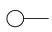
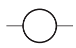
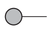
I Diagrammatic notation

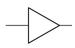
In this appendix we make some comments about the diagrammatic notation. A well-known application of this notation are Feynman diagrams used in field theory (a modification of which we use in the text). But in general, this notation is nothing other then a graphical representation of algebra of tensor objects. See for example [27] for an application to the spacetime tensors and spinors.

The main idea is to represent tensors by geometrical objects with "legs" which correspond to tensor indices. Different kind of tensor indices should be represented by different kind of legs. The contraction of tensor indices is represented by connection of corresponding legs. We can add the diagrams with the same leg structure and multiply them to obtain a diagram with more complicated leg structure which represents the tensor product of the component.

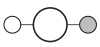
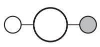
The only difference from the usual algebra is that we associate with each diagram a *symmetry factor* and we include the reciprocal of this numerical factor with the diagram. The symmetry factor is the number of ways in which the diagram can be re-arranged to obtain the identical diagram. If we want to use the diagram without the symmetry factor, we precede it with a # sign. To illustrate this convention we give some examples. Let a, b, k, H, and O be tensors represented by following diagrams

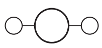
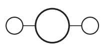
$a^n \leftrightarrow$  ; $k_{mn} \leftrightarrow$  $k_{mn} = k_{nm}$

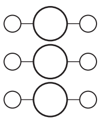
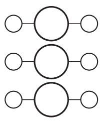
$b^n \leftrightarrow$  ; $H^{mn} \leftrightarrow$  $H^{mn} = H^{nm}$ (I.1)

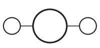
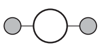
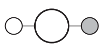
$\mathrm{O}^m_n \leftrightarrow$  .

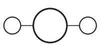
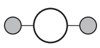
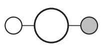
We can write

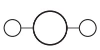
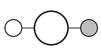
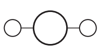
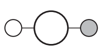
$a^m k_{mn} b^n \leftrightarrow$  $= \#$  (I.2)

$\frac{1}{2} a^m k_{mn} a^n \leftrightarrow$  $= \frac{1}{2} \#$  (I.3)

$\frac{1}{2^3} \frac{1}{3!} (a^m k_{mn} a^n)^3 \leftrightarrow$  $= \frac{1}{2^3} \frac{1}{3!} \#$  (I.4)

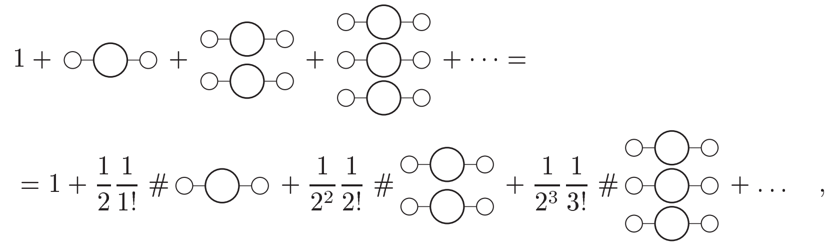
$\frac{1}{2}(a^m + b^m) k_{mn} (a^n + b^n) \leftrightarrow$  $+$  $+$  $=$

$= \frac{1}{2} \#$  $+ \frac{1}{2} \#$  $+ \#$ 

$(a^m + b^m) k_{mn} a^n \leftrightarrow 2$  $+$  $= \#$  $+ \#$  . (I.6)

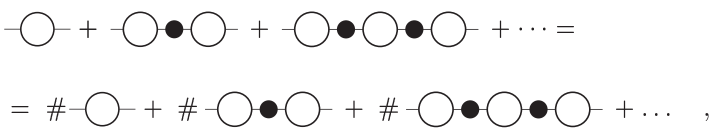
Power expansion of some functions gives

$$\exp\Big(\frac{1}{2}a^m k_{mn} a^n\Big) \quad \leftrightarrow$$



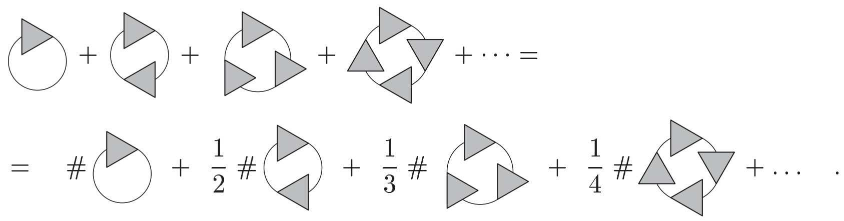
(I.7)

$$\Big(k^{-1} - H\Big)^{-1}{}_{mn} \quad \leftrightarrow$$



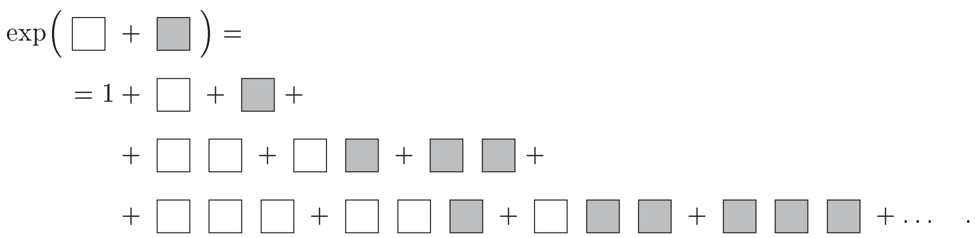
(I.8)

$$- \quad \mathrm{tr}\ln\big(\delta - \mathrm{O}\big) \quad \leftrightarrow$$



(I.9)

There are some common operation which have a nice graphical interpretation. Let have some set of elementary connected diagrams (i.e. diagrams which are not explicit product of other diagrams) without free legs. The sum of all diagrams composed of an arbitrary number of these connected diagrams is the exponential of the sum of the diagrams. For example

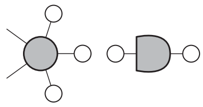


(I.10)

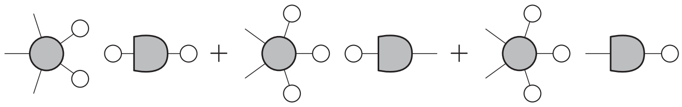
This fact is powerful in an opposite direction — if an expression is given by the sum of all possible products of elementary connected components, the sum of these connected components is given by logarithm of the expression.

In combination with (I.9) we also find that $\det(\delta - \mathrm{O})^{-\frac{1}{2}}$ is given by the sum of all possible products of loops formed using the diagram of the operator O.

Next, let us have an expression given by the product of elementary diagrams which contains the diagram representing a vector a. We can understand it as an analytical tensor-valued function of the vector a. The derivation with respect of its argument is graphically represented as sum of graphs which we get by “tearing out” the diagram a from all possible symmetrically non-equivalent positions. E.g.

$F_{\boldsymbol{mn}}(a) \quad \leftrightarrow$  ,

(I.11)

$\mathrm{d}_{\boldsymbol{k}} F_{\boldsymbol{mn}}(A) \quad \leftrightarrow$  .

Note that the numerical factors above are correct thanks to symmetry factors included in the diagrams.