) + O(v^2)\,,$$

where the infinitesimals $\xi^x(x,t)$ and $\xi^t(x,t)$ are given by

$$\xi^x(x,t) = \left.\frac{\partial\tilde{x}(x,t;v)}{\partial v}\right|_{v=0} = -t\,, \tag{6}$$

$$\xi^t(x,t) = \left.\frac{\partial\tilde{t}(x,t;v)}{\partial v}\right|_{v=0} = -\frac{x}{c^2}\,. \tag{7}$$

Thus the generator (1) is

$$X = -t\frac{\partial}{\partial x} - \frac{x}{c^2}\frac{\partial}{\partial t}\,. \tag{8}$$

2. Substituting (6) a (7) into the system (3), we get

$$\begin{aligned} \frac{\mathrm{d}\tilde{x}(\varepsilon)}{\mathrm{d}\varepsilon} &= -\tilde{t}(\varepsilon)\,, \qquad \tilde{x}(0) = x\,,\\ \frac{\mathrm{d}\tilde{t}(\varepsilon)}{\mathrm{d}\varepsilon} &= -\frac{\tilde{x}(\varepsilon)}{c^2}, \qquad \tilde{t}(0) = t\,. \end{aligned} \tag{9}$$

By taking the derivative of the first equation with respect of ε and substituting for the derivative $\mathrm{d}\tilde{t}/\mathrm{d}\varepsilon$ from the second equation, we obtain the ordinary differential equation

$$\frac{\mathrm{d}^2\tilde{x}(\varepsilon)}{\mathrm{d}\varepsilon^2} = \frac{\tilde{x}(\varepsilon)}{c^2}\,,$$

the general solution of which is

$$\tilde{x}(\varepsilon) = A\mathrm{e}^{\lambda_1\varepsilon} + B\mathrm{e}^{\lambda_2\varepsilon}, \tag{10}$$

where $\lambda_{1,2}$ are the solutions of the characteristic equation

$$\lambda^2 = \frac{1}{c^2}\,, \quad \text{thus} \quad \lambda_1 = \frac{1}{c}\,,\ \lambda_2 = -\frac{1}{c}\,.$$

For $\tilde{t}(\varepsilon)$ we get

$$\tilde{t}(\varepsilon) = -\frac{\mathrm{d}\tilde{x}(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{A}{c}\mathrm{e}^{\varepsilon/c} + \frac{B}{c}\mathrm{e}^{-\varepsilon/c}\,. \tag{11}$$

The unknown constants A, B can be determined from the initial conditions (9). We obtain the system of equations

$$\tilde{x}(\varepsilon = 0) = A + B = x\,, \quad \tilde{t}(\varepsilon = 0) = -\frac{1}{c}(A - B) = t\,,$$

the solution of which is

$$A = \frac{x - ct}{2}\,, \quad B = \frac{x + ct}{2}\,.$$

By substituting for A, B into (10) and (11) and by using the definition of the hyperbolic functions, we can express the Lorentz transformation as

$$\tilde{x}(x,t;\varepsilon) = x\cosh\frac{\varepsilon}{c} - ct\sinh\frac{\varepsilon}{c}\,, \tag{12}$$

$$c\tilde{t}(x,t;\varepsilon) = -x\sinh\frac{\varepsilon}{c} + ct\cosh\frac{\varepsilon}{c}\,. \tag{13}$$

We can see that it is a "rotation" in the $1+1$-Minkowsky space with the coordinates (x, ct). The relation between the new parametrization of the Lorentz transformation using ε and the standard parametrization using the velocity v can be obtained by comparison of the terms in front of x and t in the equations (2) and (12) for $\tilde{x}$

$$\cosh\frac{\varepsilon}{c} = \gamma\,, \quad -c\sinh\frac{\varepsilon}{c} = -\gamma v\,.$$

