Bodové symetrie ODR $\frac{d^2y}{dx^2} = 0$ (volná částice v 1D)

- hledejme všechny inf. operátory
$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$
pro které je splněno infin. kritérium $X^{(2)} y_2 \Big|_{y_2=0} = 0$, kde $y_2 = \frac{d^2y}{dx^2}$

- rozšíření do prostoru derivací:
$$\eta^{(1)}(x,y,y_1) = D_x \eta - (D_x\xi)\, y_1 = \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial y} y_1 - \frac{\partial \xi}{\partial x} y_1 - \frac{\partial \xi}{\partial y} y_1^2$$
$$\eta^{(2)}(x,y,y_1,y_2) = D_x \eta^{(1)} - (D_x\xi)\, y_2 = \frac{\partial^2 \eta}{\partial x^2} + 2\frac{\partial^2 \eta}{\partial x \partial y} y_1 + \frac{\partial^2 \eta}{\partial y^2} y_1^2 -$$
$$- \frac{\partial^2 \xi}{\partial x^2} y_1 - 2\frac{\partial^2 \xi}{\partial x \partial y} y_1^2 - \frac{\partial \xi}{\partial x} y_2 - \frac{\partial^2 \xi}{\partial y^2} y_1^3 - \frac{\partial \xi}{\partial y} 2 y_1 y_2$$

- infinitez. kritérium pak dává
$$X^{(2)} y_2 \Big|_{y_2=0} = \eta^{(2)}(x,y,y_1,y_2)\Big|_{y_2=0} = 0$$

přičemž tato podmínka musí být splněna pro všechna x, y a y_1.
Protože $\xi(x,y)$ a $\eta(x,y)$ nezávisí na y_1 a podmínka $\eta^{(2)}(x,y,y_1,y_2=0) = 0$ je polynomiální rovnice v y_1 třetího stupně, musí být každý koeficient u mocnin y_1 roven nule. Odtud

u y_1^3: $\frac{\partial^2 \xi}{\partial y^2} = 0$, u y_1^2: $\frac{\partial^2 \eta}{\partial y^2} - 2\frac{\partial^2 \xi}{\partial x \partial y} = 0$, u y_1: $2\frac{\partial^2 \eta}{\partial x \partial y} - \frac{\partial^2 \xi}{\partial x^2} = 0$

a konečně u konstantního členu v y_1: $\frac{\partial^2 \eta}{\partial x^2} = 0$

dosazení za $\frac{\partial^2 \xi}{\partial y^2} = 0$

- z prvních dvou rovnic navíc dostaneme (proderivování podle y) $\frac{\partial^3 \eta}{\partial y^3} - 2\frac{\partial^3 \xi}{\partial x \partial y^2} = \frac{\partial^3 \eta}{\partial y^3} = 0$

  a z druhých dvou obdobně $2\frac{\partial^3 \eta}{\partial x^2 \partial y} - \frac{\partial^3 \xi}{\partial x^3} = -\frac{\partial^3 \xi}{\partial x^3} = 0$

- vidíme tedy, že $\xi(x,y)$ je nejvýše lineární v y a kvadratická v x
  a $\eta(x,y)$ je nejvýše lineární v x a kvadratická v y

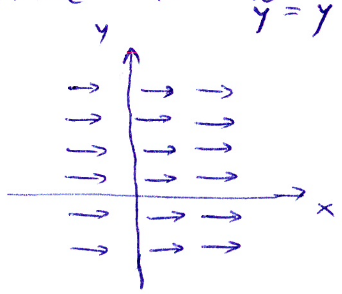
$$\Rightarrow \quad \xi(x,y) = c_1 + c_2 x + c_3 x^2 + c_4 y + c_5 xy + c_6 x^2 y$$
$$\eta(x,y) = d_1 + d_2 x + d_3 y + d_4 xy + d_5 y^2 + d_6 xy^2$$

- ovšem dosazení do $\frac{\partial^2 \eta}{\partial y^2} - 2\frac{\partial^2 \xi}{\partial x \partial y} = 0$ a $2\frac{\partial^2 \eta}{\partial x \partial y} - \frac{\partial^2 \xi}{\partial x^2} = 0$
ještě dostaneme podmínky
$$\left.\begin{aligned} 2(d_5 + d_6 x) - 2(c_5 + 2c_6 x) = 0 \\ 2(d_4 + 2d_6 y) - 2(c_3 + c_6 y) = 0 \end{aligned}\right\} \Rightarrow \quad d_5 = c_5,\quad d_4 = c_3,\quad c_6 = d_6 = 0$$
neboť to musí platit pro libovolné x a y.

