Fields of accelerated sources: Born in de Sitter*

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This paper deals thoroughly with the scalar and electromagnetic fields of uniformly accelerated charges in de Sitter spacetime. It gives details and makes various extensions of our Physical Review Letter from 2002. The basic properties of the classical Born solutions representing two uniformly accelerated charges in flat spacetime are first summarized. The worldlines of uniformly accelerated particles in de Sitter universe are defined and described in a number of coordinate frames, some of them being of cosmological significance, the other are tied naturally to the particles. The scalar and electromagnetic fields due to the accelerated charges are constructed by using conformal relations between Minkowski and de Sitter space. The properties of the generalized "cosmological" Born solutions are analyzed and elucidated in various coordinate systems. In particular, a limiting procedure is demonstrated which brings the cosmological Born fields in de Sitter space back to the classical Born solutions in Minkowski space. In an extensive Appendix, which can be used independently of the main text, nine families of coordinate systems in de Sitter spacetime are described analytically and illustrated graphically in a number of conformal diagrams.

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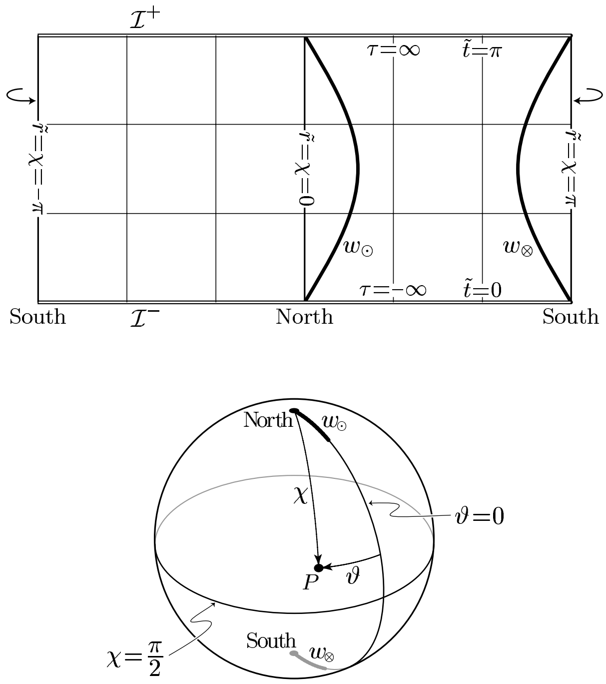

Figure 2: The spherical cosmological coordinates and a pair of uniformly accelerated particles $w_\odot$ and $w_\otimes$ in de Sitter universe: the conformal diagram (above) and projection on the spacelike cut τ = constant in the standard cosmological spherical coordinates (angle φ suppressed). The whole de Sitter spacetime could be represented by just the "right half" of the conformal diagram. For convenience, we admit negative values of radial coordinates and identify $\tilde{r}=\chi=-\pi$ and $\tilde{r}=\chi=\pi$ (see the text below Eq. (3.12) and the Appendix).

III. MANY FACES OF DE SITTER

The fields due to various types of uniformly accelerated sources in de Sitter spacetime found in [24], as well as those described briefly in Ref. [25], were constructed by employing the conformal relation between Minkowski and de Sitter spacetimes. When analyzing the worldlines of the sources in de Sitter spacetime and their relation to the corresponding worldlines in Minkowski spacetime we need to introduce appropriate coordinate systems. Suitable coordinates will later be used to exhibit various properties of the fields. An extensive literature exists on various types of coordinates in de Sitter space (e.g. [41, 42]), but we want to survey some of them in this section. In particular, we relate them to the corresponding coordinates on conformally related Minkowski spaces since this does not appear to be given elsewhere. In the next section, after identifying the worldlines of uniformly accelerated particles in de Sitter space, we shall construct new coordinate systems tied to such particles, such as Rindler-type "accelerated" coordinates, or Robinson-Trautman-type coordinates in which the null cones emanating from the particles have especially simple forms. These coordinate systems will turn out to be very useful in analyzing the fields. Here, in the main text, however, only a brief description of relevant coordinates will be given. More details, including both formulas and illustrations, are relegated to the Appendix.

As it is well-known from textbooks on general relativity (for a recent pedagogical exposition, see [43]), de Sitter spacetime, which is the solution of Einstein vacuum equations with a cosmological term $\Lambda > 0$, is best visualized as the 4-dimensional hyperboloid imbedded in flat 5-dimensional Minkowski space. It is the homogeneous space of constant curvature equal to 4Λ. Hereafter, we use the quantity

$$\ell_\Lambda = \sqrt{\frac{3}{\Lambda}} \tag{3.1}$$

(with the dimension of length) to parametrize the radius of the curvature.

The entire de Sitter spacetime can be covered by a single coordinate system—which we call *standard coordinates*—$\tau \in \mathbb{R}$, $\chi \in (0,\pi)$, $\vartheta \in (0,\pi)$, $\varphi \in (-\pi,\pi)$ in which the metric reads

$$g_{\text{dS}} = -\mathrm{d}\tau^2 + \ell_\Lambda^2 \cosh^2\frac{\tau}{\ell_\Lambda}\left(\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\omega^2\right) , \tag{3.2}$$

$$\mathrm{d}\omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2 . \tag{3.3}$$

Clearly, we can imagine the spacetime as the time evolution of a 3-sphere which shrinks from infinite extension at $\tau \to -\infty$ to a radius ℓ_Λ, and then expands again in a

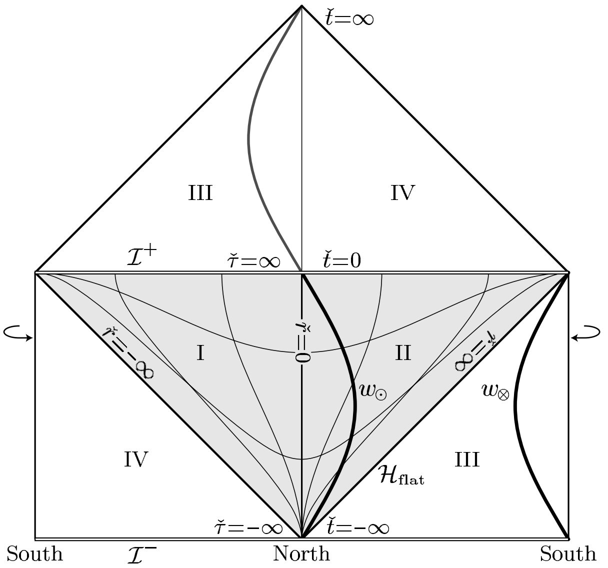

Figure 3: The flat cosmological coordinates and particles $w_\odot$, $w_\otimes$ in de Sitter space and in conformally related Minkowski space. The flat cosmological coordinates cover shaded region. Its boundary, $\check{r} = \pm\infty$, represents the horizon for observers at rest in these coordinates.

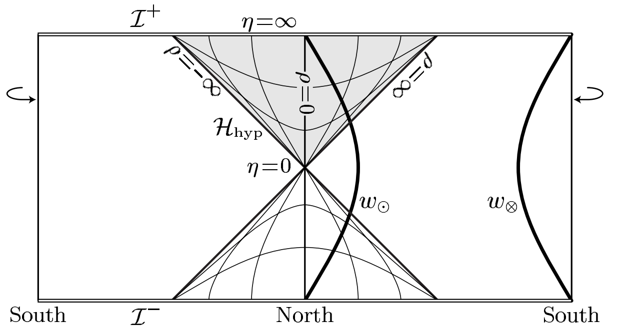

Figure 4: The hyperbolic cosmological coordinates. They cover only the shaded region and, therefore, only a part of the worldline $w_\odot$. The horizon $\mathcal{H}_{\rm hyp}$ arises for the observers who are at rest in the hyperbolic cosmological coordinates.

time-symmetric way. Hence, we also call τ, χ the *spherical cosmological coordinates*. The coordinate lines are shown in the conformal diagram, Fig. 2.

In cosmology the most popular "flat" de Sitter universe is obtained by considering only a half of de Sitter hyperboloid foliated by flat 3-dimensional spacelike hypersurfaces labeled by timelike coordinate $\check{\tau} \in \mathbb{R}$, cf. Fig. 3. Together with appropriate radial coordinate $\check{r} \in \mathbb{R}^+$, the new coordinates, which we call *flat cosmological coordinates*, are given in terms of τ, χ by

$$\begin{aligned} \check{\tau} &= \ell_\Lambda \log\Big(\sinh\frac{\tau}{\ell_\Lambda} + \cosh\frac{\tau}{\ell_\Lambda}\,\cos\chi\Big)\;, \\ \check{r} &= \ell_\Lambda \frac{\sin\chi}{\cos\chi + \tanh(\tau/\ell_\Lambda)}\;, \end{aligned} \tag{3.4}$$

implying the well-known "inflationary" metric

$$g_{\rm dS} = -\mathrm{d}\check{\tau}^2 + \exp\frac{2\check{\tau}}{\ell_\Lambda}\,\Big(\mathrm{d}\check{r}^2 + \check{r}^2\,\mathrm{d}\omega^2\Big)\;. \tag{3.5}$$

These coordinates cover only "one-half" of de Sitter space as indicated by shading in Fig. 3.

de Sitter introduced his model in what we call *hyperbolic cosmological coordinates* $\eta \in \mathbb{R}$, $\rho \in \mathbb{R}^+$ (see Fig. 4) related to τ, χ by

$$\begin{aligned} \cosh\frac{\eta}{\ell_\Lambda} &= \cosh\frac{\tau}{\ell_\Lambda}\,\cos\chi\;, \\ \tanh\frac{\rho}{\ell_\Lambda} &= \coth\frac{\tau}{\ell_\Lambda}\,\sin\chi\;. \end{aligned} \tag{3.6}$$

The metric

$$g_{\rm dS} = -\mathrm{d}\eta^2 + \sinh^2\frac{\eta}{\ell_\Lambda}\,\Big(\mathrm{d}\rho^2 + \ell_\Lambda^2\sinh^2\frac{\rho}{\ell_\Lambda}\,\mathrm{d}\omega^2\Big) \tag{3.7}$$

shows that the time slices $\eta =$ constant have the geometry of constant negative curvature, i.e., as the standard time slices in an open FRW universe.

The last commonly used coordinates in de Sitter spacetime are *static coordinates* $T \in \mathbb{R}$, $R \in (0, \ell_\Lambda)$:

$$\begin{aligned} T &= \frac{\ell_\Lambda}{2}\,\log\left|\frac{\cos\chi + \tanh(\tau/\ell_\Lambda)}{\cos\chi - \tanh(\tau/\ell_\Lambda)}\right|\;, \\ R &= \ell_\Lambda \cosh\frac{\tau}{\ell_\Lambda}\,\sin\chi\;, \end{aligned} \tag{3.8}$$

covering also only a part of the universe. The metric in these coordinates reads

$$g_{\rm dS} = -\Big(1-\frac{R^2}{\ell_\Lambda^2}\Big)\,\mathrm{d}T^2 + \Big(1-\frac{R^2}{\ell_\Lambda^2}\Big)^{-1}\mathrm{d}R^2 + R^2\,\mathrm{d}\omega^2\;, \tag{3.9}$$

revealing that $\partial/\partial T$ is a timelike Killing vector in the region $0 < R < \ell_\Lambda$.

Among the coordinates introduced until now only the standard coordinates τ, χ, ϑ, φ cover the whole de Sitter spacetime globally. One can easily extend flat cosmological coordinates to cover (though not smoothly) the whole de Sitter hyperboloid, which will be useful in discussion of the conformally related Minkowski spacetime, cf. Eq. (3.13). We shall also use extensions of the static coordinates into the whole spacetime, using definitions (3.8), but allowing $R \in \mathbb{R}^+$. In regions where $R > \ell_\Lambda$ coordinates T and R interchange their character, $\partial/\partial T$ becomes a spacelike Killing vector (analogously to $\partial/\partial t$ inside a Schwarzschild black hole). However, the static coordinates T, R are not globally smooth and uniquely valued. Namely, $T \to \infty$ at the cosmological horizons $R = \ell_\Lambda$. The static coordinates, extended to the whole de Sitter space, are illustrated in Fig. 5. Here we also indicate the regions in which $\partial/\partial T$ is spacelike by bold **F** ("future") and **P** ("past"), whereas the regions in which it is timelike are denoted by **N** (containing the "north pole" $\chi = 0$) and **S** (containing the "south pole" $\chi = \pi$). Hereafter, this notation will be used repeatedly.

The conformal structure of Minkowski and de Sitter spacetimes, their conformal relation, and their confor-

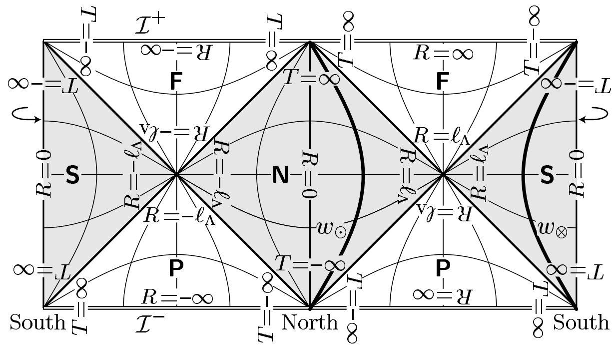

Figure 5: The static coordinates and the wordlines of particles $w_\odot$ and $w_\otimes$. These coordinates can be defined in the whole spacetime, however several coordinate patches, in diagram indicated by shaded and nonshaded regions, have to be used (cf. Appendix A 5 and A 6). These regions are separated by the cosmological horizons at $R = \ell_\Lambda$, where $T = \pm\infty$. The vector $\partial/\partial T$ is a Killing vector of de Sitter spacetime. It is timelike in the domains **N** and **S** (shaded regions) and spacelike in the domains **F** and **P**. The histories of both particles $w_\odot$ and $w_\otimes$ belong to the domains **N** and **S**.

mal relation to various regions of the Einstein static universe have been discussed extensively in literature (see, e.g., [44–47]). The complete compactified picture of these spacetimes, in particular the 3-dimensional diagram of the compactified Minkowski and de Sitter spaces $M^\#$ as parts of the Einstein universe represented by a *solid* cylinder can be found in [24]. We refer the reader especially to Section III of [24] where we explain and illustrate the compactification in detail. In the present paper we shall confine ourselves to the 2-dimensional Penrose diagrams.

The basic *standard rescaled coordinates* covering globally de Sitter spacetime including the conformal infinity are simply related to the standard coordinates as follows:

$$\tan\frac{\tilde t}{2} = \exp\frac{\tau}{\ell_\Lambda}\ , \quad \tilde r = \chi\ , \tag{3.10}$$

$\tilde t \in (0,\pi)$, $\tilde r \in (0,\pi)$. The metric (3.2) becomes

$$g_{\rm dS} = \ell_\Lambda^2\, \sin^{-2}\tilde t\, \left(-{\rm d}\tilde t^2 + {\rm d}\tilde r^2 + \sin^2\tilde r\, {\rm d}\omega^2\right)\ , \tag{3.11}$$

demonstrating explicitly the conformal relations of de Sitter spacetime to the Einstein universe:

$$g_{\rm E} = \Omega_{\rm dS}^2\, g_{\rm dS}\ , \quad \Omega_{\rm dS} = \sin\tilde t\ . \tag{3.12}$$

Therefore, we also call coordinates $\tilde t$, $\tilde r$ the *conformally Einstein coordinates*. The conformal diagram of de Sitter spacetime is illustrated in Fig. 2. The past and future infinities, $\tilde t = 0$ and $\tilde t = \pi$ are spacelike, the worldlines of the north and south poles (given by the choice of the origin of the coordinates) are described by $\tilde r = \chi = 0$ and $\tilde r = \chi = \pi$.