Dividing them, we get

$$\tanh\frac{\varepsilon}{c} = \frac{v}{c}\,. \tag{14}$$

Note that by composing two Lorentz transformations (2), or (12)–(13), the first with the parameter v_1, or ε_1, and the second with the parameter v_2, or ε_2, we get again the Lorentz transformation, this time with the parameter

$$v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}\,, \quad \text{resp.} \quad \varepsilon_3 = \varepsilon_1 + \varepsilon_2\,,$$

where the first relation is the well-known formula for composition of velocities in the special theory of relativity. We can derive it directly from the Lorentz transformation (2), or from the relation $\varepsilon_3 = \varepsilon_1 + \varepsilon_2$ by substituting for ε from (14) and using the formula

$$\tanh(\operatorname{arctanh}\alpha + \operatorname{arctanh}\beta) = \frac{\alpha+\beta}{1+\alpha\beta}\,.$$

3. In the following text, the partial derivatives will be written as indices with a preceding comma, for example $\psi_{,x} = \frac{\partial\psi}{\partial x}$, to distinguish it from other indices. The total derivative is denoted by D_x where x is the independent variable, thus

$$\mathrm{D}_x F(x,t,\psi(x,t),\psi_{,x}(x,t),\psi_{,t}(x,t)) = \frac{\partial F}{\partial x} + \psi_{,x}\frac{\partial F}{\partial \psi} + \psi_{,xx}\frac{\partial F}{\partial \psi_x} + \psi_{,tx}\frac{\partial F}{\partial \psi_t} \tag{15}$$

and similarly for functions F dependent on higher derivatives.

To check that the wave equation

$$\psi_{,xx} - \frac{1}{c^2}\psi_{,tt} = 0 \tag{16}$$

is invariant under the Lorentz transformation

$$\begin{aligned} \tilde{x} &= X(x,t,\psi;v) = \gamma(x - vt)\,,\\ \tilde{t} &= T(x,t,\psi;v) \ = \gamma(t - vx/c^2)\,,\\ \tilde{\psi} &= \Psi(x,t,\psi;v) \ = \psi\,, \end{aligned} \tag{17}$$

where $\gamma = \sqrt{1 - v^2/c^2}$, we can use three approaches.

(a) To show that the transformed solution $\psi(x,t)$ is also a solution of the given equation.

(b) To change variables of the wave equation ($\tilde{x}, \tilde{t}, \tilde{\psi} = \psi$) and to show that the new equation has the same form as the original one.

(c) To extend the infinitesimal generator (8) into the space of derivatives and to use the infinitesimal criterion.

Let us show all three approaches in detail for pedagogical reasons:

(a) Let $\psi = f(x,t)$ be an arbitrary solution of the equation (16). Under the transformation (17), it transforms into a new function, which is given implicitly by

$$\tilde{\psi} = \Psi(x,t,f(x,t),v) = \Psi(X(\tilde{x},\tilde{t},\tilde{\psi};-v), T(\tilde{x},\tilde{t},\tilde{\psi};-v), f(X(\tilde{x},\tilde{t},\tilde{\psi};-v), T(\tilde{x},\tilde{t},\tilde{\psi};-v)); v)$$

and for the Lorentz transformation we get

$$\tilde{\psi}(\tilde{x},\tilde{t}) = f(\gamma(\tilde{x} + v\tilde{t}), \gamma(\tilde{t} + v\tilde{x}/c^2)) = f(x,t)\,.$$

If the equation (16) is to be invariant under (17), the new function $\tilde{\psi}(\tilde{x},\tilde{t})$ must be a solution of the equation

$$\tilde{\psi}_{,\tilde{x}\tilde{x}} - \frac{1}{c^2}\tilde{\psi}_{,\tilde{t}\tilde{t}} = 0\,.$$