• dostali jsme tak nejobecnější ξ a η s osmi nezávislými konstantami $\Rightarrow$ 8-par. Lieova grupa transformací s osmi lin. nezávislými infin. generátory (vždy položíme jednu konstantu rovnu 1 a ostatní rovny 0):

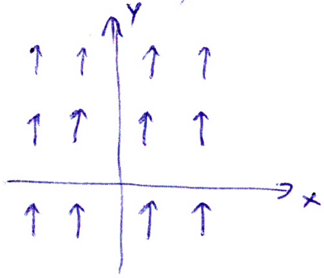
$X_1 = \frac{\partial}{\partial x}$ (pro $c_1 = 1$)

translace v x: $\tilde{x} = x + \varepsilon$, $\tilde{y} = y$



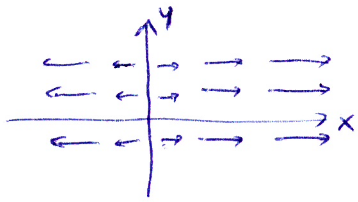
$X_2 = \frac{\partial}{\partial y}$ ($d_1 = 1$)

translace v y: $\tilde{x} = x$, $\tilde{y} = y + \varepsilon$



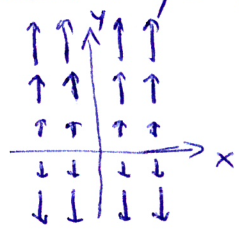
$X_3 = x\frac{\partial}{\partial x}$ ($c_2 = 1$)

škálování v x: $\tilde{x} = e^{\varepsilon}x$, $\tilde{y} = y$



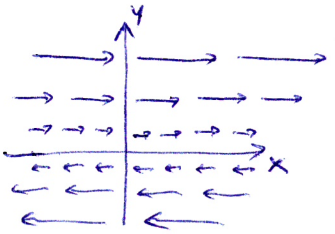
$X_4 = y\frac{\partial}{\partial y}$ ($d_3 = 1$)

škálování v y: $\tilde{x} = x$, $\tilde{y} = e^{\varepsilon}y$



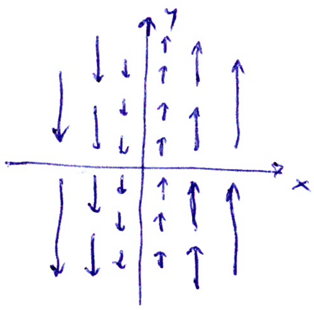
$X_5 = y\frac{\partial}{\partial x}$ ($c_4 = 1$)

Galileova transformace v x: $\tilde{x} = x + \varepsilon y$, $\tilde{y} = y$



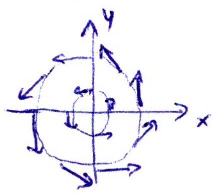
$X_6 = x\frac{\partial}{\partial y}$ ($d_2 = 1$)

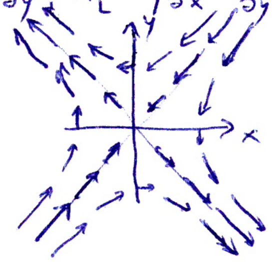
Galileova transformace v y: $\tilde{x} = x$, $\tilde{y} = y + \varepsilon x$



$\Rightarrow$ jejich lineární kombinace dávají rotace a Lorentz. transf. $\Leftarrow$

$X_R = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, $X_L = -y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$



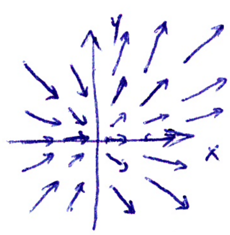


$X_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$

($c_3 = d_4 = 1$)

projektivní transformace v x:

$$\tilde{x} = \frac{x}{1-\varepsilon x}, \quad \tilde{y} = \frac{y}{1-\varepsilon x}$$



$X_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$ ($d_5 = c_5 = 1$)

projektivní transformace v y:

$$\tilde{x} = \frac{x}{1-\varepsilon y}$$

$$\tilde{y} = \frac{y}{1-\varepsilon y}$$