The whole de Sitter spacetime could be represented by just the "right half" of Fig. 2. Indeed, it is customary to draw this half only and to consider any point in the figure as a 2-sphere, except for the poles $\tilde r = 0, \pi$. As we shall see, the formulas relating coordinates on the conformally related de Sitter and Minkowski spacetimes have simpler forms if we admit negative values of the radial coordinate $\tilde r \in (-\pi, 0)$ covering the left half of the diagram. We shall thus consider the 2-dimensional diagrams as in Fig. 2 to represent the cuts of de Sitter spacetime along the axis going through the origins (through north and south poles—analogously to the cuts along the z axis in E^3). The axis, i.e., the main circle of the spatial spherical section of de Sitter spacetime, is typically chosen as $\vartheta = 0, \pi$. Thus, in the diagram the point with $\tilde r = -\tilde r_{\rm o} < 0$, $\vartheta = \vartheta_{\rm o}$, $\varphi = \varphi_{\rm o}$ is identical to that with $\tilde r = \tilde r_{\rm o}$, $\vartheta = \pi - \vartheta_{\rm o}$, and $\varphi = \varphi_{\rm o} + \pi$. We use the same convention also for other radial coordinates appearing later, as explicitly stated in the Appendix (cf. also Appendix in [24]). We admit negative radial coordinates only when describing various relations between the coordinate systems. In the expressions for the fields in the following sections only positive radial coordinates are considered.

As mentioned above, in [24] we constructed fields on de Sitter spacetime by conformally transforming the fields from Minkowski spacetime. Now "different Minkowski spaces" can be used in the conformal relation to de Sitter space, depending on which region of a Minkowski space is mapped onto which region of de Sitter space. Consider, for example, Minkowski space with metric $g_{\rm M}$ given in spherical coordinates $\check t$, $\check r$, ϑ, φ. Identify it with de Sitter space by relations

$$\check t = \frac{\ell_\Lambda \sin\tilde t}{\cos\tilde t - \cos\tilde r}\ , \quad \check r = \frac{\ell_\Lambda \sin\tilde r}{\cos\tilde r - \cos\tilde t}\ , \tag{3.13}$$

the inverse relation (A11) is given in the Appendix. In the coordinates $\check t$, $\check r$, ϑ, φ the de Sitter metric (3.11) becomes

$$g_{\rm dS} = \frac{\ell_\Lambda^2}{\check t^2}\left(-{\rm d}\check t^2 + {\rm d}\check r^2 + \check r^2\, {\rm d}\omega^2\right)\ , \tag{3.14}$$

so that

$$g_{\rm dS} = \Omega_{\check{\rm M}}^2 g_{\check{\rm M}}\ , \quad \Omega_{\check{\rm M}} = \frac{\ell_\Lambda}{\check t}\ . \tag{3.15}$$

The coordinates $\check t$, $\check r$, ϑ, φ can, of course, be used in both de Sitter and Minkowski spaces. Fig. 3 illustrates the coordinate lines. It also shows how four regions I, II, III, and IV of Minkowski space are mapped onto four regions of de Sitter space by relations (3.13). We call $\check t$, $\check r$ *rescaled flat cosmological coordinates* since their radial coordinate $\check r$ coincides with that of the flat cosmological coordinates (3.4) and the time coordinate is simply related to $\check\tau$ as

$$\check t = -\ell_\Lambda\, \exp(-\check\tau/\ell_\Lambda)\ . \tag{3.16}$$

The caron or the check (still better "háček") "∨" formed by cosmological horizon at $\check t = \pm\infty$ in de Sitter space (cf. Fig. 3) inspired our notation of these coordinates. It

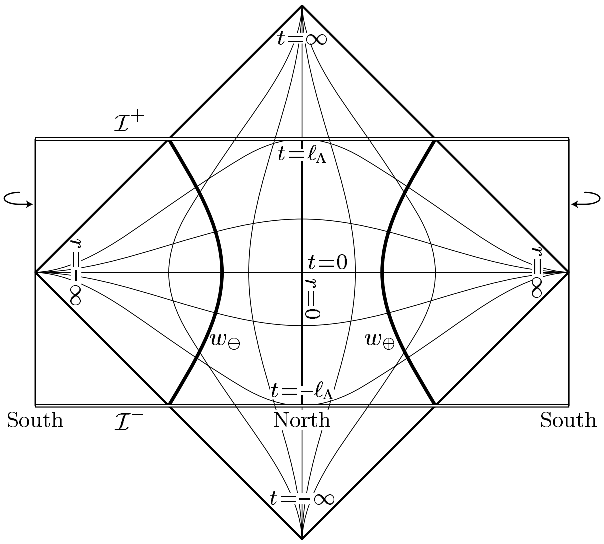

Figure 6: The conformally Minkowski coordinates. They cover the whole conformally related Minkowski space but only a part of corresponding de Sitter space. This Minkowski space is related to that in Fig. 3 by a shift "downwards" by $\pi/2$ in the direction of the conformally Einstein coordinate $\tilde{t}$.

is possible to introduce analogously the coordinates $\hat{t}$, $\hat{r}$ given in the Appendix, Eqs. (A39), (A40), that cover nicely the past conformal infinity but are not smooth at the cosmological horizon $\hat{t} = \pm\infty$; in this case they form the hat "$\wedge$" in the conformal diagram (see Fig. 16 in the Appendix).

From relations (3.13) it is explicitly seen why, when writing down mappings between de Sitter and Minkowski spaces and drawing the corresponding 2-dimensional conformal diagrams, it is advantageous to admit negative radial coordinates. If we would restrict all radial coordinates to be non-negative, we would have to consider the second relation in Eq. (3.13) with different signs for regions III and II in de Sitter space: in III $\check{r} = \ell_\Lambda \sin\tilde{r}/(\cos\tilde{r} - \cos\tilde{t})$, but in III we would have $\check{r} = -\ell_\Lambda \sin\tilde{r}/(\cos\tilde{r} - \cos\tilde{t})$.

Another mapping of Minkowski on de Sitter space will be used to advantage in the explicit manifestation that the generalized Born solution in de Sitter space goes over to the classical solution (2.13). Instead of the mapping (3.13), consider the relations

$$t = -\frac{\ell_\Lambda \cos\tilde{t}}{\cos\tilde{r} + \sin\tilde{t}}\,, \quad r = \frac{\ell_\Lambda \sin\tilde{r}}{\cos\tilde{r} + \sin\tilde{t}} \tag{3.17}$$

(see Eq. (A17) for the inverse mapping), which turn the metric (3.11) into

$$g_{\mathrm{dS}} = \Big(\frac{2\,\ell_\Lambda^2}{\ell_\Lambda^2 - t^2 + r^2}\Big)^2 \Big(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\omega^2\Big)\,. \tag{3.18}$$

We again obtain the de Sitter metric in the form explicitly conformal to the Minkowski metric with, however, a different conformal factor from that in Eq. (3.15):

$$g_{\mathrm{dS}} = \Omega_{\mathrm{M}}^2\, g_{\mathrm{M}}\,, \quad \Omega_{\mathrm{M}} = \frac{2\,\ell_\Lambda^2}{\ell_\Lambda^2 - t^2 + r^2}\,. \tag{3.19}$$

(For the use of the de Sitter metric in "atypical" form (3.18) in the work on the domain wall spacetimes, see [48].) The relation of Minkowski space to de Sitter space based on the mapping (3.17) is illustrated in Fig. 6. Clearly, the Minkowski space in this figure is shifted "downwards" by $\pi/2$ in $\tilde{t}$ coordinate, as compared with Minkowski space in Fig. 3 (Eq. (3.13)). Indeed, replacing $\tilde{t}$ by $\tilde{t} + \frac{\pi}{2}$ in Eq. (3.13), we get $\check{t} = t$, $\check{r} = r$ with t, r given by Eq. (3.17). Since coordinates t, r, ϑ, φ are not connected directly with any cosmological model and correspond to Minkowski space "centered" on de Sitter space (Fig. 6), we just call them *conformally Minkowski coordinates.*

In Ref. [24] still another Minkowski space is related to de Sitter space—one which is shifted "downward" in $\tilde{t}$ coordinate by another $\pi/2$. As mentioned below Eq. (3.16), the cosmological horizon forms hat "$\wedge$" in this case and the corresponding coordinates are accordingly denoted as $\hat{t}$, $\hat{r}$. They are given explicitly in Appendix A 3 and Fig. 16.

The three sets of coordinates $\check{t}$, $\check{r}$, t, r, and $\hat{t}$, $\hat{r}$ (with the same ϑ, φ) relating naturally "three" Minkowski spaces to de Sitter space are suitable for different purposes. The third set describes conveniently the past infinity of de Sitter space—that is why it was used extensively in [24] where we were interested in how the sources enter (are "born in") de Sitter universe. The second set will be needed in Section VII for exhibiting the flat-space limit of the generalized Born solution. The first set describes nicely the future infinity and will be employed when analyzing radiative properties of the fields.

With all the coordinates discussed above, corresponding double null coordinates can be associated; some of them will also be used in the following. Their more detailed description and illustration is presented in section A 10 of the Appendix.

Before concluding this section let us notice that the observers which are at rest in cosmological coordinate systems τ, χ, $\check{\tau}$, $\check{r}$, and η, ρ move along the geodesics with proper time τ, $\check{\tau}$, and η respectively. These geodesics are also the orbits of the conformal Killing vectors. Indeed, the symmetries of Minkowski spacetime and of the Einstein universe become conformal symmetries in conformally related de Sitter spacetime. In particular, we shall employ the fact that since $\partial/\partial\check{t}$ and $\partial/\partial t$ are timelike Killing vectors in Minkowski spacetime and $\partial/\partial\tilde{t}$ is a timelike Killing vector in the Einstein universe, the vectors

$$\frac{\partial}{\partial\tilde{t}}\,, \quad \frac{\partial}{\partial\check{t}}\,, \quad \text{and} \quad \frac{\partial}{\partial t} \tag{3.20}$$

are timelike conformal Killing vectors in de Sitter spacetime. As mentioned below Eq. (3.9), $\partial/\partial T$ is a Killing vector which is timelike for $|R| < \ell_\Lambda$.

Appendix A: THE PALETTE OF COORDINATE SYSTEMS IN DE SITTER SPACETIME

Nine families of coordinate systems are here introduced, described analytically and illustrated graphically. The corresponding forms of de Sitter metric, orthonormal tetrads and interrelations between the systems are given. All these systems are suitable for exhibiting various features of de Sitter space; two families are directly associated with uniformly accelerated particles. Although the majority (though not all) of these coordinate systems undoubtedly appeared in literature in some form already, they are scattered and, as far as we know, not summarized as comprehensively as in the following. In the main text we refer frequently to this Appendix, but the Appendix can be read independently. We hope it can serve as a catalogue useful for analyzing various aspects of physics in de Sitter universe.

By a *family of coordinate systems* we mean the systems with the *same* coordinate lines; e.g., $\{x^\mu\}$ and $\{y^\mu\}$ where $x^1 = x^1(y^1)$, $x^2 = x^2(y^2)$, etc. Seven of our families have the same spherical angular coordinates ϑ, φ, accelerated and Robinson-Trautman coordinates mix three coordinates, only azimuthal coordinate φ remains unchanged.

The homogeneous normalized metric on two-spheres (the metric "in angular direction") is denoted by

$$\mathrm{d}\omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\ . \tag{A1}$$

The radial coordinates label directions pointing out from the pole and acquire only positive values. However, transformations among coordinates take simpler forms if we allow radial coordinates to take on negative values as well. This causes no problems if, denoting by t and r the prototypes of time and radial coordinates, we adopt the convention that the following two values of coordinates describe the same point:

$$\{t,\, r,\, \vartheta,\, \varphi\}\ \leftrightarrow\ \{t,\, -r,\, \pi-\vartheta,\, \varphi+\pi\}\ . \tag{A2}$$

Hence, intuitively we may consider a point with $-r < 0$ and ϑ, φ fixed to lie on diametrically opposite side of the pole $r = 0$ with respect to the point $r > 0$, ϑ, φ.

The orthonormal tetrad e_t, e_r, e_ϑ, e_φ associated with a coordinate system is tangent to the coordinate lines and oriented (with few exceptions) in the directions of growing coordinates. It is chosen in such a way that the external product $\mathrm{e}^t \wedge \mathrm{e}^r \wedge \mathrm{e}^\vartheta \wedge \mathrm{e}^\varphi$ of 1-forms of the dual tetrad has always the same orientation. Since all forms of the metric contain the term (A1) the only component $(\mathrm{e}_\varphi)^\varphi$ of the tetrad vector e_φ in coordinate frame $\{\frac{\partial}{\partial x^\mu}\}$ is related to the ϑ-component of e_ϑ as

$$(\mathrm{e}_\varphi)^\varphi = \frac{1}{\sin\vartheta}\,(\mathrm{e}_\vartheta)^\vartheta\ , \tag{A3}$$

and we thus omit e_φ henceforth.

In the standard Newman-Penrose null complex tetrad k, l, m, $\bar{\mathrm{m}}$ with only nonvanishing inner products $\mathrm{k}\cdot\mathrm{l} = -1$, $\mathrm{m}\cdot\bar{\mathrm{m}} = 1$, the electromagnetic field F is represented by three complex components:

$$\begin{gathered}\Phi_0 = \mathrm{F}_{\alpha\beta}\,\mathrm{k}^\alpha\,\mathrm{m}^\beta\ ,\quad \Phi_2 = \mathrm{F}_{\alpha\beta}\,\bar{\mathrm{m}}^\alpha\,\mathrm{l}^\beta\ ,\\ \Phi_1 = \tfrac{1}{2}\,\mathrm{F}_{\alpha\beta}\left(\mathrm{k}^\alpha\,\mathrm{l}^\beta - \mathrm{m}^\alpha\,\bar{\mathrm{m}}^\beta\right)\ .\end{gathered} \tag{A4}$$

The null tetrad can be specified directly (as it will be done in the case of Robinson-Trautman coordinates in Eq. (A114)), or it can be associated with any orthonormal tetrad, say t, q, r, s, by relations

$$\begin{aligned}\mathrm{k} &= \tfrac{1}{\sqrt{2}}(\mathrm{t}+\mathrm{q})\ , &\qquad \mathrm{l} &= \tfrac{1}{\sqrt{2}}(\mathrm{t}-\mathrm{q})\ ,\\ \mathrm{m} &= \tfrac{1}{\sqrt{2}}(\mathrm{r}-i\,\mathrm{s})\ , &\qquad \bar{\mathrm{m}} &= \tfrac{1}{\sqrt{2}}(\mathrm{r}+i\,\mathrm{s})\ .\end{aligned} \tag{A5}$$

Here, t and q are timelike and spacelike unit vectors respectively, typically in a direction of "time" and "radial" coordinate, and r, s are spacelike unit vectors in angular directions, $\mathrm{r} = \mathrm{e}_\vartheta$, $\mathrm{s} = \mathrm{e}_\varphi$.