For derivatives, we get

$$\begin{aligned}
\frac{\partial \tilde{\psi}}{\partial \tilde{x}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \tilde{x}} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial \tilde{x}} = \gamma\left(\frac{\partial f}{\partial x} + \frac{v}{c^2}\frac{\partial f}{\partial t}\right), \\
\frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}^2} &= \gamma^2\left(\frac{\partial^2 f}{\partial x^2} + 2\frac{v}{c^2}\frac{\partial^2 f}{\partial x \partial t} + \frac{v^2}{c^4}\frac{\partial^2 f}{\partial t^2}\right), \\
\frac{\partial \tilde{\psi}}{\partial \tilde{t}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \tilde{t}} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial \tilde{t}} = \gamma\left(v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}\right), \\
\frac{\partial^2 \tilde{\psi}}{\partial \tilde{t}^2} &= \gamma^2\left(v^2\frac{\partial^2 f}{\partial x^2} + 2v\frac{\partial^2 f}{\partial x \partial t} + \frac{\partial^2 f}{\partial t^2}\right)
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}^2} - \frac{1}{c^2}\frac{\partial^2 \tilde{\psi}}{\partial \tilde{t}^2} &= \gamma^2\left[\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2 f}{\partial x^2} + 2\left(\frac{v}{c^2}-\frac{v}{c^2}\right)\frac{\partial^2 f}{\partial x \partial t} + \left(\frac{v^2}{c^4}-\frac{1}{c^2}\right)\frac{\partial^2 f}{\partial t^2}\right] = \\
&= \gamma^2\left(1-\frac{v^2}{c^2}\right)\left[\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}\right] = 0\,.
\end{aligned}$$

The last equality follows from the assumption that $f(x,t)$ is the solution of the wave equation (16). We showed that the function $\tilde{\psi}(\tilde{x},\tilde{t})$ is also the solution of the wave equation and thus this equation is invariant under the Lorentz transformation.

(b) To write the equation (16) in new variables $\tilde{x}, \tilde{t}, and \tilde{\psi}$ given by (17), we have to express all derivatives of ψ with respect to x and t using the derivatives of $\tilde{\psi}$ with respect to $\tilde{x}$ and $\tilde{t}$. We start with the inverse transformations

$$x = X(\tilde{x},\tilde{t},\tilde{\psi};-v) = \gamma(\tilde{x}+v\tilde{t})\,, \tag{18}$$

$$t = T(\tilde{x},\tilde{t},\tilde{\psi};-v) = \gamma(\tilde{t}+v\tilde{x}/c^2)\,, \tag{19}$$

$$\psi = \Psi(\tilde{x},\tilde{t},\tilde{\psi};-v) = \tilde{\psi} \tag{20}$$

and write the total differential of the function ψ

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial x}\mathrm{d}x + \frac{\partial \psi}{\partial t}\mathrm{d}t \tag{21}$$

using (20) as

$$\mathrm{d}\psi = \mathrm{D}_{\tilde{x}}\Psi \mathrm{d}\tilde{x} + \mathrm{D}_{\tilde{t}}\Psi \mathrm{d}\tilde{t}\,, \tag{22}$$

where we used the total derivatives instead of partial derivatives because the function Ψ in the equation (20) is, in general, a function of $\tilde{x}, \tilde{t}$, and $\tilde{\psi}(\tilde{x},\tilde{t})$. The differentials $\mathrm{d}x$ and $\mathrm{d}t$ can be expressed in a similar way from the equations (18) and (19)

$$\mathrm{d}x = \mathrm{D}_{\tilde{x}}X \mathrm{d}\tilde{x} + \mathrm{D}_{\tilde{t}}X \mathrm{d}\tilde{t}\,, \tag{23}$$

$$\mathrm{d}t = \mathrm{D}_{\tilde{x}}T \mathrm{d}\tilde{x} + \mathrm{D}_{\tilde{t}}T \mathrm{d}\tilde{t} \tag{24}$$

and by substituting into (21) and by comparing the expressions in front of $\mathrm{d}\tilde{x}$ and $\mathrm{d}\tilde{t}$ with those in the equation (22) we finally get a system of equations for the derivatives of ψ with respect to x and t

$$\mathrm{D}_{\tilde{x}}X\frac{\partial \psi}{\partial x} + \mathrm{D}_{\tilde{x}}T\frac{\partial \psi}{\partial t} = \mathrm{D}_{\tilde{x}}\Psi\,, \tag{25}$$

$$\mathrm{D}_{\tilde{t}}X\frac{\partial \psi}{\partial x} + \mathrm{D}_{\tilde{t}}T\frac{\partial \psi}{\partial t} = \mathrm{D}_{\tilde{t}}\Psi\,. \tag{26}$$

