STATICKÝ PROSTOROČAS

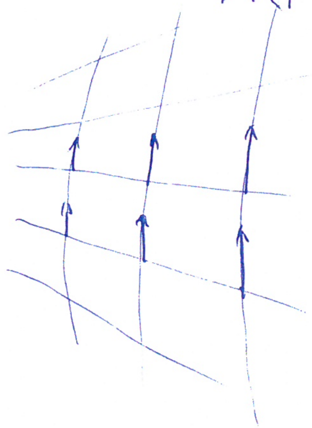
Killingův vektor ξ kolmý k nadploše

$\Rightarrow$ lze zvolit 3+1 rozštěpení přizpůsobené ξ

$\vec{t} = \xi$ t = konst kolmé na $\vec{t}$ $\Rightarrow$ $\vec{N} = 0$ $N = |\xi^a \xi^b g_{ab}|^{\frac{1}{2}}$

žádná veličina nezávisí na t

$\dot{N} = 0$ $\dot{q} = 0$ $\dot{V} = 0$



prostorové bi-distribuce (na Σ)

$\tilde{q} = \frac{1}{N} g^{\frac{1}{2}} \delta$ $\dot{\tilde{q}} = 0$

$\tilde{v} = \tilde{d}_a \cdot_{\Sigma} (N q^{ab} g^{\frac{1}{2}} \delta) \cdot \tilde{d}_b + N V g^{\frac{1}{2}} \delta$ $\dot{\tilde{v}} = 0$

$\tilde{p} = 0$

Hamiltonián

$$H = \frac{1}{2}(\pi \cdot \tilde{q}^{-1} \cdot \pi + \varphi \cdot \tilde{v} \cdot \varphi)$$

$$= \frac{1}{2}\int_{\Sigma} (\pi^2 g^{-1} + d_a\varphi \, d_b\varphi \, q^{ab} + V \varphi^2) N g^{\frac{1}{2}}$$

pohybové rovnice

$$\dot{\varphi} = \frac{\delta H}{\delta \pi} = \tilde{q}^{-1} \cdot \pi$$

$$\dot{\pi} = -\frac{\delta H}{\delta \varphi} = -\tilde{v} \cdot \varphi$$

$\Downarrow$ $(\tilde{q} \cdot \dot{\varphi})^{\cdot} = -\tilde{v} \cdot \varphi$

$\Downarrow$ $\ddot{\varphi} + \Omega^2 \cdot \varphi = 0$ $\Omega^2 = \tilde{q}^{-1} \cdot \tilde{v}$ frekv. operátor

obecné řešení

$$\varphi(t) = \cos(\Omega t) \cdot \varphi_0 + \Omega^{-1} \cdot \sin(\Omega t) \cdot \tilde{q} \cdot \pi_0$$

$\varphi|_{t=0} = \varphi_0$ $\pi|_{t=0} = \pi_0$

definice $\cos(\Omega t)$, $\sin(\Omega t)$ pomocí módů

Frekvenční operátor

$$\Omega^2 = \tilde{q}^{-1} \cdot \tilde{v}$$
$$= -N g^{\frac{1}{2}} \overleftrightarrow{d_a} \cdot [N q^{ab} g^{\frac{1}{2}} \overleftrightarrow{d_b}] + N V g^{\frac{1}{2}} \delta$$
$$= N^2 [-N^2 \mathring{\Delta} + V] \delta = N^2 [-\Delta + \lambda^a d_a + V] \delta$$

$\mathring{q}^{ab} \mathring{\nabla}_a \mathring{\nabla}_b$ $\qquad$ $q^{ab} \nabla_a \nabla_b$ $\qquad$ $q^{ab} d_b \ln N$

$\mathring{q}_{ab} = N^2 q_{ab}$

Ω^2 pos. def. (pro $N>0$ $V>0$)

symetr. vůči $\tilde{q}$

$$\tilde{q} \cdot \Omega^2 = \tilde{v} = \Omega^2 \cdot \tilde{q} \qquad \text{tj.} \quad (\varphi, \Omega^2 \psi)^{\sim} = \varphi \cdot \tilde{v} \cdot \psi = (\Omega^2 \cdot \varphi, \psi)^{\sim}$$

lze defi-ovat fce $F(\Omega^2)$, tj. i Ω

$$\Omega = [N^2 [-\Delta + \lambda^a d_a + V] \delta]^{\frac{1}{2}}$$

vlastní funkce

$$\Omega \cdot \vartheta_k = \omega_k \vartheta_k \iff \Omega^2 \cdot \vartheta_k = \omega_k^2 \vartheta_k$$