For each coordinate family we give the diagram illustrating section ϑ, $\varphi =$ constant with the radial coordinate taking on *both* positive and negative values. The diagrams thus represent the history of the *entire* main circle of the spatial spherical section of de Sitter universe. The left and right edges of the diagrams represent the south pole and should be considered as identified; the central vertical line describes the history of the north pole. Recalling the meaning of the negative radial coordinate we could eliminate the left half of each of the diagrams by transforming it into the right one by replacements $\{\vartheta,\, \varphi\} \to \{\pi-\vartheta,\, \varphi+\pi\}$. However, it is instructive to keep both halves for better understanding of the spatial topology of the sections. All diagrams are compactified—they are adapted to the standard rescaled coordinates $\tilde{t}$, $\tilde{r}$ (see below). The past and future conformal infinities are drawn as double lines. The ranges of time and radial coordinates are shown, the orientation of coordinate labels indicates the directions of the growth of corresponding coordinates.

We will also introduce several sign factors. The values of these factors in different domains of spacetime are indicated in Fig. 13.

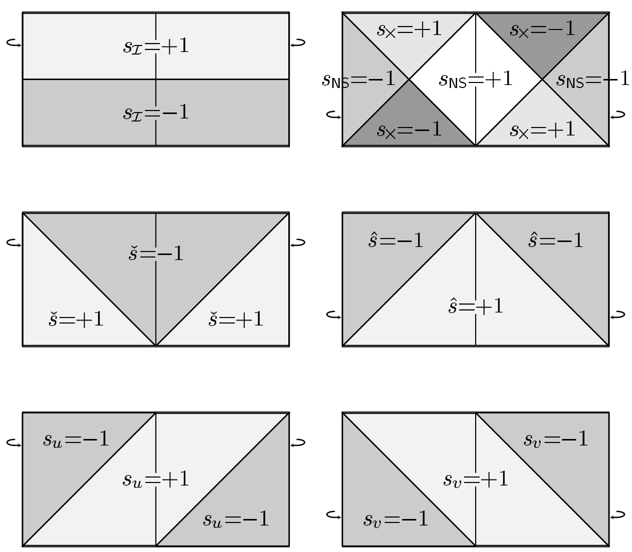

Figure 13: The values of the factors $s_\mathcal{I}$, s_{NS}, $s_\times$, $\check{s}$, $\hat{s}$, s_u and s_v in various regions of de Sitter space. The factors are defined in Eqs. (A73), (A61), (A74), (A21), (A36) and (A128), respectively. The factor $s_\times$ is used only in the expressions for static coordinates in the region where the Killing vector is spacelike. Therefore, we indicated the values of $s_\times$ only in those regions, although Eq. (A74) defines $s_\times$ everywhere. The factors $s_\times$, s_u, and s_v are defined only for any given section $\vartheta =$ constant, but not as unique functions on the whole spacetime (they are not symmetric with respect to the pole). This is related to our convention using negative radial coordinates, cf. the text below Eq. (A1).

1. The spherical cosmological family

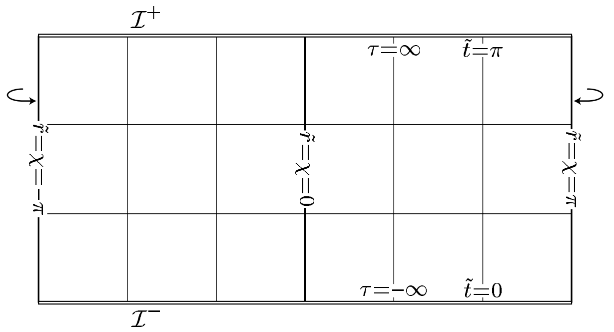

Figure 14: The spherical cosmological family of coordinates.

The first family consists of the *standard* or *spherical cosmological coordinates* τ, χ, ϑ, φ, and of the *standard rescaled* or *conformally Einstein coordinates* $\tilde{t}$, $\tilde{r}$, ϑ, φ (where $\tilde{r}\equiv\chi$). These coordinates cover de Sitter spacetime globally. They are associated with cosmological observers with homogeneous spatial sections of positive spatial curvature. The coordinates are adjusted to the spherical symmetry of the spatial sections: χ, ϑ, and φ are standard angular coordinates. The coordinate τ is a proper time along the worldlines of the cosmological observers given by χ, ϑ, $\varphi=$ constant. The vector $\partial/\partial\tau$ is a conformal Killing vector which is everywhere timelike. The rescaled coordinates $\tilde{t}$, $\tilde{r}$, ϑ, φ can also be viewed as the standard coordinates of the conformally related Einstein universe; they cover smoothly both conformal infinities $\mathcal{I}^\pm$ of de Sitter spacetime.

Metric and relation between coordinates

$$g=-\mathrm{d}\tau^2+\ell_\Lambda^2\,\cosh^2(\tau/\ell_\Lambda)\,\big(\mathrm{d}\chi^2+\sin^2\chi\,\mathrm{d}\omega^2\big)\,, \tag{A6}$$

$$g=\ell_\Lambda^2\,\sin^{-2}\tilde{t}\,\big(-\mathrm{d}\tilde{t}^2+\mathrm{d}\tilde{r}^2+\sin^2\tilde{r}\,\mathrm{d}\omega^2\big)\,, \tag{A7}$$

$$\begin{aligned}\tan\frac{\tilde{t}}{2}&=\exp\frac{\tau}{\ell_\Lambda}\,,\quad \cot\tilde{t}=-\sinh\frac{\tau}{\ell_\Lambda}\,,\\ \sin\tilde{t}&=\cosh^{-1}\frac{\tau}{\ell_\Lambda}\,,\quad \cos\tilde{t}=-\tanh\frac{\tau}{\ell_\Lambda}\,,\end{aligned} \tag{A8a}$$

$$\tilde{r}=\chi\,. \tag{A8b}$$

The ranges of coordinates are

$$\begin{aligned}&\tau\in\mathbb{R}\,,\quad \chi\in(-\pi,\pi)\,,\\ &\tilde{t}\in(0,\pi)\,,\quad \tilde{r}\in(-\pi,\pi)\,,\end{aligned} \tag{A9}$$

with negative values of radial coordinates χ, $\tilde{r}$ interpreted in accordance with Eq. (A2).

Orthonormal tetrad

$$\mathrm{e}_\tau=\frac{\partial}{\partial\tau}=\frac{1}{\ell_\Lambda}\sin\tilde{t}\,\frac{\partial}{\partial\tilde{t}}\,,$$

$$\mathrm{e}_\chi=\frac{1}{\ell_\Lambda}\cosh^{-1}\frac{\tau}{\ell_\Lambda}\,\frac{\partial}{\partial\chi}=\frac{1}{\ell_\Lambda}\sin\tilde{t}\,\frac{\partial}{\partial\tilde{r}}\,, \tag{A10}$$

$$\mathrm{e}_\vartheta=\frac{1}{\ell_\Lambda}\,\frac{1}{\cosh(\tau/\ell_\Lambda)\sin\chi}\,\frac{\partial}{\partial\vartheta}=\frac{1}{\ell_\Lambda}\,\frac{\sin\tilde{t}}{\sin\tilde{r}}\,\frac{\partial}{\partial\vartheta}\,.$$

Relation to flat cosmological family

$$\begin{aligned}\tan\tilde{t}&=\frac{2\,\ell_\Lambda\hat{t}}{\ell_\Lambda^2-\hat{t}^2+\hat{r}^2}=\frac{2\,\ell_\Lambda\check{t}}{\ell_\Lambda^2-\check{t}^2+\check{r}^2}\,,\\ \tan\tilde{r}&=\frac{2\,\ell_\Lambda\hat{r}}{\ell_\Lambda^2+\hat{t}^2-\hat{r}^2}=\frac{2\,\ell_\Lambda\check{r}}{\ell_\Lambda^2+\check{t}^2-\check{r}^2}\,.\end{aligned} \tag{A11}$$

Relation to hyperbolic cosmological coordinates

$$\begin{aligned}\cot\tilde{t}&=-\sinh\frac{\eta}{\ell_\Lambda}\,\cosh\frac{\rho}{\ell_\Lambda}\,,\\ \tan\tilde{r}&=\tanh\frac{\eta}{\ell_\Lambda}\,\sinh\frac{\rho}{\ell_\Lambda}\,.\end{aligned} \tag{A12}$$

Relation to static family in timelike domains **N**, **S**

$$\begin{aligned}\tan\tilde{t}&=-s_{\mathsf{NS}}\,\frac{\ell_\Lambda}{\sqrt{\ell_\Lambda^2-R^2}}\,\sinh^{-1}\frac{T}{\ell_\Lambda}\,,\\ \tan\tilde{r}&=s_{\mathsf{NS}}\,\frac{R}{\sqrt{\ell_\Lambda^2-R^2}}\,\cosh^{-1}\frac{T}{\ell_\Lambda}\,,\end{aligned} \tag{A13}$$

$$\tan\tilde{t}=-s_{\mathsf{NS}}\,\frac{\cosh\frac{\bar{r}}{\ell_\Lambda}}{\sinh\frac{\bar{t}}{\ell_\Lambda}}\,,\quad \tan\tilde{r}=s_{\mathsf{NS}}\,\frac{\sinh\frac{\bar{r}}{\ell_\Lambda}}{\cosh\frac{\bar{t}}{\ell_\Lambda}}\,, \tag{A14}$$

where $s_{\mathsf{NS}}=+1$ (-1) in domain **N** (**S**), cf. Eq. (A61).

Relation to static family in spacelike domains **F**, **P**

$$\begin{aligned}\tan\tilde{t}&=\frac{-s_{\mathcal{I}}\,\ell_\Lambda}{\sqrt{R^2-\ell_\Lambda^2}}\,\cosh^{-1}\frac{T}{\ell_\Lambda}\,,\\ \tan\tilde{r}&=\frac{s_{\mathcal{I}}\,R}{\sqrt{R^2-\ell_\Lambda^2}}\,\sinh^{-1}\frac{T}{\ell_\Lambda}\,,\end{aligned} \tag{A15}$$

$$\tan\tilde{t}=s_\times\,\frac{\sinh\frac{\bar{r}}{\ell_\Lambda}}{\cosh\frac{\bar{t}}{\ell_\Lambda}}\,,\quad \tan\tilde{r}=-s_\times\,\frac{\cosh\frac{\bar{r}}{\ell_\Lambda}}{\sinh\frac{\bar{t}}{\ell_\Lambda}}\,, \tag{A16}$$

where $s_{\mathcal{I}}=-\operatorname{sign}\cos\tilde{t}$ and $s_\times=-s_{\mathcal{I}}\operatorname{sign}\tilde{r}$, cf. Eqs. (A73) and (A74).

Relation to conformally Minkowski coordinates

$$\cot\tilde{t}=\frac{2\ell_\Lambda t}{t^2-r^2-\ell_\Lambda^2}\,,\quad \tan\tilde{r}=\frac{2\ell_\Lambda r}{t^2-r^2+\ell_\Lambda^2}\,. \tag{A17}$$

2. The flat cosmological family, type "∨"

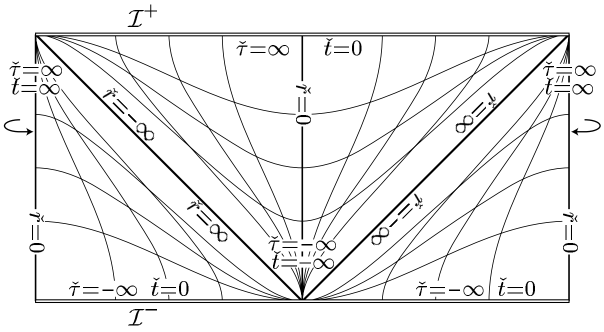

Figure 15: The flat cosmological family, type "∨".

The first flat cosmological coordinate family consists of the *flat cosmological coordinates* $\check{\tau}$, $\check{r}$, ϑ, φ and of the *rescaled flat cosmological coordinates* $\check{t}$, $\check{r}$, ϑ, φ. Hypersurfaces $\check{\tau}=$ constant are homogeneous flat spaces and coordinate lines $\check{r}$, ϑ, $\varphi=$ constant are worldlines of cosmological observers orthogonal to these hypersurfaces. They are geodesic with proper time $\check{\tau}$, the vector $\partial/\partial\check{\tau}$ is a conformal Killing vector. The coordinates cover de Sitter spacetime smoothly, except for the past cosmological horizon, $\tilde{r}=\tilde{t}$, of the north pole where $\check{r}$, $\check{t}\to\pm\infty$. The coordinates thus split into two coordinate patches—"above" and "below" the horizon. The domain above the horizon has a cosmological interpretation of an exponentially expanding flat three-space. The rescaled coordinates can be viewed as inertial coordinates in the conformally related Minkowski space $\check{M}$, cf. Fig. 3; the domain above the horizon corresponds to the "lower half", $\check{t}<0$, of $\check{M}$, the domain below corresponds to the "upper half", $\check{t}>0$.

Metric and relation between coordinates

$$g=\frac{\ell_\Lambda^2}{\check{t}^2}\Big(-\mathrm{d}\check{t}^{\,2}+\mathrm{d}\check{r}^2+\check{r}^2\,\mathrm{d}\omega^2\Big)\,, \tag{A18}$$

$$g=-\mathrm{d}\check{\tau}^2+\exp\!\big(-\check{s}\,2\,\check{\tau}/\ell_\Lambda\big)\,\big(\mathrm{d}\check{r}^2+\check{r}^2\,\mathrm{d}\omega^2\big)\,. \tag{A19}$$

$$\check{t}=\check{s}\,\ell_\Lambda\exp\!\Big(\check{s}\frac{\check{\tau}}{\ell_\Lambda}\Big)\,, \tag{A20}$$

$$\check{s}=\operatorname{sign}\check{t}\,. \tag{A21}$$

The ranges of coordinates are

$$\begin{aligned}&\check{\tau}\in\mathbb{R}\,,\quad \check{t}\in\mathbb{R}^-\,,\quad \check{r}\in\mathbb{R}\quad\text{above the horizon}\,,\\ &\check{\tau}\in\mathbb{R}\,,\quad \check{t}\in\mathbb{R}^+\,,\quad \check{r}\in\mathbb{R}\quad\text{below the horizon}\,,\end{aligned} \tag{A22}$$

with negative values of radial coordinate $\check{r}$ interpreted as described in Eq. (A2).