Using the Jacobian of the transformations (18)–(19)

$$J = \begin{pmatrix} \mathrm{D}_{\tilde{x}}X & \mathrm{D}_{\tilde{x}}T \\ \mathrm{D}_{\tilde{t}}X & \mathrm{D}_{\tilde{t}}T \end{pmatrix},$$

we can find the solution of the system (25)–(26) if we know the inverse matrix J^{-1}. For the Lorentz transformation we get

$$J = \begin{pmatrix} \gamma & v\gamma/c^2 \\ v\gamma & \gamma \end{pmatrix}, \qquad J^{-1} = \begin{pmatrix} \gamma & -v\gamma/c^2 \\ -v\gamma & \gamma \end{pmatrix},$$

and thus

$$\begin{pmatrix} \frac{\partial\psi}{\partial x} \\ \frac{\partial\psi}{\partial t} \end{pmatrix} = J^{-1} \begin{pmatrix} \mathrm{D}_{\tilde{x}}\Psi \\ \mathrm{D}_{\tilde{t}}\Psi \end{pmatrix} = \begin{pmatrix} \gamma\frac{\partial\tilde{\psi}}{\partial\tilde{x}} - \gamma\frac{v}{c^2}\frac{\partial\tilde{\psi}}{\partial\tilde{t}} \\ -v\gamma\frac{\partial\tilde{\psi}}{\partial\tilde{x}} + \gamma\frac{\partial\tilde{\psi}}{\partial\tilde{t}} \end{pmatrix}.$$

In a similar way, we can find the second derivatives $\psi_{,xx}$ and $\psi_{,xt}$ from the transformation

$$\psi_{,x} = \Psi_x(\tilde{x}, \tilde{t}, \tilde{\psi}, \tilde{\psi}_{,\tilde{x}}, \tilde{\psi}_{,\tilde{t}}) = \gamma\tilde{\psi}_{,\tilde{x}} - \gamma\frac{v}{c^2}\tilde{\psi}_{,\tilde{t}}$$

and $\psi_{,tx}$ and $\psi_{,tt}$ from the transformation

$$\psi_{,t} = \Psi_t(\tilde{x}, \tilde{t}, \tilde{\psi}, \tilde{\psi}_{,\tilde{x}}, \tilde{\psi}_{,\tilde{t}}) = -v\gamma\tilde{\psi}_{,\tilde{x}} + \gamma\tilde{\psi}_{,\tilde{t}}\,.$$

Again by comparing the differentials and by solving systems similar to (25)–(26), we found the formulas

$$\begin{pmatrix} \psi_{,xx} \\ \psi_{,xt} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2\psi}{\partial x^2} \\ \frac{\partial^2\psi}{\partial x\partial t} \end{pmatrix} = J^{-1}\begin{pmatrix} \mathrm{D}_{\tilde{x}}\Psi_x \\ \mathrm{D}_{\tilde{t}}\Psi_x \end{pmatrix} = \begin{pmatrix} \gamma^2\left[\tilde{\psi}_{,\tilde{x}\tilde{x}} - 2\frac{v}{c^2}\tilde{\psi}_{,\tilde{x}\tilde{t}} + \frac{v^2}{c^4}\tilde{\psi}_{,\tilde{t}\tilde{t}}\right] \\ \gamma^2\left[-v\tilde{\psi}_{,\tilde{x}\tilde{x}} + \left(1+\frac{v^2}{c^2}\right)\tilde{\psi}_{,\tilde{x}\tilde{t}} - \frac{v}{c^2}\tilde{\psi}_{,\tilde{t}\tilde{t}}\right] \end{pmatrix},$$