ϑ_k normaliz. báze $\qquad \omega_k \geq 0$

reálné módy

prostor $\mathcal{V}[\Sigma]$ $\qquad (\varphi, \psi)^{\sim} = \varphi \cdot \tilde{q} \cdot \psi = \int_\Sigma \varphi(x) \psi(x) \frac{1}{N} g^{\frac{1}{2}}$

Ω symetrické

$$(\vartheta_k, \vartheta_l)^{\sim} = \delta_{kl} \qquad \sum_k \vartheta_k \vartheta_k = \tilde{q}^{-1}$$

komplexní módy

prostor $\mathcal{V}[\Sigma]^{\mathbb{C}}$ $\qquad (\varphi, \psi)^{\sim}_{\mathbb{C}} = \varphi^* \cdot \tilde{q} \cdot \psi = \int \varphi^* \psi \frac{1}{N} g^{\frac{1}{2}}$

Ω hermitovské, reálné

ϑ_k ϑ_k^* stejné vl. čísla

nechť jsou lin. nez., tj. $\vartheta_k, \vartheta_k^* \leftrightarrow \operatorname{Re} \vartheta_k, \operatorname{Im} \vartheta_k$

obsahuje stejnou informaci

$$(\vartheta_k, \vartheta_l)^{\sim}_{\mathbb{C}} = \delta_{kl} \qquad (\vartheta_k^*, \vartheta_l)^{\sim}_{\mathbb{C}} = 0$$
$$\Updownarrow \qquad\qquad \Updownarrow$$
$$\vartheta_k^* \cdot \tilde{q} \cdot \vartheta_l = \delta_{kl} \qquad \vartheta_k \cdot \tilde{q} \cdot \vartheta_l = 0$$
$$\tilde{q}^{-1} = \sum_k (\vartheta_k^* \vartheta_k + \vartheta_k \vartheta_k^*)$$

Rozštěpení na pos/neg frekvenční řešení

prostory $\Phi^\pm$ invariantní vůči čas. posunu

invariance mód po módu $\quad \phi_k^\pm = \mathrm{diff}_{\xi}(\Delta t)\,\phi_k^\pm$ – příliš silné

stačí proporcionalita po módu $\quad \mathrm{diff}_\xi(\Delta t)\,\phi_k^\pm \sim \phi_k^\pm$

separace časové a prost. proměnné.

báze pos/neg řešení

$$\varphi_k^+(t,x) = c_k \exp(-i\omega_k t)\, \vartheta_k(x) \qquad \phi_k^+$$

$$\varphi_k^-(t,x) = c_k^* \exp(i\omega_k t)\, \vartheta_k^*(x) \qquad \phi_k^-$$

časový posun $\quad t \to t+\Delta t$

$$\varphi_k^\pm \to \exp(\mp i\omega_k \Delta t)\,\varphi_k^\pm$$

normalizace

$$\vartheta_k\cdot\hat{q}\cdot\vartheta_l = \delta_{kl} \qquad (\text{resp. } \vartheta_k^*\cdot\hat{q}\cdot\vartheta_l = \delta_{kl} \qquad \vartheta_k\cdot\tilde{q}\cdot\vartheta_l = 0\,)$$

$$\delta_{kl} = \langle \phi_k, \phi_l\rangle = -i\,(\pi_k^-\cdot\varphi_l^+ - \varphi_k^-\cdot\pi_l^+) =$$

$$= (\omega_k+\omega_l)\, c_k^* c_l\, \vartheta_k^*\cdot\hat{q}\cdot\vartheta_l = 2\omega_k |c_k|^2 \delta_{kl}$$

$$c_k = \frac{1}{\sqrt{2\omega_k}}$$

pro reálné módy $\vartheta_k^* = \vartheta_k \quad \vartheta_k\cdot\hat{q}\cdot\vartheta_l = \delta_{kl}$
pro komplexní módy $\vartheta_k^*\cdot\hat{q}\cdot\vartheta_l = \delta_{kl}$ } podobně i jinde

$$\varphi_k^+(t) = \frac{1}{\sqrt{2\omega_k}} \exp(-i\omega_k t)\,\vartheta_k$$

reálné módy v Φ

$$\varphi_k = \varphi_k^+ + \varphi_k^- = \sqrt{\frac{2}{\omega_k}}\cos(\omega_k t)\,\vartheta_k \qquad = \frac{1}{\sqrt{\omega_k}}\cos(\omega_k t)\,\vartheta_k^R + \frac{1}{\sqrt{\omega_k}}\sin(\omega_k t)\,\vartheta_k^I$$