Orthonormal tetrad

$$\begin{gathered}\mathrm{e}_{\check{t}}=\frac{\partial}{\partial\check{\tau}}=\frac{\check{s}\,\check{t}}{\ell_\Lambda}\frac{\partial}{\partial\check{t}}\,,\quad \mathrm{e}_{\check{r}}=\exp\frac{\check{s}\check{\tau}}{\ell_\Lambda}\frac{\partial}{\partial\check{r}}=\frac{\check{s}\,\check{t}}{\ell_\Lambda}\frac{\partial}{\partial\check{r}}\,,\\ \mathrm{e}_{\vartheta}=-\frac{\check{s}}{\check{r}}\exp\frac{\check{s}\check{\tau}}{\ell_\Lambda}\frac{\partial}{\partial\vartheta}=-\frac{1}{\ell_\Lambda}\frac{\check{t}}{\check{r}}\frac{\partial}{\partial\vartheta}\,.\end{gathered} \tag{A23}$$

Relation to spherical cosmological family

$$\check{t}=\frac{-\ell_\Lambda\cosh^{-1}(\tau/\ell_\Lambda)}{\cos\chi+\tanh(\tau/\ell_\Lambda)}\,,\quad \check{r}=\frac{\ell_\Lambda\cosh^{-1}(\tau/\ell_\Lambda)}{\cos\chi+\tanh(\tau/\ell_\Lambda)}\,. \tag{A24}$$

$$\check{t}=\frac{\ell_\Lambda\sin\tilde{t}}{\cos\tilde{t}-\cos\tilde{r}}\,,\quad \check{r}=\frac{\ell_\Lambda\sin\tilde{r}}{\cos\tilde{r}-\cos\tilde{t}}\,. \tag{A25}$$

Relation to flat cosmological family, type "∧"

$$\check{t}=-\frac{\hat{t}\,\ell_\Lambda^2}{\hat{t}^2-\hat{r}^2}\,,\quad \check{r}=\frac{\hat{r}\,\ell_\Lambda^2}{\hat{t}^2-\hat{r}^2}\,, \tag{A26}$$

$$\begin{gathered}\check{t}\,\hat{r}+\hat{t}\,\check{r}=0\,,\quad \hat{t}\,\check{t}+\hat{r}\,\check{r}=-\ell_\Lambda^2\,,\\ \big(-\hat{t}^2+\hat{r}^2\big)\big(-\check{t}^2+\check{r}^2\big)=\ell_\Lambda^4\,,\\ (\hat{t}+\hat{r})(\check{t}+\check{r})=(\hat{t}-\hat{r})(\check{t}-\check{r})=-\ell_\Lambda^2\,.\end{gathered} \tag{A27}$$

Relation to static family in timelike domains **N**, **S**

$$\begin{aligned}\frac{\check{t}}{\ell_\Lambda}&=-s_{\mathsf{NS}}\frac{\ell_\Lambda}{\sqrt{\ell_\Lambda^2-R^2}}\exp\Big(-\frac{T}{\ell_\Lambda}\Big)\,,\\ \frac{\check{r}}{\ell_\Lambda}&=s_{\mathsf{NS}}\frac{R}{\sqrt{\ell_\Lambda^2-R^2}}\exp\Big(-\frac{T}{\ell_\Lambda}\Big)\,,\end{aligned} \tag{A28}$$

$$\begin{aligned}\check{t}&=-s_{\mathsf{NS}}\,\ell_\Lambda\exp\Big(-\frac{\bar{t}}{\ell_\Lambda}\Big)\cosh\frac{\bar{r}}{\ell_\Lambda}\,,\\ \check{r}&=s_{\mathsf{NS}}\ell_\Lambda\exp\Big(-\frac{\bar{t}}{\ell_\Lambda}\Big)\sinh\frac{\bar{r}}{\ell_\Lambda}\,,\end{aligned} \tag{A29}$$

where $s_{\mathsf{NS}}=+1$ (-1) in domain **N** (**S**), cf. Eq. (A61).

Relation to static family in spacelike domains **F**, **P**

$$\begin{aligned}\frac{\check{t}}{\ell_\Lambda}&=s_\times\frac{\ell_\Lambda}{\sqrt{R^2-\ell_\Lambda^2}}\exp\Big(-\frac{T}{\ell_\Lambda}\Big)\,,\\ \frac{\check{r}}{\ell_\Lambda}&=-s_\times\frac{R}{\sqrt{R^2-\ell_\Lambda^2}}\exp\Big(-\frac{T}{\ell_\Lambda}\Big)\,,\end{aligned} \tag{A30}$$

$$\begin{aligned}\check{t}&=s_\times\ell_\Lambda\exp\Big(-\frac{\bar{t}}{\ell_\Lambda}\Big)\sinh\frac{\bar{r}}{\ell_\Lambda}\,,\\ \check{r}&=-s_\times\ell_\Lambda\exp\Big(-\frac{\bar{t}}{\ell_\Lambda}\Big)\cosh\frac{\bar{r}}{\ell_\Lambda}\,,\end{aligned} \tag{A31}$$

where $s_\times=\operatorname{sign}\bar{r}\,\operatorname{sign}\cos\tilde{t}$, cf. Eqs. (A73) and (A74).

Relation to conformally Minkowski coordinates

$$\frac{\check{t}}{\ell_\Lambda}=-\frac{\ell_\Lambda^2-t^2+r^2}{(\ell_\Lambda+t)^2-r^2}\,,\quad \frac{\check{r}}{\ell_\Lambda}=\frac{2\,\ell_\Lambda r}{(\ell_\Lambda+t)^2-r^2}\,. \tag{A32}$$

3. The flat cosmological family, type "∧"

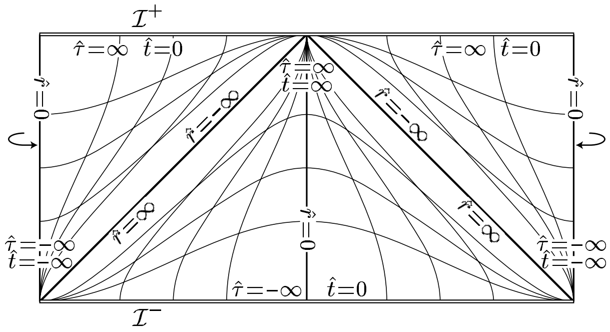

Figure 16: The flat cosmological family, type "∧".

The second flat cosmological coordinate family consists of the *flat cosmological coordinates* $\hat\tau$, $\hat r$, ϑ, φ and of the *rescaled flat cosmological coordinates* $\hat t$, $\hat r$, ϑ, φ. They can be built analogously to the flat coordinates introduced above, with north and south poles interchanged only. They thus have similar properties: Hypersurfaces $\hat t =$ constant are homogeneous flat three-spaces, coordinate lines $\hat r$, ϑ, $\varphi =$ constant are geodesics with proper time $\hat\tau$, and $\partial/\partial\hat\tau$ is a conformal Killing vector. The coordinates cover de Sitter spacetime everywhere except the future cosmological horizon, $\tilde r = \pi - \tilde t$, of the north pole (i.e., the past horizon of the south pole), and the rescaled coordinates can be viewed as inertial coordinates in the conformally related Minkowski space $\hat M$.

Metric and relation between coordinates

$$g = \frac{\ell_\Lambda^2}{\hat t^2}\left(-\mathrm{d}\hat t^{\,2} + \mathrm{d}\hat r^2 + \hat r^2\,\mathrm{d}\omega^2\right)\,, \tag{A33}$$

$$g = -\mathrm{d}\hat\tau^2 + \exp\!\left(-\hat s\,2\,\hat\tau/\ell_\Lambda\right)\left(\mathrm{d}\hat r^2 + \hat r^2\,\mathrm{d}\omega^2\right)\,, \tag{A34}$$

$$\hat t = \hat s\,\ell_\Lambda \exp\!\left(\hat s\frac{\hat\tau}{\ell_\Lambda}\right)\,, \tag{A35}$$

where

$$\hat s = \operatorname{sign}\hat t\,. \tag{A36}$$

The ranges of coordinates are

$$\begin{aligned} &\hat\tau\in\mathbb{R}\,,\quad \hat t\in\mathbb{R}^-\,,\quad \hat r\in\mathbb{R}\quad \text{above the horizon}\,,\\ &\hat\tau\in\mathbb{R}\,,\quad \hat t\in\mathbb{R}^+\,,\quad \hat r\in\mathbb{R}\quad \text{below the horizon}\,, \end{aligned} \tag{A37}$$

with negative values of radial coordinate $\hat r$ interpreted as described in Eq. (A2).

Orthonormal tetrad

$$\begin{aligned} \mathrm{e}_{\hat t} &= \frac{\partial}{\partial\hat\tau} = \frac{\hat s\,\hat t}{\ell_\Lambda}\frac{\partial}{\partial\hat t}\,,\quad \mathrm{e}_{\hat r} = \exp\frac{\hat s\hat\tau}{\ell_\Lambda}\frac{\partial}{\partial\hat r} = \frac{\hat s\,\hat t}{\ell_\Lambda}\frac{\partial}{\partial\hat r}\,,\\ \mathrm{e}_\vartheta &= \frac{\hat s}{\hat r}\exp\frac{\hat s\hat\tau}{\ell_\Lambda}\frac{\partial}{\partial\vartheta} = \frac{1}{\ell_\Lambda}\frac{\hat t}{\hat r}\frac{\partial}{\partial\vartheta}\,. \end{aligned} \tag{A38}$$

Relation to spherical cosmological family

$$\hat t = \frac{\ell_\Lambda\cosh^{-1}(\tau/\ell_\Lambda)}{\cos\chi - \tanh(\tau/\ell_\Lambda)}\,,\quad \hat r = \frac{\ell_\Lambda\cosh^{-1}(\tau/\ell_\Lambda)}{\cos\chi - \tanh(\tau/\ell_\Lambda)}\,. \tag{A39}$$

$$\hat t = \frac{\ell_\Lambda\sin\tilde t}{\cos\tilde t + \cos\tilde r}\,,\quad \hat r = \frac{\ell_\Lambda\sin\tilde r}{\cos\tilde r + \cos\tilde t}\,. \tag{A40}$$

Relation to flat cosmological family, type "∧"

$$\hat t = -\frac{\check t\,\ell_\Lambda^2}{\check t^2 - \check r^2}\,,\quad \hat r = \frac{\check r\,\ell_\Lambda^2}{\check t^2 - \check r^2}\,, \tag{A41}$$

$$\begin{gathered} \check t\,\hat r + \hat t\,\check r = 0\,,\quad \hat t\,\check t + \hat r\,\check r = -\ell_\Lambda^2\,,\\ \left(-\hat t^2 + \hat r^2\right)\left(-\check t^2 + \check r^2\right) = \ell_\Lambda^4\,,\\ (\hat t + \hat r)(\check t + \check r) = (\hat t - \hat r)(\check t - \check r) = -\ell_\Lambda^2\,. \end{gathered} \tag{A42}$$

Relation to static family in timelike domains **N**, **S**

$$\begin{aligned} \frac{\hat t}{\ell_\Lambda} &= s_{\mathsf{NS}}\frac{\ell_\Lambda}{\sqrt{\ell_\Lambda^2 - R^2}}\exp\frac{T}{\ell_\Lambda}\,,\\ \frac{\hat r}{\ell_\Lambda} &= s_{\mathsf{NS}}\frac{R}{\sqrt{\ell_\Lambda^2 - R^2}}\exp\frac{T}{\ell_\Lambda}\,, \end{aligned} \tag{A43}$$

$$\begin{aligned} \hat t &= s_{\mathsf{NS}}\,\ell_\Lambda\exp\frac{\bar t}{\ell_\Lambda}\cosh\frac{\bar r}{\ell_\Lambda}\,,\\ \hat r &= s_{\mathsf{NS}}\,\ell_\Lambda\exp\frac{\bar t}{\ell_\Lambda}\sinh\frac{\bar r}{\ell_\Lambda}\,, \end{aligned} \tag{A44}$$

where $s_{\mathsf{NS}} = +1$ (-1) in domain **N** (**S**), cf. Eq. (A61).

Relation to static family in spacelike domains **F**, **P**

$$\begin{aligned} \frac{\hat t}{\ell_\Lambda} &= s_\times\frac{\ell_\Lambda}{\sqrt{R^2 - \ell_\Lambda^2}}\exp\frac{T}{\ell_\Lambda}\,,\\ \frac{\hat r}{\ell_\Lambda} &= s_\times\frac{R}{\sqrt{R^2 - \ell_\Lambda^2}}\exp\frac{T}{\ell_\Lambda}\,, \end{aligned} \tag{A45}$$

$$\begin{aligned} \hat t &= s_\times\,\ell_\Lambda\exp\frac{\bar t}{\ell_\Lambda}\sinh\frac{\bar r}{\ell_\Lambda}\,,\\ \hat r &= s_\times\,\ell_\Lambda\exp\frac{\bar t}{\ell_\Lambda}\cosh\frac{\bar r}{\ell_\Lambda}\,, \end{aligned} \tag{A46}$$

where $s_\times = \operatorname{sign}\tilde r\,\operatorname{sign}\cos\tilde t$, cf. Eqs. (A73) and (A74).

Relation to conformally Minkowski coordinates

$$\frac{\hat t}{\ell_\Lambda} = \frac{\ell_\Lambda^2 - t^2 + r^2}{(\ell_\Lambda - t)^2 - r^2}\,,\quad \frac{\hat r}{\ell_\Lambda} = \frac{2\,\ell_\Lambda r}{(\ell_\Lambda - t)^2 - r^2}\,. \tag{A47}$$

4. The conformally Minkowski family

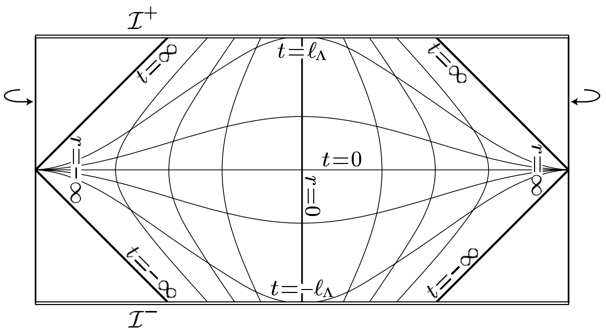

Figure 17: The conformally Minkowski family of coordinates.

The *conformally Minkowski coordinates* t, r, ϑ, φ can be understood as spherical coordinates in the conformally related Minkowski space M. The coordinates do not cover de Sitter spacetime globally—they cover only a region around north pole, see Fig. 17. The boundary of this region is given by the conformal infinity of the Minkowski spacetime. These coordinates are useful for studying the limit $\Lambda \to 0$.

The metric

$$g = \left(\frac{2\,\ell_\Lambda^2}{\ell_\Lambda^2 - t^2 + r^2}\right)^2 \left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\omega^2\right)\,, \tag{A48}$$

the ranges of coordinates

$$t \in \mathbb{R}\,,\quad r \in \mathbb{R}\,,\quad \text{such that}\quad t^2 - r^2 < \ell_\Lambda^2\,, \tag{A49}$$

with negative values of radial coordinate r interpreted as described in Eq. (A2).