$$\begin{pmatrix} \psi_{,tx} \\ \psi_{,tt} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2\psi}{\partial t\partial x} \\ \frac{\partial^2\psi}{\partial t\partial t} \end{pmatrix} = J^{-1}\begin{pmatrix} \mathrm{D}_{\tilde{x}}\Psi_t \\ \mathrm{D}_{\tilde{t}}\Psi_t \end{pmatrix} = \begin{pmatrix} \gamma^2\left[-v\tilde{\psi}_{,\tilde{x}\tilde{x}} + \left(1+\frac{v^2}{c^2}\right)\tilde{\psi}_{,\tilde{x}\tilde{t}} - \frac{v}{c^2}\tilde{\psi}_{,\tilde{t}\tilde{t}}\right] \\ \gamma^2\left[v^2\tilde{\psi}_{,\tilde{x}\tilde{x}} - 2v\tilde{\psi}_{,\tilde{x}\tilde{t}} + \tilde{\psi}_{,\tilde{t}\tilde{t}}\right] \end{pmatrix},$$

where $\psi_{,xt}$ and $\psi_{,tx}$ transform in the same way, as one can expect.
Finally, by substituting for $\psi_{,xx}$ and $\psi_{,tt}$ into the wave equation (16), we get

$$0 = \psi_{,xx} - \frac{1}{c^2}\psi_{,tt} = \gamma^2\left[\left(1 - \frac{v^2}{c^2}\right)\tilde{\psi}_{,\tilde{x}\tilde{x}} + \left(\frac{v^2}{c^4} - \frac{1}{c^2}\right)\tilde{\psi}_{,\tilde{t}\tilde{t}}\right] = \tilde{\psi}_{,\tilde{x}\tilde{x}} - \frac{1}{c^2}\tilde{\psi}_{,\tilde{t}\tilde{t}}\,.$$

Under the Lorentz transformation the wave equation (16) changes into the same equation and thus it is invariant under this transformation.

(c) Because the wave equation (16) is of the second order, we need the extension of the infinitesimal generator (8) up to the space of the second derivatives which is of the general form

$$X^{(2)} = \xi^x\frac{\partial}{\partial x} + \xi^t\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial\psi} + \eta_x^{(1)}\frac{\partial}{\partial\psi_{,x}} + \eta_t^{(1)}\frac{\partial}{\partial\psi_{,t}} + \eta_{xx}^{(2)}\frac{\partial}{\partial\psi_{,xx}} + \eta_{xt}^{(2)}\frac{\partial}{\partial\psi_{,xt}} + \eta_{tt}^{(2)}\frac{\partial}{\partial\psi_{,tt}},$$

where $\eta_x^{(1)}$ and $\eta_t^{(1)}$ determine the extension into the space of the first derivatives via

$$\begin{aligned} \psi'_{,x'} &= \psi_{,x} + \varepsilon\eta_x^{(1)}(x, t, \psi, \psi_{,x}, \psi_{,t}) + O(\varepsilon^2)\,, \\ \psi'_{,t'} &= \psi_{,t} + \varepsilon\eta_t^{(1)}(x, t, \psi, \psi_{,x}, \psi_{,t}) + O(\varepsilon^2)\,. \end{aligned}$$

and similarly $\eta_{xx}^{(2)}$, $\eta_{xt}^{(2)}$, and $\eta_{tt}^{(2)}$ determine the extension into the space of the second derivatives. The infinitesimal criterion for the wave equation (16) gives

$$X^{(2)}(\psi_{,xx} - \frac{1}{c^2}\psi_{,tt}) = \eta_{xx}^{(2)} - \frac{1}{c^2}\eta_{tt}^{(2)} = 0\,, \tag{27}$$

which must be satisfied for each solution $\psi(x,t)$ of the wave equation (16). Thus we have to determine only $\eta_{xx}^{(2)}$ and $\eta_{tt}^{(2)}$. To do that we use the general formulas

$$\begin{aligned} \eta_{x_i}^{(1)} &= \mathrm{D}_{x_i}\eta - \sum_k(\mathrm{D}_{x_i}\xi^{x_k})\psi_{,x_k}\,, \\ \eta_{x_ix_j}^{(2)} &= \mathrm{D}_{x_j}\eta_{x_i}^{(1)} - \sum_k(\mathrm{D}_{x_j}\xi^{x_k})\psi_{,x_ix_k}\,. \end{aligned}$$