$$J\cdot\varphi_k = i\varphi_k^+ - i\varphi_k^- = \sqrt{\frac{2}{\omega_k}}\sin(\omega_k t)\,\vartheta_k \qquad = \frac{1}{\sqrt{\omega_k}}\sin(\omega_k t)\,\vartheta_k^R - \frac{1}{\sqrt{\omega_k}}\cos(\omega_k t)\,\vartheta_k^I$$

komplex ϑ_k $\qquad \vartheta_k = \frac{1}{\sqrt{2}}(\vartheta_k^R + i\vartheta_k^I)$

$$(\vartheta_k,\vartheta_l)_{\mathbb{C}}^\sim = 0 \quad (\vartheta_k^*,\vartheta_l)_{\mathbb{C}}^\sim = \delta_{kl} \iff (\vartheta_k^{R,I},\vartheta_l^{R,I})^\sim = \delta_{kl} \quad (\vartheta_k^R,\vartheta_l^I)^\sim = 0$$

obecné řešení

$$\varphi(t) = \cos(\Omega t)\cdot\varphi_0 + \Omega^{-1}\cdot\sin(\Omega t)\cdot\tilde{q}^{-1}\cdot\pi_0$$

$$= \sum_k \left(\varphi_{0k}\cos(\omega_k t)\,\vartheta_k + \pi_{0k}\,\omega_k^{-1}\sin(\omega_k t)\,\vartheta_k\right)$$

kde $\quad \varphi_0 = \sum_k \varphi_{0k}\vartheta_k \qquad \tilde{q}^{-1}\cdot\pi_0 = \sum_k \underbrace{\pi_{0k}}_{\dot{\varphi}_{0k}}\vartheta_k$

Charakteristika pomocí pč. dat. (reálné ϑ_k)

$$\varphi_k^{\pm}(t) = \frac{1}{\sqrt{2\omega_k}} \exp(\mp i\omega_k t)\, \vartheta_k$$

$$\Downarrow$$

$$\hat{q}^{-1}\cdot\pi_k^{\pm}(t) = \dot{\varphi}_k^{\pm}(t) = \mp i\,\omega_k\, \varphi_k^{\pm}(t)$$

$$\Downarrow$$

$$\phi \in \mathcal{S}^{\pm} \Longleftrightarrow \dot{\varphi}(t) \equiv \hat{q}^{-1}\cdot\pi(t) = \mp i\,\Omega\cdot\varphi(t)$$

vskutku, obecné řeš. má tvar

$$\varphi = \sum_k \varphi_{0k}\cos(\omega_k t)\,\vartheta_k + \pi_{0k}\,\omega_k^{-1}\sin(\omega_k t)\,\vartheta_k$$

$$= \underbrace{\sum_k \tfrac{1}{2}\Big(\varphi_{0k} - \tfrac{i}{\omega_k}\pi_{0k}\Big)\exp(i\omega_k t)\,\vartheta_k}_{\varphi^-} + \underbrace{\sum_k \tfrac{1}{2}\Big(\varphi_{0k} + \tfrac{i}{\omega_k}\pi_{0k}\Big)\exp(-i\omega_k t)\,\vartheta_k}_{\varphi^+}$$

$$= \tfrac{1}{2}\exp(i\Omega t)\cdot\big(\varphi_0 - i\Omega^{-1}\cdot\hat{q}^{-1}\cdot\pi_0\big) + \tfrac{1}{2}\exp(-i\Omega t)\cdot\big(\varphi_0 + i\Omega^{-1}\cdot\hat{q}^{-1}\cdot\pi_0\big)$$

$$\Big[\frac{d}{dt} - i\Omega\Big]\varphi^- = 0 \qquad\qquad \Big[\frac{d}{dt} + i\Omega\Big]\cdot\varphi^+ = 0$$