Orthonormal tetrad

$$\begin{aligned} \mathrm{e}_t &= \frac{\ell_\Lambda^2 - t^2 + r^2}{2\ell_\Lambda^2}\,\frac{\partial}{\partial t}\,,\\ \mathrm{e}_r &= \frac{\ell_\Lambda^2 - t^2 + r^2}{2\ell_\Lambda^2}\,\frac{\partial}{\partial r}\,,\\ \mathrm{e}_\vartheta &= \frac{\ell_\Lambda^2 - t^2 + r^2}{2\ell_\Lambda^2}\,\frac{1}{r}\,\frac{\partial}{\partial t}\,. \end{aligned} \tag{A50}$$

Relation to spherical cosmological family

$$\begin{aligned} t &= -\frac{\ell_\Lambda \cos\tilde{t}}{\cos\tilde{r} + \sin\tilde{t}}\,,\\ r &= \frac{\ell_\Lambda \sin\tilde{r}}{\cos\tilde{r} + \sin\tilde{t}}\,. \end{aligned} \tag{A51}$$

Relation to flat cosmological family

$$\begin{aligned} \frac{t}{\ell_\Lambda} &= -\frac{\ell_\Lambda^2 - \hat{t}^2 + \hat{r}^2}{(\ell_\Lambda + \hat{t})^2 - \hat{r}^2} = \frac{\ell_\Lambda^2 - \check{t}^2 + \check{r}^2}{(\ell_\Lambda - \check{t})^2 - \check{r}^2}\,,\\ \frac{r}{\ell_\Lambda} &= \frac{2\,\ell_\Lambda \hat{r}}{(\ell_\Lambda + \hat{t})^2 - \hat{r}^2} = \frac{2\,\ell_\Lambda \check{r}}{(\ell_\Lambda - \check{t})^2 - \check{r}^2}\,. \end{aligned} \tag{A52}$$

Relation to hyperbolic cosmological coordinates

$$\begin{aligned} \frac{t}{\ell_\Lambda} &= \tanh\frac{\eta}{2\ell_\Lambda}\,\cosh\frac{\beta}{\ell_\Lambda}\,,\\ \frac{r}{\ell_\Lambda} &= \tanh\frac{\eta}{2\ell_\Lambda}\,\sinh\frac{\beta}{\ell_\Lambda}\,. \end{aligned} \tag{A53}$$

Relation to static family in timelike domains **N**, **S**

$$\begin{aligned} \frac{t}{\ell_\Lambda} &= \frac{\sinh\frac{\bar{t}}{\ell_\Lambda}}{\cosh\frac{\bar{t}}{\ell_\Lambda} + s_{\mathsf{NS}}\cosh\frac{\bar{r}}{\ell_\Lambda}}\,,\\ \frac{r}{\ell_\Lambda} &= \frac{\sinh\frac{\bar{r}}{\ell_\Lambda}}{\cosh\frac{\bar{r}}{\ell_\Lambda} + s_{\mathsf{NS}}\cosh\frac{\bar{t}}{\ell_\Lambda}}\,, \end{aligned} \tag{A54}$$

$$\begin{aligned} \frac{t}{\ell_\Lambda} &= \frac{\sqrt{\ell_\Lambda^2 - R^2}\,\sinh\frac{T}{\ell_\Lambda}}{s_{\mathsf{NS}}\ell_\Lambda + \sqrt{\ell_\Lambda^2 - R^2}\,\cosh\frac{T}{\ell_\Lambda}}\,,\\ \frac{r}{\ell_\Lambda} &= \frac{R}{\ell_\Lambda + s_{\mathsf{NS}}\sqrt{\ell_\Lambda^2 - R^2}\,\cosh\frac{T}{\ell_\Lambda}}\,, \end{aligned} \tag{A55}$$

where $s_{\mathsf{NS}} = +1$ (-1) in domain **N** (**S**), cf. Eq. (A61).

Relation to static family in spacelike domains **F**, **P**

$$\begin{aligned} \frac{t}{\ell_\Lambda} &= \frac{\cosh\frac{\bar{t}}{\ell_\Lambda}}{\sinh\frac{\bar{t}}{\ell_\Lambda} - s_\times \sinh\frac{\bar{r}}{\ell_\Lambda}}\,,\\ \frac{r}{\ell_\Lambda} &= \frac{\cosh\frac{\bar{r}}{\ell_\Lambda}}{\sinh\frac{\bar{r}}{\ell_\Lambda} - s_\times \cosh\frac{\bar{t}}{\ell_\Lambda}}\,, \end{aligned} \tag{A56}$$

$$\begin{aligned} \frac{t}{\ell_\Lambda} &= \frac{\sqrt{R^2 - \ell_\Lambda^2}\,\cosh\frac{T}{\ell_\Lambda}}{-s_\times \ell_\Lambda + \sqrt{R^2 - \ell_\Lambda^2}\,\sinh\frac{T}{\ell_\Lambda}}\,,\\ \frac{r}{\ell_\Lambda} &= \frac{R}{\ell_\Lambda - s_\times \sqrt{R^2 - \ell_\Lambda^2}\,\sinh\frac{T}{\ell_\Lambda}}\,, \end{aligned} \tag{A57}$$

with $s_\times = \operatorname{sign}\tilde{r}\,\operatorname{sign}\cos\tilde{t}$, cf. Eqs. (A73) and (A74).

5. The static family in timelike domains **N** and **S**

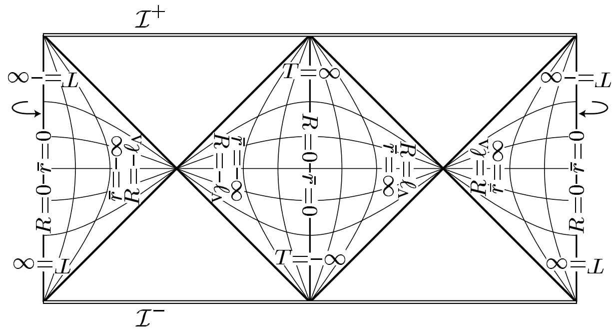

Figure 18: The static family of coordinates, timelike domains.

This family consists of the *static coordinates* T, R, ϑ, φ and the *"tortoidal" static coordinates* $\bar{t}$, $\bar{r}$, ϑ, φ. The metric does not depend on time coordinate $T=\bar{t}$—the coordinates are associated with a Killing vector. Since the Killing vector changes its character, the coordinates do not cover the spacetime smoothly. We first describe the static coordinates in domains **N** and **S**, where the Killing vector is timelike. In domain **N** the orbits of the Killing vector (corresponding to the worldlines of static observers) start and end at the north pole, in domain **S**—at the south pole. They are orthogonal to slices $T=$ constant, each of which consists of two hemispheres (one in domain **N**, the other in **S**) with homogeneous spherical 3-metric. The distances between static observers (measured within these slices) do not change. Since the static observers must overcome first the cosmological contraction and then the expansion, they move with a (uniform) acceleration.

Metric and relation between coordinates

$$g = \cosh^{-2}\frac{\bar{r}}{\ell_\Lambda}\left(-\mathrm{d}\bar{t}^2 + \mathrm{d}\bar{r}^2 + \ell_\Lambda^2 \sinh^2\frac{\bar{r}}{\ell_\Lambda}\,\mathrm{d}\omega^2\right)\,, \tag{A58}$$

$$g = -\Big(1-\frac{R^2}{\ell_\Lambda^2}\Big)\,\mathrm{d}T^2 + \Big(1-\frac{R^2}{\ell_\Lambda^2}\Big)^{-1}\mathrm{d}R^2 + R^2\,\mathrm{d}\omega^2, \tag{A59}$$

$$T = \bar{t}\,, \tag{A60a}$$

$$\begin{aligned} \exp\frac{\bar{r}}{\ell_\Lambda} &= \sqrt{\frac{\ell_\Lambda+R}{\ell_\Lambda-R}}\,, & \sinh\frac{\bar{r}}{\ell_\Lambda} &= \frac{R}{\sqrt{\ell_\Lambda^2-R^2}}\,,\\ \tanh\frac{\bar{r}}{\ell_\Lambda} &= \frac{R}{\ell_\Lambda}\,, & \cosh\frac{\bar{r}}{\ell_\Lambda} &= \frac{\ell_\Lambda}{\sqrt{\ell_\Lambda^2-R^2}}\,, \end{aligned} \tag{A60b}$$

$$s_{\text{NS}} = \begin{cases} +1 & \text{in domain } \mathbf{N}\,,\\ -1 & \text{in domain } \mathbf{S}\,. \end{cases} \tag{A61}$$

The ranges of coordinates are

$$\begin{gathered} T\in\mathbb{R}\,,\quad R\in(-\ell_\Lambda,\ell_\Lambda)\,,\\ \bar{t}\in\mathbb{R}\,,\quad \bar{r}\in\mathbb{R}\,, \end{gathered} \tag{A62}$$

with negative values of radial coordinate R and $\bar{r}$ interpreted as described in Eq. (A2).

Orthonormal tetrad

$$\begin{aligned} \mathrm{e}_T &= \Big(1-\frac{R^2}{\ell_\Lambda^2}\Big)^{-1/2}\frac{\partial}{\partial T} = \cosh\frac{\bar{r}}{\ell_\Lambda}\,\frac{\partial}{\partial\bar{t}}\,,\\ \mathrm{e}_R &= \Big(1-\frac{R^2}{\ell_\Lambda^2}\Big)^{1/2}\frac{\partial}{\partial R} = \cosh^{-1}\frac{\bar{r}}{\ell_\Lambda}\,\frac{\partial}{\partial\bar{r}}\,,\\ \mathrm{e}_\vartheta &= \frac{1}{R}\,\frac{\partial}{\partial\vartheta} = \frac{1}{\ell_\Lambda}\coth\frac{\bar{r}}{\ell_\Lambda}\,\frac{\partial}{\partial\vartheta}\,. \end{aligned} \tag{A63}$$

Relation to spherical cosmological family

$$T = \frac{\ell_\Lambda}{2}\log\frac{\cos\tilde{r}-\cos\tilde{t}}{\cos\tilde{r}+\cos\tilde{t}}\,,\quad R = \ell_\Lambda\,\frac{\sin\tilde{r}}{\sin\tilde{t}}\,, \tag{A64}$$

$$\bar{t} = \frac{\ell_\Lambda}{2}\log\Big(\tan\frac{\tilde{t}+\tilde{r}}{2}\,\tan\frac{\tilde{t}-\tilde{r}}{2}\Big)\,, \tag{A65a}$$

$$\begin{aligned} \exp\frac{\bar{t}}{\ell_\Lambda} &= \sqrt{\frac{\cos\tilde{r}-\cos\tilde{t}}{\cos\tilde{r}+\cos\tilde{t}}}\,, & \sinh\frac{\bar{t}}{\ell_\Lambda} &= \frac{-s_{\text{NS}}\cos\tilde{t}}{\sqrt{\cos^2\tilde{r}-\cos^2\tilde{t}}}\,,\\ \tanh\frac{\bar{t}}{\ell_\Lambda} &= -\frac{\cos\tilde{t}}{\cos\tilde{r}}\,, & \cosh\frac{\bar{t}}{\ell_\Lambda} &= \frac{s_{\text{NS}}\cos\tilde{r}}{\sqrt{\cos^2\tilde{r}-\cos^2\tilde{t}}}\,, \end{aligned}$$

$$\bar{r} = \frac{\ell_\Lambda}{2}\log\Big(\tan\frac{\tilde{t}+\tilde{r}}{2}\,\cot\frac{\tilde{t}-\tilde{r}}{2}\Big)\,, \tag{A65b}$$

$$\begin{aligned} \exp\frac{\bar{r}}{\ell_\Lambda} &= \sqrt{\frac{\sin\tilde{t}+\sin\tilde{r}}{\sin\tilde{t}-\sin\tilde{r}}}\,, & \sinh\frac{\bar{r}}{\ell_\Lambda} &= \frac{\sin\tilde{r}}{\sqrt{\sin^2\tilde{t}-\sin^2\tilde{r}}}\,,\\ \tanh\frac{\bar{r}}{\ell_\Lambda} &= \frac{\sin\tilde{r}}{\sin\tilde{t}}\,, & \cosh\frac{\bar{r}}{\ell_\Lambda} &= \frac{\sin\tilde{t}}{\sqrt{\sin^2\tilde{t}-\sin^2\tilde{r}}}\,. \end{aligned}$$

Relation to flat cosmological family

$$\begin{aligned} \bar{t} &= \frac{\ell_\Lambda}{2}\log\frac{\hat{t}^2-\hat{r}^2}{\ell_\Lambda^2} = -\frac{\ell_\Lambda}{2}\log\frac{\check{t}^2-\check{r}^2}{\ell_\Lambda^2}\,,\\ \bar{r} &= \frac{\ell_\Lambda}{2}\log\frac{\hat{t}+\hat{r}}{\hat{t}-\hat{r}} = \frac{\ell_\Lambda}{2}\log\frac{\check{t}-\check{r}}{\check{t}+\check{r}}\,, \end{aligned} \tag{A66}$$

$$\begin{aligned} \frac{T}{\ell_\Lambda} &= \frac{1}{2}\log\frac{\hat{t}^2-\hat{r}^2}{\ell_\Lambda^2} = -\frac{1}{2}\log\frac{\check{t}^2-\check{r}^2}{\ell_\Lambda^2}\,,\\ \frac{R}{\ell_\Lambda} &= \frac{\hat{r}}{\hat{t}} = -\frac{\check{r}}{\check{t}}\,. \end{aligned} \tag{A67}$$

Relation to conformally Minkowski coordinates

$$\tanh\frac{T}{\ell_\Lambda} = \frac{2\,\ell_\Lambda t}{\ell_\Lambda^2+t^2-r^2}\,,\quad \frac{R}{\ell_\Lambda} = \frac{2\,\ell_\Lambda r}{\ell_\Lambda^2+r^2-t^2}\,, \tag{A68}$$

$$\bar{t} = \frac{\ell_\Lambda}{2}\log\frac{(\ell_\Lambda+t)^2-r^2}{(\ell_\Lambda-t)^2-r^2}\,,\quad \bar{r} = \frac{\ell_\Lambda}{2}\log\frac{(\ell_\Lambda+r)^2-t^2}{(\ell_\Lambda-r)^2-t^2}\,. \tag{A69}$$

6. The static family in spacelike domains F and P

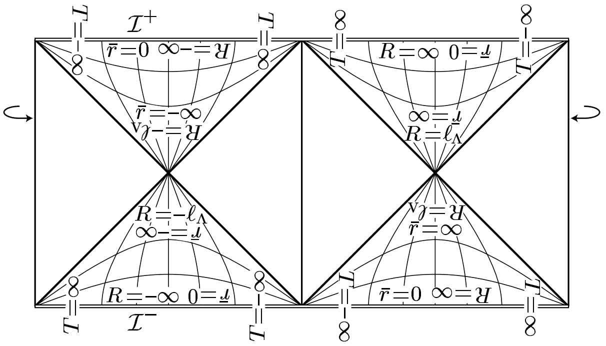

Figure 19: The static family of coordinates, spacelike domains.

Here we describe the *static coordinates* T, R, ϑ, φ and the *"tortoidal" static coordinates* $\bar{t}$, $\bar{r}$, ϑ, φ from the preceding section in domains **F** and **S** where the Killing vector is spacelike. These "non-static" domains extend up to infinity, namely, domain **F** up to $\mathcal{I}^+$, domain **P** up to $\mathcal{I}^-$. The orbits of the Killing vector start at the south pole and end at the north pole in **F**, and they point in opposite direction in **P**. The motion along them could thus be characterized as a "translation" from one pole to the other. The Lorentzian hypersurfaces $T = \text{constant}$ are homogeneous spaces with positive curvature, i.e., 3-dimensional de Sitter spacetimes.