In our case we have $x_1 = x, x_2 = t, \xi^x = -t, \xi^t = -x/c^2, \eta = 0$ and we get

$$\begin{aligned}
\eta_x^{(1)} &= -(\mathrm{D}_x \xi^t)\psi_{,t} = \frac{1}{c^2}\psi_{,t}\,, \\
\eta_{xx}^{(2)} &= \mathrm{D}_x \eta_x^{(1)} - (\mathrm{D}_x \xi^t)\psi_{,xt} = \frac{2}{c^2}\psi_{,xt}\,, \\
\eta_t^{(1)} &= -(\mathrm{D}_t \xi^x)\psi_{,x} = \psi_{,x}\,, \\
\eta_{tt}^{(2)} &= \mathrm{D}_t \eta_t^{(1)} - (\mathrm{D}_t \xi^x)\psi_{,tx} = 2\psi_{,xt}\,,
\end{aligned}$$

By substituting for $\eta_{xx}^{(2)}$ and $\eta_{tt}^{(2)}$ into the condition (27), we finally obtain

$$X^{(2)}(\psi_{,xx} - \frac{1}{c^2}\psi_{,tt}) = \frac{2}{c^2}\psi_{,xt} - \frac{2}{c^2}\psi_{,xt} = 0$$

for solutions of the wave equation (16) which is thus invariant under the Lorentz transformation.

To find at least some other point symmetries of the wave equation (16) it is not necessary to use the infinitesimal criterion (27) with general *ansatz* for ξ^x, ξ^t, and η nad to solve the resulting system of partial differential equations (even though it is possible in the case).

It is enough to realize that the wave equation (16) is not explicitly dependent on the variables x, t, or ψ, and thus it must be invariant under translations in all these variables. Moreover, it is a homogeneous linear equation which depends only on the second derivatives and thus it must be also invariant under scaling of the dependent variable ($\tilde{\psi} = \alpha\psi$) and also under simultaneous scaling of the independent variables ($\tilde{x} = \beta x, \tilde{t} = \beta t$). The infinitesimal generators corresponding to these point transformations are

$$X_1 = \frac{\partial}{\partial x}\,, \tag{28}$$

$$X_2 = \frac{\partial}{\partial t}\,, \tag{29}$$

$$X_3 = \frac{\partial}{\partial \psi}\,, \tag{30}$$

$$X_4 = \psi\frac{\partial}{\partial \psi}\,, \tag{31}$$

$$X_5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\,. \tag{32}$$

4. If the point symmetry of the wave equation (16) is to be also the variational symmetry, it has to satisfy the infinitesimal criterion of the invariance of the corresponding variational functional

$$X^{(1)}L + L(\mathrm{D}_x\xi^x + \mathrm{D}_t\xi^t) = 0\,, \tag{33}$$

where L is the Lagrangian

$$L[\psi] = \frac{1}{2}\left(\psi_{,x}^2 - \frac{1}{c^2}\psi_{,t}^2\right), \tag{34}$$

and $X^{(1)}$ is the first extension of the given infinitesimal generator. Without going into details (derivation of $X^{(1)}$ is completely analogous to that in the paragraph 4c), we state that the condition (33) is satisfied for the generators (8), (28), (29), (30), and (32). The point symmetry generated by the operator (31) is not variational symmetry, because

$$X_4^{(1)} = \psi\frac{\partial}{\partial\psi} + \psi_{,x}\frac{\partial}{\partial\psi_{,x}} + \psi_{,t}\frac{\partial}{\partial\psi_{,t}}$$