Komplexní struktura v 3+1 rozštěpení

$$\phi \in \mathcal{S}^{\pm} \iff \pi = \mp i\, \tilde{q} \cdot \Omega \cdot \varphi$$

$$\phi \longleftrightarrow \begin{pmatrix} \varphi \\ \pi \end{pmatrix}$$

$$\phi^{\pm} \longleftrightarrow \begin{pmatrix} \varphi^{\pm} \\ \pi^{\pm} \end{pmatrix} = \begin{pmatrix} \varphi^{\pm} \\ \mp i\, \tilde{q} \cdot \Omega \cdot \varphi^{\pm} \end{pmatrix}$$

$$\phi = \phi^{+} + \phi^{-} \longleftrightarrow \begin{pmatrix} \varphi \\ \pi \end{pmatrix} = \begin{pmatrix} \varphi^{+} + \varphi^{-} \\ -i\, \tilde{q} \cdot \Omega \cdot (\varphi^{+} - \varphi^{-}) \end{pmatrix}$$

$$J \cdot \phi^{\pm} = \pm i\, \phi^{\pm} \longleftrightarrow \begin{pmatrix} \pm i\, \varphi^{\pm} \\ \tilde{q} \cdot \Omega \cdot \varphi^{\pm} \end{pmatrix}$$

$$J \cdot \phi = i(\phi^{+} - \phi^{-}) \longleftrightarrow \begin{pmatrix} i(\varphi^{+} - \varphi^{-}) \\ \hat{q} \cdot \Omega \cdot (\varphi^{+} + \varphi^{-}) \end{pmatrix} = \begin{pmatrix} -\Omega^{-1} \cdot \tilde{q}^{-1} \cdot \pi \\ \hat{q} \cdot \Omega \cdot \varphi \end{pmatrix} = \begin{pmatrix} 0 & -\Omega^{-1} \cdot \tilde{q}^{-1} \\ \hat{q} \cdot \Omega & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \pi \end{pmatrix}$$

$$J \longleftrightarrow \begin{pmatrix} 0 & -\Omega^{-1} \cdot \tilde{q}^{-1} \\ \tilde{q} \cdot \Omega & 0 \end{pmatrix}$$

$$P^{\pm} \longleftrightarrow \frac{1}{2} \begin{pmatrix} \delta & \pm i\, \Omega^{-1} \cdot \hat{q}^{-1} \\ \mp i\, \tilde{q} \cdot \Omega & \delta \end{pmatrix}$$

Diagonalizace Hamiltoniánu

$$\varphi^{\pm} \iff \phi^{\pm}_k = \frac{1}{\sqrt{2\omega_k}} \exp(\mp i\omega_k t)\, \vartheta_k$$

1) toto rozštěpení zaručí, že

$$\hat{H} = :H(\hat{\Phi}): = \sum_k \omega_k \hat{n}_k$$

tj. Hamilt. je diag. v bázi z módů ϕ_k

2) požadavek diagonalizace jednozn. určuje $\varphi^{\pm}$

Diagonalizace

připomínka:

$$\varphi^{\pm}_k = \frac{1}{\sqrt{2\omega_k}} \exp(\mp i\omega_k t)\, \vartheta_k$$

$$\pi^{\pm}_k = \mp i\omega_k \tilde{q} \cdot \varphi^{\pm}_k$$

$$\pi^{\pm} = \mp i \tilde{q} \cdot \Omega \cdot \varphi^{\pm}$$

$$H(\varphi) = \frac{1}{2}\left(\pi \cdot \tilde{q}^{-1} \cdot \pi + \varphi \cdot \hat{\omega} \cdot \varphi\right) = \frac{1}{2}\left(\pi \cdot \tilde{q}^{-1} \cdot \pi + \varphi \cdot \Omega \cdot \tilde{q} \cdot \Omega \cdot \varphi\right)$$

$$\mathcal{H}(\phi^-, \phi^+) = H(\phi^- + \phi^+) =$$

$$= \frac{1}{2}\left(\pi^- \cdot \tilde{q}^{-1} \cdot \pi^- + \varphi^- \cdot \Omega \cdot \tilde{q} \cdot \Omega \cdot \varphi^-\right)^{\nearrow 0} + \frac{1}{2}\left(\pi^+ \cdot \tilde{q}^{-1} \cdot \pi^+ + \varphi^+ \cdot \Omega \cdot \tilde{q} \cdot \Omega \cdot \varphi^+\right)^{\nearrow 0}$$

$$+ \left(\pi^- \cdot \tilde{q}^{-1} \cdot \pi^+ + \varphi^- \cdot \Omega \cdot \tilde{q} \cdot \Omega \cdot \varphi^+\right)$$