Metric and relation between coordinates

$$g = \sinh^{-2}\frac{\bar{r}}{\ell_\Lambda}\left(-\mathrm{d}\bar{r}^2 + \mathrm{d}\bar{t}^2 + \ell_\Lambda^2\cosh^2\frac{\bar{r}}{\ell_\Lambda}\,\mathrm{d}\omega^2\right), \quad \text{(A70)}$$

$$g = -\left(1-\frac{R^2}{\ell_\Lambda^2}\right)\mathrm{d}T^2 + \left(1-\frac{R^2}{\ell_\Lambda^2}\right)^{-1}\mathrm{d}R^2 + R^2\,\mathrm{d}\omega^2, \quad \text{(A71)}$$

$$T = \bar{t}\,, \quad \text{(A72a)}$$

$$\begin{aligned} &\exp\frac{\bar{r}}{\ell_\Lambda} = \sqrt{\frac{R+\ell_\Lambda}{R-\ell_\Lambda}}\,, \quad \left|\sinh\frac{\bar{r}}{\ell_\Lambda}\right| = \frac{\ell_\Lambda}{\sqrt{R^2-\ell_\Lambda^2}}\,,\\ &\tanh\frac{\bar{r}}{\ell_\Lambda} = \frac{\ell_\Lambda}{R}\,, \qquad \cosh\frac{\bar{r}}{\ell_\Lambda} = \frac{|R|}{\sqrt{R^2-\ell_\Lambda^2}}\,. \end{aligned} \quad \text{(A72b)}$$

The signature factors $s_{\mathcal{I}}$ and $s_\times$ are defined as

$$s_{\mathcal{I}} = \begin{cases} +1 & \text{in domain } \mathbf{F}\,,\\ -1 & \text{in domain } \mathbf{P}\,, \end{cases} \quad \text{(A73)}$$

and

$$s_\times = -s_{\mathcal{I}}\,\operatorname{sign}\bar{r}\,. \quad \text{(A74)}$$

The coordinates ranges are

$$\begin{gathered} T\in\mathbb{R}\,,\quad |R|\in(\ell_\Lambda,\infty)\,,\\ \bar{t}\in\mathbb{R}\,,\quad \bar{r}\in\mathbb{R}\,, \end{gathered} \quad \text{(A75)}$$

with negative values of radial coordinate R and $\bar{r}$ interpreted as described in Eq. (A2).

Orthonormal tetrad

$$\begin{aligned} \mathrm{e}_T &= \left(\frac{R^2}{\ell_\Lambda^2}-1\right)^{-1/2}\frac{\partial}{\partial T} = \left|\sinh\frac{\bar{r}}{\ell_\Lambda}\right|\frac{\partial}{\partial\bar{t}}\,,\\ \mathrm{e}_R &= \left(\frac{R^2}{\ell_\Lambda^2}-1\right)^{1/2}\frac{\partial}{\partial R} = -\left|\sinh^{-1}\frac{\bar{r}}{\ell_\Lambda}\right|\frac{\partial}{\partial\bar{r}}\,,\\ \mathrm{e}_\vartheta &= \frac{1}{R}\frac{\partial}{\partial\vartheta} = \frac{1}{\ell_\Lambda}\left|\tanh\frac{\bar{r}}{\ell_\Lambda}\right|\frac{\partial}{\partial\vartheta}\,. \end{aligned} \quad \text{(A76)}$$

Relation to spherical cosmological family

$$T = \frac{\ell_\Lambda}{2}\log\frac{\cos\tilde{t}-\cos\tilde{r}}{\cos\tilde{t}+\cos\tilde{r}}\,,\quad R = \ell_\Lambda\frac{\sin\tilde{r}}{\sin\tilde{t}}\,, \quad \text{(A77)}$$

$$\bar{t} = \frac{\ell_\Lambda}{2}\log\left(-\tan\frac{\tilde{t}+\tilde{r}}{2}\,\tan\frac{\tilde{t}-\tilde{r}}{2}\right), \quad \text{(A78a)}$$

$$\begin{aligned} &\exp\frac{\bar{t}}{\ell_\Lambda} = \sqrt{\frac{\cos\tilde{t}-\cos\tilde{r}}{\cos\tilde{t}+\cos\tilde{r}}}\,,\quad \sinh\frac{\bar{t}}{\ell_\Lambda} = \frac{s_{\mathcal{I}}\cos\tilde{r}}{\sqrt{\cos^2\tilde{t}-\cos^2\tilde{r}}}\,,\\ &\tanh\frac{\bar{t}}{\ell_\Lambda} = -\frac{\cos\tilde{r}}{\cos\tilde{t}}\,,\qquad \cosh\frac{\bar{t}}{\ell_\Lambda} = \frac{-s_{\mathcal{I}}\cos\tilde{t}}{\sqrt{\cos^2\tilde{t}-\cos^2\tilde{r}}}\,, \end{aligned}$$

$$\bar{r} = \frac{\ell_\Lambda}{2}\log\left(-\tan\frac{\tilde{t}+\tilde{r}}{2}\,\cot\frac{\tilde{t}-\tilde{r}}{2}\right), \quad \text{(A78b)}$$

$$\begin{aligned} &\exp\frac{\bar{r}}{\ell_\Lambda} = \sqrt{\frac{\sin\tilde{r}+\sin\tilde{t}}{\sin\tilde{r}-\sin\tilde{t}}}\,,\quad \left|\sinh\frac{\bar{r}}{\ell_\Lambda}\right| = \frac{\sin\tilde{t}}{\sqrt{\sin^2\tilde{r}-\sin^2\tilde{t}}}\,,\\ &\tanh\frac{\bar{r}}{\ell_\Lambda} = \frac{\sin\tilde{t}}{\sin\tilde{r}}\,,\qquad \cosh\frac{\bar{r}}{\ell_\Lambda} = \frac{|\sin\tilde{r}|}{\sqrt{\sin^2\tilde{r}-\sin^2\tilde{t}}}\,. \end{aligned}$$

Relation to flat cosmological family

$$\begin{aligned} \bar{t} &= \frac{\ell_\Lambda}{2}\log\frac{-\hat{t}^2+\hat{r}^2}{\ell_\Lambda^2} = -\frac{\ell_\Lambda}{2}\log\frac{-\check{t}^2+\check{r}^2}{\ell_\Lambda^2}\,,\\ \bar{r} &= \frac{\ell_\Lambda}{2}\log\frac{\hat{r}+\hat{t}}{\hat{r}-\hat{t}} = \frac{\ell_\Lambda}{2}\log\frac{\check{r}-\check{t}}{\check{r}+\check{t}}\,, \end{aligned} \quad \text{(A79)}$$

$$\begin{aligned} \frac{T}{\ell_\Lambda} &= \frac{1}{2}\log\frac{-\hat{t}^2+\hat{r}^2}{\ell_\Lambda^2} = -\frac{1}{2}\log\frac{-\check{t}^2+\check{r}^2}{\ell_\Lambda^2}\,,\\ \frac{R}{\ell_\Lambda} &= \frac{\hat{r}}{\hat{t}} = -\frac{\check{r}}{\check{t}}\,. \end{aligned} \quad \text{(A80)}$$

Relation to conformally Minkowski coordinates

$$\coth\frac{T}{\ell_\Lambda} = \frac{2\,\ell_\Lambda t}{\ell_\Lambda^2+t^2-r^2}\,,\quad \frac{R}{\ell_\Lambda} = \frac{2\,\ell_\Lambda r}{\ell_\Lambda^2+r^2-t^2}\,, \quad \text{(A81)}$$

$$\begin{aligned} \bar{t} &= \frac{\ell_\Lambda}{2}\log\left(-\frac{(\ell_\Lambda+t)^2-r^2}{(\ell_\Lambda-t)^2-r^2}\right),\\ \bar{r} &= \frac{\ell_\Lambda}{2}\log\left(-\frac{(\ell_\Lambda+r)^2-t^2}{(\ell_\Lambda-r)^2-t^2}\right), \end{aligned} \quad \text{(A82)}$$

7. The hyperbolic cosmological family

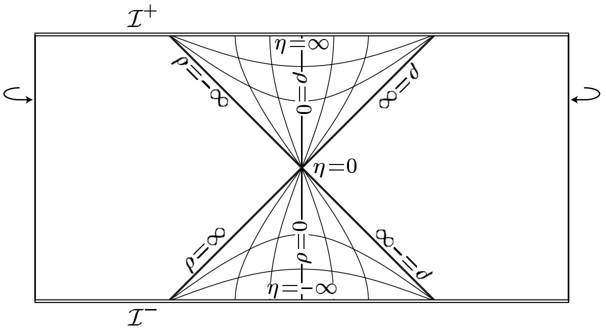

Figure 20: The hyperbolic cosmological family of coordinates.

The third type of cosmological coordinates are the *hyperbolic cosmological coordinates* η, ρ, ϑ, φ. The hypersurfaces $\eta = \text{constant}$ are homogeneous spaces with negative curvature, coordinate lines $\rho, \vartheta, \varphi = \text{constant}$ correspond to the worldlines of cosmological observers orthogonal to these slices, and the vector $\partial/\partial\eta$ is a timelike conformal Killing vector. The coordinates cover spacetime only partially—they can be introduced in two disconnected domains near the north pole, namely, in the past of the event $\tilde{t} = \pi/2$, $\tilde{r} = 0$ (where $\eta < 0$), and in the future of this event (where $\eta > 0$).

The metric

$$g = -\mathrm{d}\eta^2 + \sinh^2\tfrac{\eta}{\ell_\Lambda}\left(\mathrm{d}\rho^2 + \ell_\Lambda^2 \sinh^2\tfrac{\rho}{\ell_\Lambda}\,\mathrm{d}\omega^2\right) , \qquad \text{(A83)}$$

The ranges of coordinates and the signature factor $s_\mathcal{I}$ are

$$\eta \in \mathbb{R}^+ , \quad \rho \in \mathbb{R} , \quad s_\mathcal{I} = +1 \quad \text{in the future patch} ,$$
$$\eta \in \mathbb{R}^- , \quad \rho \in \mathbb{R} , \quad s_\mathcal{I} = -1 \quad \text{in the past patch} , \qquad \text{(A84)}$$

with negative values of radial coordinate ρ interpreted as described in Eq. (A2).

Orthonormal tetrad

$$\mathrm{e}_\eta = \frac{\partial}{\partial\eta} , \quad \mathrm{e}_\rho = \sinh^{-1}\frac{\eta}{\ell_\Lambda}\,\frac{\partial}{\partial\rho} , \qquad \text{(A85)}$$
$$\mathrm{e}_\vartheta = \sinh^{-1}\frac{\eta}{\ell_\Lambda}\,\sinh^{-1}\frac{\rho}{\ell_\Lambda}\,\frac{\partial}{\partial\rho} .$$

Relation to spherical cosmological family

$$\tanh\frac{\eta}{2\ell_\Lambda} = s_\mathcal{I}\sqrt{\frac{\cos\tilde{r} - \sin\tilde{t}}{\cos\tilde{r} + \sin\tilde{t}}} , \quad \tanh\frac{\rho}{\ell_\Lambda} = -\frac{\sin\tilde{r}}{\cos\tilde{t}} . \qquad \text{(A86)}$$

Relation to conformally Minkowski coordinates

$$\tanh\frac{\eta}{2\ell_\Lambda} = s_\mathcal{I}\frac{\sqrt{t^2 - r^2}}{\ell_\Lambda} , \quad \tanh\frac{\rho}{\ell_\Lambda} = \frac{r}{t} . \qquad \text{(A87)}$$

8. The accelerated coordinate family

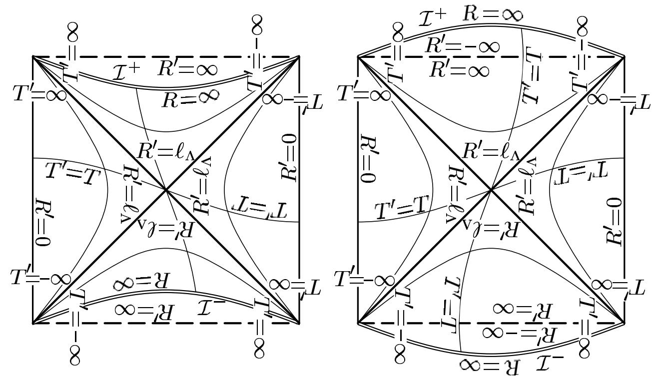

Figure 21: The accelerated family of coordinates.

This family consists of the *accelerated coordinates* T', R', ϑ', φ, and the *C-metric-like coordinates* τ, υ, ξ, φ (τ being different from τ of the standard coordinates). Contrary to the previous cases the accelerated coordinates are centered on uniformly accelerate origins: $R' = 0$ corresponds to two worldlines with acceleration $|a_\mathrm{o}|$. The transformation relations to the systems introduced above mix these three coordinates in general.

The accelerated coordinates are closely related to the static system. Their time coordinates coincide, $T'{=}T$, and coordinate lines $R', \vartheta', \varphi = \text{constant}$ are the same as those with $R, \vartheta, \varphi = \text{constant}$. Both coordinate systems are identical for $a_\mathrm{o} = 0$. Sections $T, T', \varphi{=}\text{constant}$ with $R, R' < \ell_\Lambda$ have geometry of 2-sphere with parallels and meridians given by the coordinate lines of the static coordinates R, ϑ. The lines of coordinates R', ϑ' are the deformed version of static ones, their poles are shifted along meridian $\vartheta = 0$ towards each other, cf. Fig. 11.

Two conformal diagrams of sections $\vartheta', \varphi = \text{constant}$ ($\vartheta' < \pi/2$ on the right, $\vartheta' > \pi/2$ on the left), adapted to the accelerated coordinates, are depicted in Fig. 21. The shape of the diagram varies with different values of ϑ'; indeed, the position of infinity is given by $R' = -\ell_\Lambda^2/R_\mathrm{o}\,\cos^{-1}\vartheta'$. See also Fig. 10 for sections $\vartheta' = 0, \pi$.

The C-metric-like coordinates rescale only the values of the accelerated coordinates and regularize the coordinate singularity $R' = \pm\infty$. de Sitter metric in these coordinates is a zero-mass limit of the C-metric (the metric describing accelerated black holes; see, e.g., [28, 29]).