and thus

$$X_4^{(1)}L + L(\mathrm{D}_x\xi^x + \mathrm{D}_t\xi^t) = 2L \neq 0\,.$$

We determine the conservation laws corresponding to translations of the independent variables. The general form of the conservation law for the wave equation (16) is given by

$$\mathrm{Div}P = \mathrm{D}_x P^x + \mathrm{D}_t P^t = 0\,, \tag{35}$$

where quantities P^x and P^t are some functions of variables x, t, ψ and derivatives of ψ with respect to x and t. From Noether's theorem, we get for these quantities

$$P^x = (\xi^x \psi_{,x} + \xi^t \psi_{,t} - \eta) \frac{\partial L}{\partial \psi_{,x}} - L\xi^x \,, \tag{36}$$

$$P^t = (\xi^x \psi_{,x} + \xi^t \psi_{,t} - \eta) \frac{\partial L}{\partial \psi_{,t}} - L\xi^t \,. \tag{37}$$

For the generators (28) and (29), for which $\xi^x = 1, \xi^t = 0, \eta = 0$, and $\xi^x = 0, \xi^t = 1, \eta = 0$ respectively, we get

$$P_1^x = \psi_{,x} \frac{\partial L}{\partial \psi_{,x}} - L\xi^x = \frac{1}{2}\left(\psi_{,x}^2 + \frac{1}{c^2}\psi_{,t}^2\right) \,, \tag{38}$$

$$P_1^t = \psi_{,x} \frac{\partial L}{\partial \psi_{,t}} = -\frac{1}{c^2}\psi_{,x}\psi_{,t} \,, \tag{39}$$

and

$$P_2^x = \psi_{,t} \frac{\partial L}{\partial \psi_{,x}} = \psi_{,x}\psi_{,t} \,, \tag{40}$$

$$P_2^t = \psi_{,t} \frac{\partial L}{\partial \psi_{,t}} - L\xi^t = -\frac{1}{2}\left(\psi_{,x}^2 + \frac{1}{c^2}\psi_{,t}^2\right) \,. \tag{41}$$

These quantities satisfy the equation (35) as one can easily check. Note that by collecting these quantities into the matrix

$$T_\mu^\nu = \begin{pmatrix} P_1^x & P_1^t \\ P_2^x & P_2^t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\psi_{,x}^2 + \frac{1}{c^2}\psi_{,t}^2\right) & -\frac{1}{c^2}\psi_{,x}\psi_{,t} \\ \psi_{,x}\psi_{,t} & -\frac{1}{2}\left(\psi_{,x}^2 + \frac{1}{c^2}\psi_{,t}^2\right) \end{pmatrix} ,$$

we obtain the "tensor of energy and momentum" know from the theory of relativity for the scalar field $\psi(x, t)$, for which the conservation law is usually written as

$$T_{\mu\,,\nu}^{\nu} = 0 \,.$$

In a similar way, we could find other conservation laws for other variational symmetries.

5. To find the particular solution $\psi = f(x, t)$ of the wave equation

$$\psi_{,xx} - \frac{1}{c^2}\psi_{,tt} = 0 \tag{42}$$

invarian under the Lorentz transformation ($\xi^x = -t, \xi^t = -x/c^2, \eta = 0$), i.e. satisfying the condition

$$X(\psi - f(x,t))|_{\psi=f(x,t)} = \eta - \xi^x \frac{\partial f}{\partial x} - \xi^t \frac{\partial f}{\partial t}\bigg|_{\psi=f(x,t)} = t\frac{\partial f}{\partial x} + \frac{x}{c^2}\frac{\partial f}{\partial t} = 0 \,, \tag{43}$$

we can use two methods.

(a) *Invariant-form method:* we first find a general solution of the condition (43) and then we substitute this solution into the wave equation (42) and solve the resulting equation.

(b) *Method of direct substitution:* we first reduce the number of independent variables in the wave equation (42) using the condition (43), getting the ordinary differential equation, the solution of which will be dependent on arbitrary functions of the excluded variable, these functions are then determined by substituting into (43).