$-i\tilde{q}\cdot\Omega\cdot\varphi^+$

$-i\tilde{q}\cdot\Omega\cdot\varphi^+$

$i\tilde{q}\cdot\Omega\cdot\varphi^-$

$$= 2\, \varphi^- \cdot \Omega \cdot \tilde{q} \cdot \Omega \cdot \varphi^+$$

$$\mathcal{H}(\phi^-_k, \phi^+_l) = \frac{1}{\sqrt{\omega_k \omega_l}} \exp(i(\omega_k - \omega_l)t)\, \vartheta^*_k \cdot \Omega \cdot \tilde{q} \cdot \Omega \cdot \vartheta_l$$

$$= \omega_k \delta_{kl}$$

$$\hat{H} = :H(\hat{\Phi}): = :\mathcal{H}\Big(\sum_k \hat{a}^+_k \phi^-_k, \sum_l \hat{a}_l \phi^+_l\Big):$$

$$= \sum_{k,l} \hat{a}^+_k \hat{a}_l\, \mathcal{H}(\phi^-_k, \phi^+_l) = \sum_k \omega_k \hat{a}^+_k \hat{a}_k$$

Jednoznačnost částicové interpretace diagonalizující $\hat{H}$

podmínka, že $\hat{H}$ je diagonální v částicových stavech jednoznačně určuje částicovou interpr., tj. volbu J

- nejde nutně o statický případ
- stačí uvažovat Hamiltonián na nadploše Σ

předpoklad diagonality:

$$\hat{H} = \sum_k \omega_k \hat{n}_k = \sum_k \hat{a}_k^+ \omega_k \hat{a}_k = \sum_k \langle \hat{\Phi}, \phi_k \rangle \omega_k \langle \phi_k, \hat{\Phi} \rangle =$$

$$= \langle \hat{\Phi}, \Omega \cdot \hat{\Phi} \rangle = -i\, \hat{\Phi}^- \cdot \partial\mathcal{F} \cdot \Omega \cdot \hat{\Phi}^+ = -\frac{1}{2} : \hat{\Phi} \cdot \partial\mathcal{F} \cdot \underline{J \cdot \Omega} \cdot \hat{\Phi} :$$

operátor Ω (různý od $\Omega = (\tilde{q}^{-1} \cdot \tilde{\pi})^{\frac{1}{2}}$)

$$\Omega \cdot \phi_k = \omega_k \phi_k \qquad [\Omega, J] = 0 \qquad \Omega \cdot \partial\mathcal{F} = \partial\mathcal{F} \cdot \Omega$$

vlastní vektory J-linearita J-hermiticita

předpoklad lokalizace na Σ

$$H(\phi) = \frac{1}{2} \phi \cdot \mathcal{H} \cdot \phi \qquad \mathcal{H} = \mathcal{H}[\Sigma] \qquad H = H[\Sigma]$$

$$\hat{H} = : H(\hat{\Phi}) : = \quad \text{kvantový Hamilt. daný norm. uspoř. klasického}$$

$$= \frac{1}{2} : \hat{\Phi} \cdot \mathcal{H} \cdot \hat{\Phi} : = -\frac{1}{2} : \hat{\Phi} \cdot \partial\mathcal{F} \cdot \underline{G_c \cdot \mathcal{H}} \cdot \hat{\Phi} :$$

porovnáním dostáváme

$$G_c \cdot \mathcal{H} = J \cdot \Omega = \Omega \cdot J$$

polární dekompozice operátoru $G_c \cdot \mathcal{H}$ definuje J a Ω

$$\Omega = |G_c \cdot \mathcal{H}| \qquad J = \operatorname{sign}(G_c \cdot \mathcal{H}) \qquad \text{tj.}$$

$$\Omega^2 = -G_c \cdot \mathcal{H} \cdot G_c \cdot \mathcal{H} \qquad J = \Omega^{-1} \cdot G_c \cdot \mathcal{H}$$

kde se využije, že $(G_c \cdot \mathcal{H})^T = -G_c \cdot \mathcal{H}$ pro transpozici T danou pos. def. bi-lineární formou $\mathcal{H}$

polární dekompozice

mějme reálný Hilb. pr. se sk. souč. daným bi-lin formou B

B definuje transpozici T: $\quad A^T = B^{-1} \cdot A \cdot B$

pokud pro operátor A platí $[A, A^T] = 0$, pak existuje jednoznačná polární dekompozice na abs. hodnotu $|A|$ a signum $\operatorname{sign} A$

$$A = (\operatorname{sign} A) \cdot |A| = |A| \cdot (\operatorname{sign} A)$$

$|A|$ symetr. pos. def. $|A| = |A|^T$, $\quad \operatorname{sign} A$ ortogonální $(\operatorname{sign} A) \cdot (\operatorname{sign} A)^T = \delta$

kde

$$|A|^2 = A \cdot A^T \qquad \operatorname{sign} A = |A|^{-1} \cdot A = A \cdot |A|^{-1}$$