Finally, we use four parameters a_o, α_o, R_o, b_o to parametrize the acceleration.They are related as follows:

$$\begin{aligned}
\sinh\alpha_\mathrm{o} &= \frac{R_\mathrm{o}}{\sqrt{\ell_\Lambda^2 - R_\mathrm{o}^2}} = \frac{b_\mathrm{o}^2 - \ell_\Lambda^2}{2\,\ell_\Lambda b_\mathrm{o}} = -a_\mathrm{o}\ell_\Lambda ,\\
\cosh\alpha_\mathrm{o} &= \frac{\ell_\Lambda}{\sqrt{\ell_\Lambda^2 - R_\mathrm{o}^2}} = \frac{b_\mathrm{o}^2 + \ell_\Lambda^2}{2\,\ell_\Lambda b_\mathrm{o}} = \sqrt{1 + a_\mathrm{o}^2\ell_\Lambda^2} ,\\
\tanh\alpha_\mathrm{o} &= \frac{R_\mathrm{o}}{\ell_\Lambda} = \frac{b_\mathrm{o}^2 - \ell_\Lambda^2}{b_\mathrm{o}^2 + \ell_\Lambda^2} = -\frac{a_\mathrm{o}\ell_\Lambda}{\sqrt{1 + a_\mathrm{o}^2\ell_\Lambda^2}} ,\\
\exp\alpha_\mathrm{o} &= \sqrt{\frac{\ell_\Lambda + R_\mathrm{o}}{\ell_\Lambda - R_\mathrm{o}}} = \frac{b_\mathrm{o}}{\ell_\Lambda} = \sqrt{1 + a_\mathrm{o}^2\ell_\Lambda^2} - a_\mathrm{o}\ell_\Lambda .
\end{aligned} \qquad \text{(A88)}$$

Metric and relation between coordinates

$$g=\Omega^2\Big[-\Big(1-\frac{R'^2}{\ell_\Lambda^2}\Big)\,\mathrm{d}T'^2+\Big(1-\frac{R'^2}{\ell_\Lambda^2}\Big)^{-1}\mathrm{d}R'^2+R'^2\,\mathrm{d}\omega^{2\prime}\Big], \tag{A89}$$

$$g=\mathfrak{r}^2\Big[-(\upsilon^2-1)\,\mathrm{d}\tau^2+\frac{1}{\upsilon^2-1}\,\mathrm{d}\upsilon^2+\frac{1}{1-\xi^2}\,\mathrm{d}\xi^2+(1-\xi^2)\,\mathrm{d}\varphi^2\Big], \tag{A90}$$

where

$$\mathrm{d}\omega^{2\prime}=\big(\mathrm{d}\vartheta'^2+\sin^2\vartheta'\,\mathrm{d}\varphi^2\big)\;, \tag{A91}$$

$$\Omega=\frac{\sqrt{1-R_o^2/\ell_\Lambda^2}}{1+(R'R_o/\ell_\Lambda^2)\cos\vartheta'}=\frac{\mathfrak{r}}{R'}=\frac{\mathfrak{r}\,\upsilon}{\ell_\Lambda}\;, \tag{A92}$$

$$\mathfrak{r}=\frac{\ell_\Lambda}{\upsilon\cosh\alpha_o-\xi\sinh\alpha_o}=\Omega\,R'=\Omega\,\frac{\ell_\Lambda}{\upsilon}\;. \tag{A93}$$

$$\tau=\frac{T'}{\ell_\Lambda}\qquad \upsilon=\frac{\ell_\Lambda}{R'}\qquad \xi=-\cos\vartheta'\;, \tag{A94}$$

Orthonormal tetrad

$$\begin{aligned}
\mathrm{e}_{T'}&=|\Omega|^{-1}\Big(1-\frac{R'^2}{\ell_\Lambda^2}\Big)^{-1/2}\frac{\partial}{\partial T'}=\frac{1}{\mathfrak{r}\sqrt{\upsilon^2-1}}\,\frac{\partial}{\partial T'}\;,\\
\mathrm{e}_{R'}&=|\Omega|^{-1}\Big(1-\frac{R'^2}{\ell_\Lambda^2}\Big)^{1/2}\frac{\partial}{\partial R'}=\frac{1}{\mathfrak{r}}\sqrt{\upsilon^2-1}\,\frac{\partial}{\partial R'}\;,\\
\mathrm{e}_{\vartheta'}&=\frac{1}{\Omega R'}\,\frac{\partial}{\partial\vartheta'}=\frac{1}{\mathfrak{r}}\,\frac{\partial}{\partial\vartheta'}\;.
\end{aligned} \tag{A95}$$

Relation to static coordinates

$$T=T'$$

$$\begin{aligned}
R\cos\vartheta&=\frac{R'\cos\vartheta'+R_o}{1+(R'R_o/\ell_\Lambda^2)\cos\vartheta'}\;,\\
R\sin\vartheta&=\frac{R'\sin\vartheta'\sqrt{1-\frac{R_o^2}{\ell_\Lambda^2}}}{1+(R'R_o/\ell_\Lambda^2)\cos\vartheta'}\;,\\
\frac{R^2}{\ell_\Lambda^2}&=1-\frac{\big(1-R'^2/\ell_\Lambda^2\big)\big(1-R_o^2/\ell_\Lambda^2\big)}{\big(1+(R'R_o/\ell_\Lambda^2)\cos\vartheta'\big)^2}\;,\\
\tan\vartheta&=\frac{R'\sin\vartheta'\sqrt{1-\frac{R_o^2}{\ell_\Lambda^2}}}{R'\cos\vartheta'+R_o}\;.
\end{aligned} \tag{A96}$$

The inverse relations have the same form with T, R, ϑ and T', R', ϑ' interchanged only and α_o replaced by $-\alpha_o$.

$$\Omega=\frac{\sqrt{1-R_o^2/\ell_\Lambda^2}}{1+(R'R_o/\ell_\Lambda^2)\cos\vartheta'}=\frac{1-(RR_o/\ell_\Lambda^2)\cos\vartheta}{\sqrt{1-R_o^2/\ell_\Lambda^2}}\;, \tag{A97}$$

$$\Big(1+\frac{R'R_o}{\ell_\Lambda^2}\cos\vartheta'\Big)\Big(1-\frac{RR_o}{\ell_\Lambda^2}\cos\vartheta\Big)=1-\frac{R_o^2}{\ell_\Lambda^2}\;, \tag{A98}$$

$$\frac{1-R'^2/\ell_\Lambda^2}{1+(R'R_o/\ell_\Lambda^2)\cos\vartheta'}=\frac{1-R^2/\ell_\Lambda^2}{1-(RR_o/\ell_\Lambda^2)\cos\vartheta}\;. \tag{A99}$$

Relation to Robinson-Trautman coordinates

$$\begin{aligned}
T'&=\mathfrak{u}\cosh\alpha_o-\frac{\ell_\Lambda}{2}\log\left|\frac{\ell_\Lambda-\mathfrak{r}\,(\sinh\alpha_o\cos\vartheta'+\cosh\alpha_o)}{\ell_\Lambda-\mathfrak{r}\,(\sinh\alpha_o\cos\vartheta'-\cosh\alpha_o)}\right|\;,\\
R'&=\frac{\mathfrak{r}\,\cosh\alpha_o}{1-(\mathfrak{r}/\ell_\Lambda)\sinh\alpha_o\cos\vartheta'}\;,\\
\Big|\tan\frac{\vartheta'}{2}\Big|&=\exp\Big(\psi-\frac{\mathfrak{u}}{\ell_\Lambda}\sinh\alpha_o\Big)\;,
\end{aligned} \tag{A100}$$

$$\begin{aligned}
\tau&=\frac{\mathfrak{u}}{\ell_\Lambda}\cosh\alpha_o-\frac{1}{2}\log\left|\frac{\ell_\Lambda-\mathfrak{r}\,(\sinh\alpha_o\cos\vartheta'+\cosh\alpha_o)}{\ell_\Lambda-\mathfrak{r}\,(\sinh\alpha_o\cos\vartheta'-\cosh\alpha_o)}\right|\;,\\
\upsilon&=\frac{\ell_\Lambda}{\mathfrak{r}\cosh\alpha_o}-\tanh\alpha_o\cos\vartheta'\;,\\
\xi&=\tanh\Big(\psi-\frac{\mathfrak{u}}{\ell_\Lambda}\sinh\alpha_o\Big)\;,
\end{aligned} \tag{A101}$$

where $\cos\vartheta'=-\xi$ is given in terms of the Robinson-Trautman coordinates by the last equation.

Relation to flat cosmological family

If we introduce the spherical coordinates $\check{t}'$, $\check{r}'$, ϑ', φ boosted with respect to the flat cosmological coordinates $\check{t}$, $\check{r}$, ϑ, φ by a boost α_o (in the sense of Minkowski space $\check{M}$), we find that the accelerated coordinates T', R' are related to $\hat{t}'$, $\hat{r}'$ in exactly the same way as the static coordinates T, R are related to the coordinates $\check{t}$, $\check{r}$. The boost $\check{t}'=\check{t}\cosh\alpha_o+\check{z}\sinh\alpha_o$, $\check{x}'=\check{x}$, $\check{y}'=\check{y}$, $\check{z}'=\check{t}\sinh\alpha_o+\check{z}\cosh\alpha_o$, rewritten in the spherical coordinates $\check{r}'\cos\vartheta'=\check{z}'$, $\check{r}'\sin\vartheta'=\sqrt{\check{x}'^2+\check{y}'^2}$, reads

$$\begin{aligned}
\check{t}'&=\check{t}\cosh\alpha_o+\check{r}\cos\vartheta\sinh\alpha_o\;,\\
\check{r}'\cos\vartheta'&=\check{t}\sinh\alpha_o+\check{r}\cos\vartheta\cosh\alpha_o\;,\\
\check{r}'\sin\vartheta'&=\check{r}\sin\vartheta\;,
\end{aligned} \tag{A102}$$

and relations analogous to Eqs. (A67) and (A80) are:

$$T'=-\frac{\ell_\Lambda}{2}\log\Big|\frac{\check{t}'^2-\check{r}'^2}{\ell_\Lambda^2}\Big|\;,\quad R'=-\ell_\Lambda\frac{\check{r}'}{\check{t}'}\;. \tag{A103}$$

Similarly, the formulas relating the accelerated coordinates to the coordinates $\hat{t}$, $\hat{r}$, ϑ are:

$$\begin{aligned}
\hat{t}'&=\hat{t}\cosh\alpha_o-\hat{r}\cos\vartheta\sinh\alpha_o\;,\\
\hat{r}'\cos\vartheta'&=-\hat{t}\sinh\alpha_o+\hat{r}\cos\vartheta\cosh\alpha_o\;,\\
\hat{r}'\sin\vartheta'&=\hat{r}\sin\vartheta\;,
\end{aligned} \tag{A104}$$

$$T'=\frac{\ell_\Lambda}{2}\log\Big|\frac{\hat{t}'^2-\hat{r}'^2}{\ell_\Lambda^2}\Big|\;,\quad R'=\ell_\Lambda\frac{\hat{r}'}{\hat{t}'}\;. \tag{A105}$$

The conformal factor takes the form

$$\Omega=\frac{\check{t}'}{\check{t}}=\frac{\hat{t}'}{\hat{t}}=\cosh\alpha_o-\frac{R}{\ell_\Lambda}\sinh\alpha_o\cos\vartheta\;. \tag{A106}$$

9. The Robinson-Trautman coordinates

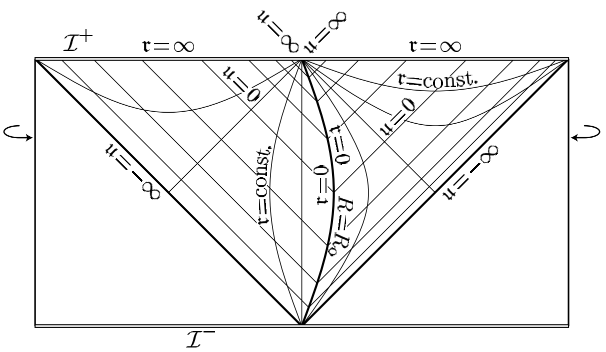

Figure 22: The Robinson-Trautman coordinates.

In the *Robinson-Trautman coordinates* $\mathfrak{u}$, $\mathfrak{r}$, ψ, φ (or in their complex version $\mathfrak{u}$, $\mathfrak{r}$, ζ, $\bar{\zeta}$), de Sitter metric takes the standard Robinson-Trautman form [50]. The coordinate $\mathfrak{u}$ is null, the "radial" coordinate $\mathfrak{r}$ is an affine parameter along coordinate lines $\mathfrak{u}, \psi, \varphi =$ constant. These lines are null geodesics generating light cones with vertices at the origin $\mathfrak{r} = 0$. The coordinates ψ, φ (or ζ, $\bar{\zeta}$) are angular coordinates, however, they are not functions of the accelerated angular coordinates ϑ', φ only (cf. Eq. (A112)). Because ϑ', φ have a clearer geometrical meaning, we list some formulas also in the mixed coordinate system $\mathfrak{u}$, $\mathfrak{r}$, ϑ', φ.

The origin $\mathfrak{r} = 0$ of the Robinson-Trautman coordinates is centered on the worldline of the uniformly accelerated observer moving with the acceleration $|a_o| = \left|\ell_\Lambda^{-1} \sinh\alpha_o\right|$. The coordinates are thus closely related to the accelerated coordinates.

The coordinates $\mathfrak{u}$, $\mathfrak{r}$, ψ, φ do not cover the whole spacetime smoothly. They can be introduced smoothly in the future of the north pole, or in the past of the south pole. At the boundary of these two domains, $\mathfrak{u} \to \pm\infty$.

Metric and relation between coordinates

$$g = -H\,\mathrm{d}\mathfrak{u}^2 - \mathrm{d}\mathfrak{u} \vee \mathrm{d}\mathfrak{r} + \frac{\mathfrak{r}^2}{P^2}\left(\mathrm{d}\psi^2 + \mathrm{d}\varphi^2\right) , \tag{A107}$$

$$g = -H\,\mathrm{d}\mathfrak{u}^2 - \mathrm{d}\mathfrak{u} \vee \mathrm{d}\mathfrak{r} + \frac{\mathfrak{r}^2}{P^2}\,\mathrm{d}\zeta \vee \mathrm{d}\bar{\zeta} , \tag{A108}$$

$$\begin{aligned} g = &-\cosh^2\!\alpha_o\,\frac{\mathfrak{r}^2}{\ell_\Lambda^2}\,(\upsilon^2 - 1)\,\mathrm{d}\mathfrak{u}^2 - \mathrm{d}\mathfrak{u} \vee \mathrm{d}\mathfrak{r} \\ &+ \cosh\alpha_o\,\frac{\mathfrak{r}^2}{\ell_\Lambda}\,\sin\vartheta'\,\mathrm{d}\mathfrak{u} \vee \mathrm{d}\vartheta' + \mathfrak{r}^2\left(\mathrm{d}\vartheta'^2 + \sin^2\!\vartheta'\,\mathrm{d}\varphi^2\right) , \end{aligned} \tag{A109}$$

$$\begin{aligned} H &= -\frac{\mathfrak{r}^2}{\ell_\Lambda^2} + 2\,\frac{\mathfrak{r}}{\ell_\Lambda}\,\sinh\alpha_o\,\tanh\!\Big(\psi - \frac{\mathfrak{u}}{\ell_\Lambda}\,\sinh\alpha_o\Big) + 1 \\ &= -\frac{\mathfrak{r}^2}{\ell_\Lambda^2} - 2\,\frac{\mathfrak{r}}{\ell_\Lambda}\,\sinh\alpha_o\,\cos\vartheta' + 1 , \end{aligned} \tag{A110}$$

$$P = \cosh\!\Big(\psi - \frac{\mathfrak{u}}{\ell_\Lambda}\,\sinh\alpha_o\Big) = \frac{1}{\sin\vartheta'} . \tag{A111}$$

$$\begin{aligned} \psi &= \frac{\mathfrak{u}}{\ell_\Lambda}\,\sinh\alpha_o + \log\left|\tan\frac{\vartheta'}{2}\right| , \\ \left|\tan\frac{\vartheta'}{2}\right| &= \exp\!\Big(\psi - \frac{\mathfrak{u}}{\ell_\Lambda}\,\sinh\alpha_o\Big) , \end{aligned} \tag{A112}$$

$$\begin{aligned} \zeta &= \frac{1}{\sqrt{2}}(\psi - i\varphi) , \quad \psi = \frac{1}{\sqrt{2}}(\zeta + \bar{\zeta}) , \\ \bar{\zeta} &= \frac{1}{\sqrt{2}}(\psi + i\varphi) , \quad \varphi = \frac{i}{\sqrt{2}}(\zeta - \bar{\zeta}) . \end{aligned} \tag{A113}$$