We use both methods for pedagogical reasons:

(a) **Invariant-form method**

The condition (43) is a linear partial differential equation and its solution can be find using the method of characteristics. The characteristic equation

$$\frac{\mathrm{d}x}{t} = c^2 \frac{\mathrm{d}t}{x} \,,$$

which is an ordinary differential equation, has the solution

$$x^2 - c^2t^2 = \text{const} = z\,,$$

and thus the general solution of the equation (43) is

$$f(x,t) = F(z) = F(x^2 - c^2t^2)\,,$$

where F is an arbitrary function. By substituting into the wave equation (42), we get the following ordinary differential equation for $F(z)$

$$\frac{\mathrm{d}F}{\mathrm{d}z} + z\frac{\mathrm{d}^2F}{\mathrm{d}z^2} = 0\,.$$

Its general solution (set $\frac{\mathrm{d}^2F}{\mathrm{d}z^2} = G(z)$ and solve for $G(z) first$) is

$$F(z) = A\ln z + B\,,$$

where A and B are arbitrary constants. The invariant solution of the wave equation (42) thus is given by

$$\psi(x,t) = A\ln(x^2 - c^2t^2) + B\,. \tag{44}$$

(b) **Method of direct substitution**

We rewrite the condition (43) as

$$\frac{\partial f}{\partial x} = -\frac{x}{tc^2}\frac{\partial f}{\partial t} \tag{45}$$

and we take the derivative with respect to x

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{tc^2}\frac{\partial f}{\partial t} - \frac{x}{tc^2}\frac{\partial^2 f}{\partial t\partial x} = -\left(\frac{1}{tc^2} + \frac{x^2}{t^3c^4}\right)\frac{\partial f}{\partial t} + \frac{x^2}{t^2c^4}\frac{\partial^2 f}{\partial t^2}. \tag{46}$$

The last equality follows from the substitution for $\partial f/\partial x$ from the equation (45). By substituting into the wave equation (42) from (46), we get

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \left(\frac{x^2}{t^2c^4} - \frac{1}{c^2}\right)\frac{\partial^2 f}{\partial t^2} - \left(\frac{1}{tc^2} + \frac{x^2}{t^3c^4}\right)\frac{\partial f}{\partial t} = 0\,.$$

In this equation, x is a parameter and we can solve it as an ordinary differential equation for $f(t;x)$ parametrically dependent on x. Its solution can be found by letting $\partial f(t;x)/\partial t = g(t;x)$ and integrating

$$\ln g(t;x) = \int \frac{x^2 + t^2c^2}{x^2 - t^2c^2}\frac{\mathrm{d}t}{t} = \ln\frac{t}{x^2 - t^2c^2} + c(x)\,,$$

where the integration constant can in general be an arbitrary functions of x. By letting $c(x) = \ln C(x)$ we get

$$g(t;x) = \frac{C(x)t}{x^2 - t^2c^2}\,.$$

The function $f(t;x)$ is then determined by another integration

$$f(t;x) = \int g(t;x)\mathrm{d}t = -\frac{C(x)}{2c^2}\ln(x^2 - t^2c^2) + B(x) = A(x)\ln(x^2 - t^2c^2) + B(x)$$

with two arbitrary functions $A(x)$ and $B(x)$. If the function $f(x,t)$ is to be the invariant solution of the wave equation (42) we must to determine $A(x)$ and $B(x)$ to be satisfied the condition (43). We obtain

$$t\frac{\partial f}{\partial x} + \frac{x}{c^2}\frac{\partial f}{\partial t} = t\frac{\mathrm{d}A(x)}{\mathrm{d}x}\ln(x^2 - t^2c^2) + t\frac{\mathrm{d}B(x)}{\mathrm{d}x} = 0\,.$$

This equation must be satisfied for arbitrary t and thus

$$\frac{\mathrm{d}A(x)}{\mathrm{d}x} = 0\,, \qquad \frac{\mathrm{d}B(x)}{\mathrm{d}x} = 0\,.$$

We see that the functions $A(x)$ and $B(x)$ must be constant. Again, we get the invariant solution given by (44).