Termální stavy

smíšený stav o teplotě $T = \frac{1}{\beta}$ při fixované střední energii

$$\hat{D}_\beta = \frac{1}{Z} \exp(-\beta \hat{H})$$

$$Z = \mathrm{Tr} \exp(-\beta \hat{H}) \qquad \mathrm{Tr}\, \hat{D}_\beta = 1$$

Hamiltonián (normálně uspořádaný)

$$\hat{H} = \sum_k \omega_k \hat{n}_k$$

střední hodnota ve stavu $\hat{D}_\beta$

$$\langle \hat{A} \rangle_\beta = \mathrm{Tr}(\hat{D}_\beta \hat{A})$$

vakuum

$$\hat{D}_\infty = |vac\rangle\langle vac|$$

Platí:

$$Z = \prod_k \frac{1}{1-\exp(-\beta\omega_k)} = \det{}^{-1}(1-\exp(-\beta\Omega))$$

$$\langle \hat{a}_k^\dagger \hat{a}_k \rangle_\beta = \frac{\exp(-\beta\omega_k)}{1-\exp(-\beta\omega_k)} = \langle \hat{n}_k \rangle$$

$$\langle \hat{a}_k \hat{a}_k^\dagger \rangle_\beta = \frac{1}{1-\exp(-\beta\omega_k)}$$

$$\langle \hat{a}_k \hat{a}_l \rangle_\beta = \langle \hat{a}_k^\dagger \hat{a}_l^\dagger \rangle_\beta = \underset{k \neq l}{\langle \hat{a}_k^\dagger \hat{a}_l \rangle_\beta} = \underset{k \neq l}{\langle \hat{a}_k \hat{a}_l^\dagger \rangle_\beta} = 0$$

odvození:

$$Z = \mathrm{Tr}\, \exp\left(-\beta \sum_{k} \omega_k \hat{n}_k\right) \qquad \Leftarrow \quad \mathrm{Tr} \equiv \sum_{m} \langle u:m| \ldots |u:m\rangle$$

$$= \sum_{m} \exp\left(-\beta \sum_{k} \omega_k m_k\right)$$

$$= \prod_{k} \sum_{m_k=0}^{\infty} \exp(-\beta \omega_k m_k)$$

$$= \prod_{k} \frac{1}{1-\exp(-\beta\omega_k)}$$

$$= \det{}^{-1}\left(1-\exp(-\beta\Omega)\right)$$

$$\langle \hat{a}_k^{+} \hat{a}_k \rangle_\beta = \mathrm{Tr}\left(\hat{D}_\beta \hat{n}_k\right) = \frac{1}{Z} \mathrm{Tr}\left(\hat{n}_k \exp(-\beta \hat{H})\right)$$

$$= \frac{1}{Z} \sum_{m} m_k \exp\left(-\beta \sum_{l} m_l \omega_l\right) =$$

$$= \left(1-\exp(-\beta\omega_k)\right)\left(\sum_{m_k} m_k \exp(-\beta m_k \omega_k)\right) \times$$

$$\times \prod_{l\neq k} \underbrace{\left(1-\exp(-\beta\omega_l)\right)\left(\sum_{m_l} \exp(-\beta m_l \omega_l)\right)}_{1}$$

$$= -\left(1-\exp(-\beta\omega_k)\right) \frac{1}{\omega_k} \frac{\partial}{\partial\beta} \sum_{m_k} \exp(-\beta m_k \omega_k)$$

$$= -\left(1-\exp(-\beta\omega_k)\right) \frac{1}{\omega_k} \frac{\partial}{\partial\beta} \frac{1}{1-\exp(-\beta\omega_k)}$$

$$= \frac{1-\exp(-\beta\omega_k)}{\left(1-\exp(-\beta\omega_k)\right)^2} \frac{\omega_k}{\omega_k} \exp(-\beta\omega_k)$$

$$= \frac{\exp(-\beta\omega_k)}{1-\exp(-\beta\omega_k)}$$

$$\langle \hat{a}_k \hat{a}_k^{+} \rangle_\beta = \langle \hat{a}_k^{+} \hat{a}_k + \hat{1} \rangle_\beta = \frac{\exp(-\beta\omega_k)}{1-\exp(-\beta\omega_k)} + 1 = \frac{1}{1-\exp(-\beta\omega_k)}$$