Null tetrad

Since the Robinson-Trautman coordinates are closely related to the congruence of null geodesics, it is convenient to introduce the null tetrad which is parallelly transported along these geodesics $\mathfrak{u}, \psi, \varphi =$ constant:

$$\begin{aligned} \mathrm{k}_{\mathrm{RT}} &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial\mathfrak{r}} , \quad \mathrm{l}_{\mathrm{RT}} = -\frac{1}{\sqrt{2}}H\,\frac{\partial}{\partial\mathfrak{r}} + \sqrt{2}\,\frac{\partial}{\partial\mathfrak{u}} , \\ \mathrm{m}_{\mathrm{RT}} &= \frac{1}{\sqrt{2}}\,\frac{P}{\mathfrak{r}}\left(\frac{\partial}{\partial\psi} - i\frac{\partial}{\partial\varphi}\right) , \\ \bar{\mathrm{m}}_{\mathrm{RT}} &= \frac{1}{\sqrt{2}}\,\frac{P}{\mathfrak{r}}\left(\frac{\partial}{\partial\psi} + i\frac{\partial}{\partial\varphi}\right) . \end{aligned} \tag{A114}$$

Relation to accelerated coordinate family

$$\begin{aligned} \mathfrak{r} &= \frac{R'\sqrt{1 - R_o^2/\ell_\Lambda^2}}{1 + (R'R_o/\ell_\Lambda^2)\cos\vartheta'} , \\ \mathfrak{u} &= \sqrt{1 - \frac{R_o^2}{\ell_\Lambda^2}}\left(T' + \frac{\ell_\Lambda}{2}\log\left|\frac{R' - \ell_\Lambda}{R' + \ell_\Lambda}\right|\right) , \\ \psi &= \frac{R_o}{\ell_\Lambda}\left(\frac{T'}{\ell_\Lambda} + \frac{1}{2}\log\left|\frac{R' - \ell_\Lambda}{R' + \ell_\Lambda}\right|\right) + \log\left|\tan\frac{\vartheta'}{2}\right| , \end{aligned} \tag{A115}$$

$$\begin{aligned} \mathfrak{r} &= \frac{\ell_\Lambda}{\upsilon\cosh\alpha_o - \xi\sinh\alpha_o} , \\ \mathfrak{u} &= \frac{\ell_\Lambda}{\cosh\alpha_o}\left(\tau + \frac{1}{2}\log\left|\frac{1 - \upsilon}{1 + \upsilon}\right|\right) , \\ \psi &= \tanh\alpha_o\left(\tau + \frac{1}{2}\log\left|\frac{1 - \upsilon}{1 + \upsilon}\right|\right) + \frac{1}{2}\log\left|\frac{1 + \xi}{1 - \xi}\right| . \end{aligned} \tag{A116}$$

Relation to static family

$$\mathfrak{r} = \frac{\ell_\Lambda}{\sqrt{1 - R_o^2/\ell_\Lambda^2}}\left[\Big(1 - \frac{RR_o}{\ell_\Lambda^2}\cos\vartheta\Big)^2 - \Big(1 - \frac{R^2}{\ell_\Lambda^2}\Big)\Big(1 - \frac{R_o^2}{\ell_\Lambda^2}\Big)\right]^{\frac{1}{2}} , \tag{A117}$$

$$\mathfrak{r}\sin\vartheta' = R\sin\vartheta , \quad \mathfrak{r}\cos\vartheta' = \frac{R\cos\vartheta - R_o}{\sqrt{1 - R_o^2/\ell_\Lambda^2}} . \tag{A118}$$

$$\begin{aligned} R\sin\vartheta &= \mathfrak{r}\sin\vartheta' , \\ R\cos\vartheta &= \mathfrak{r}\cos\vartheta'\sqrt{1 - R_o^2/\ell_\Lambda^2} + R_o . \end{aligned} \tag{A119}$$

10. The null family

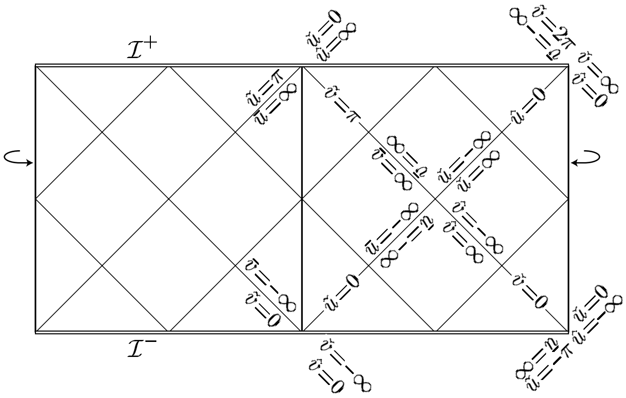

Figure 23: The null family of coordinates

Finally, we return back to the coordinate systems which employ standard coordinates ϑ, φ. Time and radial coordinates can be transformed into two null coordinates. Such null coordinates can be associated with most coordinate families introduced above. Coordinates $\tilde{u}$, $\tilde{v}$ are related to the standard coordinates; $\check{u}$, $\check{v}$ and $\hat{u}$, $\hat{v}$ to the flat cosmological coordinates; u, v to the conformally Minkowski; and $\bar{u}$, $\bar{v}$ to the static coordinates. Coordinate vectors $\{\partial/\partial\tilde{u},\, \partial/\partial\tilde{v}\}$, $\{\partial/\partial\check{u},\, \partial/\partial\check{v}\}$, etc., are the pairs of independent null vectors in the radial 2-slices $\vartheta, \varphi = \text{constant}$. We do not allow the radial coordinate to be negative in the definitions of null coordinates because this would interchange the meaning of u and v. The null coordinates are thus drawn in the right half of Fig. 23 only.

Metric and relation to other coordinates

$$g = \frac{\ell_\Lambda^2}{1-\cos(\tilde{u}+\tilde{v})}\Big(-\mathrm{d}\tilde{u}\vee\mathrm{d}\tilde{v} + \big(1-\cos(\tilde{u}-\tilde{v})\big)\,\mathrm{d}\omega^2\Big)\,, \tag{A120}$$

$$g = \frac{\ell_\Lambda^2}{(\hat{u}+\hat{v})^2}\Big(-2\,\mathrm{d}\hat{u}\vee\mathrm{d}\hat{v} + (\hat{u}-\hat{v})^2\,\mathrm{d}\omega^2\Big)\,, \tag{A121}$$

$$g = \frac{\ell_\Lambda^2}{(\check{u}+\check{v})^2}\Big(-2\,\mathrm{d}\check{u}\vee\mathrm{d}\check{v} + (\check{u}-\check{v})^2\,\mathrm{d}\omega^2\Big)\,, \tag{A122}$$

$$g = \Big(\frac{\ell_\Lambda^2}{\ell_\Lambda^2-uv}\Big)^2\Big(-2\,\mathrm{d}u\vee\mathrm{d}v + (u-v)^2\,\mathrm{d}\omega^2\Big)\,, \tag{A123}$$

$$\begin{aligned} g = {}&\Big(\exp\frac{\bar{u}}{\ell_\Lambda} + \exp\frac{\bar{v}}{\ell_\Lambda}\Big)^{-2} \\ &\times\bigg(-2\exp\frac{\bar{u}+\bar{v}}{\ell_\Lambda}\,\mathrm{d}\bar{u}\vee\mathrm{d}\bar{v} + \ell_\Lambda^2\Big(\exp\frac{\bar{u}}{\ell_\Lambda}-\exp\frac{\bar{v}}{\ell_\Lambda}\Big)^2\mathrm{d}\omega^2\bigg)\,. \end{aligned} \tag{A124}$$

The relation of time and radial coordinates $\overset{*}{t}$, $\overset{*}{r}$ to the corresponding null coordinates $\overset{*}{u}$, $\overset{*}{v}$ is given by usual formulas:

$$\begin{aligned} \overset{*}{t} &= \tfrac{1}{2}(\overset{*}{v}+\overset{*}{u})\,, & \overset{*}{u} &= \overset{*}{t}-\overset{*}{r}\,,\\ \overset{*}{r} &= \tfrac{1}{2}(\overset{*}{v}-\overset{*}{u})\,, & \overset{*}{v} &= \overset{*}{t}+\overset{*}{r}\,. \end{aligned} \tag{A125}$$

Here $\{\overset{*}{t},\, \overset{*}{r}\}$ stands for $\{\tilde{t},\, \tilde{r}\}$, $\{\check{t},\, \check{r}\}$, $\{\hat{t},\, \hat{r}\}$, $\{t,\, r\}$, and $\{\bar{t},\, \bar{r}\}$ respectively; similarly with $\{\overset{*}{u},\, \overset{*}{v}\}$.

Relation between null coordinates

The coordinates $\hat{u}$, $\hat{v}$, u, v, and $\check{u}$, $\check{v}$ can be viewed as null coordinates in the conformally related Minkowski spaces $\hat{M}$, M, and $\check{M}$; these are shifted with respect to each other by $\frac{\pi}{2}$ in the direction of the conformally Einstein time coordinate $\tilde{t}$, or associated null coordinates:

$$\begin{aligned} \frac{\hat{u}}{\ell_\Lambda} &= \tan\frac{\tilde{u}}{2}\,, & \frac{\hat{v}}{\ell_\Lambda} &= \tan\frac{\tilde{v}}{2}\,,\\ \frac{u}{\ell_\Lambda} &= \tan\Big(\frac{\tilde{u}}{2}-\frac{\pi}{4}\Big)\,, & \frac{v}{\ell_\Lambda} &= \tan\Big(\frac{\tilde{v}}{2}-\frac{\pi}{4}\Big)\,,\\ \frac{\check{u}}{\ell_\Lambda} &= \tan\Big(\frac{\tilde{u}}{2}-\frac{\pi}{2}\Big)\,, & \frac{\check{v}}{\ell_\Lambda} &= \tan\Big(\frac{\tilde{v}}{2}-\frac{\pi}{2}\Big)\,. \end{aligned} \tag{A126}$$

The remaining coordinates $\bar{u}$, $\bar{v}$ are related to the conformally Einstein null coordinates $\tilde{u}$, $\tilde{v}$ by the "compactification transformation":

$$\tan\frac{\tilde{u}}{2} = s_u\exp\frac{\bar{u}}{\ell_\Lambda}\,,\quad \tan\frac{\tilde{v}}{2} = s_v\exp\frac{\bar{v}}{\ell_\Lambda}\,. \tag{A127}$$

Here the sign factors s_u and s_v are given by

$$s_u = \operatorname{sign}\tan\frac{\tilde{u}}{2}\,,\quad s_v = \operatorname{sign}\tan\frac{\tilde{v}}{2}\,. \tag{A128}$$

Relations (A126), (A127) between null coordinates can also be rewritten as follows:

$$\tan\frac{\tilde{u}}{2} = s_u\exp\frac{\bar{u}}{\ell_\Lambda} = \frac{\hat{u}}{\ell_\Lambda} = -\frac{\ell_\Lambda}{\check{u}} = \frac{\ell_\Lambda+u}{\ell_\Lambda-u}\,, \tag{A129}$$

$$\tan\tilde{u} = -s_u\sinh^{-1}\frac{\bar{u}}{\ell_\Lambda} = \frac{2\hat{u}\ell_\Lambda}{\ell_\Lambda^2-\hat{u}^2} = \frac{2\check{u}\ell_\Lambda}{\ell_\Lambda^2-\check{u}^2} = \frac{u^2-\ell_\Lambda^2}{2u\ell_\Lambda}\,,$$

$$\sin\tilde{u} = s_u\cosh^{-1}\frac{\bar{u}}{\ell_\Lambda} = \frac{2\hat{u}\ell_\Lambda}{\ell_\Lambda^2+\hat{u}^2} = \frac{-2\check{u}\ell_\Lambda}{\ell_\Lambda^2+\check{u}^2} = \frac{\ell_\Lambda^2-u^2}{\ell_\Lambda^2+u^2}\,,$$

$$\cos\tilde{u} = -\tanh\frac{\bar{u}}{\ell_\Lambda} = \frac{\ell_\Lambda^2-\hat{u}^2}{\ell_\Lambda^2+\hat{u}^2} = \frac{\hat{u}^2-\ell_\Lambda^2}{\hat{u}^2+\ell_\Lambda^2} = \frac{-2u\ell_\Lambda}{\ell_\Lambda^2+u^2}\,,$$

$$\frac{\hat{u}}{\ell_\Lambda} = \tan\frac{\tilde{u}}{2} = s_u\exp\frac{\bar{u}}{\ell_\Lambda} = -\frac{\ell_\Lambda}{\check{u}} = \frac{\ell_\Lambda+u}{\ell_\Lambda-u}\,, \tag{A130}$$

$$-\frac{\check{u}}{\ell_\Lambda} = \cot\frac{\tilde{u}}{2} = s_u\exp\Big(-\frac{\bar{u}}{\ell_\Lambda}\Big) = \frac{\ell_\Lambda}{\hat{u}} = \frac{\ell_\Lambda-u}{\ell_\Lambda+u}\,, \tag{A131}$$

$$\begin{aligned} \frac{u}{\ell_\Lambda} &= -\frac{1-\sin\tilde{u}}{\cos\tilde{u}} = -\frac{\cos\tilde{u}}{1+\sin\tilde{u}}\\ &= \Big(\tanh\frac{\bar{u}}{2\ell_\Lambda}\Big)^{s_u} = \frac{\hat{u}-\ell_\Lambda}{\hat{u}+\ell_\Lambda} = \frac{\ell_\Lambda+\check{u}}{\ell_\Lambda-\check{u}}\,, \end{aligned} \tag{A132}$$

$$\begin{aligned} \frac{\bar{u}}{\ell_\Lambda} &= \log\Big|\tan\frac{\tilde{u}}{2}\Big| = \log\Big|\frac{\hat{u}}{\ell_\Lambda}\Big| = \log\Big|\frac{\ell_\Lambda}{\check{u}}\Big|\\ &= \log\Big|\frac{\ell_\Lambda+u}{\ell_\Lambda-u}\Big| = 2\operatorname{arctanh}\Big(\frac{u}{\ell_\Lambda}\Big)^{s_u}\,. \end{aligned} \tag{A133}$$

$$\hat{u}\check{u} = -\ell_\Lambda^2\,,\quad \frac{\hat{u}}{\ell_\Lambda}+\frac{\ell_\Lambda}{\check{u}} = 0\,. \tag{A134}$$

The same relations hold for coordinates v.