Special Topic Lectures UTF
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Part I
Surface gravity and temperature in dynamical spacetimes

1 Introduction

1.1 What is surface gravity?

- Newtonian picture: surface gravity is *the acceleration due to the force of gravity.* That is, for a large spherical body of mass M, the acceleration due to the force of gravity is
$$a = \frac{GM}{r^2}, \tag{1}$$
where G is the gravitational constant and r is the radius to the centre of mass. On Earth, this becomes the familiar $g \approx 9.81$ ms^{-1} of Newtonian mechanics.

- More broadly: surface gravity is the acceleration required to keep a point particle (of negligible mass) in place on a given surface. This is consistent when we have in mind a large astronomical body such as a planet but what of a (static) black hole?

- Relativistic picture: Instead of physical surface, we have the abstract 'surface' of an event horizon. This horizon is generated by the failure of null rays to reach infinity which obscures information to a distant observer. Crucially, however, the acceleration blows up as the radius r approaches zero. To get around this, we must introduce the concept of a Killing vector which we will turn to shortly.

General trajectory of course:

- Surface gravity and temperature for a black hole → Surface gravity and temperature for cosmology

- Mode decomposition in cosmology → Unruh-DeWitt particle detector model
- Combine and analyse the behaviour of a detector in cosmological spacetimes.

References

- *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* by Eric Poisson (Cambridge University Press, 2004)
- *Cosmological and Black Hole Apparent Horizons* by Valerio Faraoni (Springer 2015)
- *Dynamical surface gravity* by A. B. Nielsen and J. H. Yoon, (Classical Quantum Gravity 25, 085010, 2008) arXiv:0711.1445.
- *Unruh-DeWitt detectors in cosmological spacetimes* by Aindriú Conroy (Phys.Rev.D 105 (2022) 12, 123513) arXiv:2204.00359

1.2 Preliminaries

Some things to note before we begin:

1. A stationary (or static) spacetime allows for the existence of a timelike Killing vector χ^μ which, by definition, satisfies the Killing equation

$$\nabla_{(\mu}\chi_{\nu)} = \frac{1}{2}\left(\nabla_\mu\chi_\nu + \nabla_\nu\chi_\mu\right) = 0. \tag{2}$$

   Note: both static and stationary spacetimes are independent of the time coordinate t while static spacetime also have no rotation. For example, compare an object in orbit (stationary) with an object remaining in a fixed position (static).

2. In the region where χ^μ is timelike, the *norm* $\chi^\sigma\chi_\sigma < 0$, while a *Killing horizon* is formed on the surface where $\chi^\sigma\chi_\sigma = 0$, i.e. a Killing horizon is formed where timelike and null Killing vectors coincide.

3. On a null (hyper)surface, any null vector that is normal to a null surface is also tangent to it, see diagrams.

4. This implies that the *gradient* of the norm $\nabla_\mu(\chi^\sigma\chi_\sigma)$ will be directed along χ_μ, i.e. they are proportional to each other so that we can write

$$\nabla_\mu(\chi^\sigma\chi_\sigma) = -2\kappa\chi_\mu, \tag{3}$$

   where κ is a constant and the factor of -2 is for convenience with the benefit of hindsight.

2 Surface gravity on a Killing horizon

Before we turn to the open question of how to define surface gravity on a dynamical horizon, let us first review the situation in a static or stationary spacetime. The calculation proceeds as follows. We have already deduced that $\nabla_\mu(\chi^\sigma\chi_\sigma) = -2\kappa\chi_\mu$. Unpacking yields

$$\begin{aligned}(\nabla_\mu\chi^\sigma)\chi_\sigma + \chi^\sigma(\nabla_\mu\chi_\sigma) &= -2\kappa\chi_\mu,\\ \chi^\sigma\nabla_\mu\chi_\sigma &= -\kappa\chi_\mu.\end{aligned} \tag{4}$$

The Killing equation states that $\nabla_\mu\chi_\sigma = -\nabla_\sigma\chi_\mu$ so that

$$\chi^\sigma\nabla_\sigma\chi_\mu = -\kappa\chi_\mu. \tag{5}$$

Then

$$\kappa^2\chi_\mu\chi^\mu = (\chi^\sigma\nabla_\sigma\chi_\mu)(\chi^\lambda\nabla_\lambda\chi_\mu) \quad \text{i.e.} \quad \kappa^2 = V^{-2}(\chi^\sigma\nabla_\sigma\chi_\mu)(\chi^\lambda\nabla_\lambda\chi^\mu), \tag{6}$$

where $V = \sqrt{|\chi_\mu\chi^\mu|}$ is the *red-shift factor.* We can write this in terms of the *four-acceleration* $a_\mu = u^\sigma\nabla_\sigma u_\mu$ like so

$$\kappa^2 = V^2 a^\mu a_\mu, \quad \text{or} \quad \kappa|_{r=r_H} = V\cdot A|_{r=r_H}. \tag{7}$$

where $A = \sqrt{|a_\mu a^\mu|}$. One way of seeing this is by noting that $a_\mu = u^\sigma\nabla_\sigma u_\mu = \nabla_\mu \ln V$ (*Exercise*: see Appendix A) so that

$$\begin{aligned}a_\mu &= \nabla_\mu \ln V\\ &= \frac{\nabla_\mu V}{V},\\ &= V^{-1}\nabla_\mu\sqrt{\chi_\sigma\chi^\sigma}\\ &= \frac{1}{2}V^{-2}\nabla_\mu(\chi_\sigma\chi^\sigma)\\ &= -\kappa V^{-2}\chi_\mu\end{aligned} \tag{8}$$

from which we obtain

$$A^2 = \frac{\kappa^2}{V^2} \quad \implies \quad \kappa|_{r=r_H} = V\cdot A|_{r=r_H}. \tag{9}$$

Interpretation:

- We interpret A as the locally-applied force required to hold a particle in position at some radius r.
- This quantity diverges on the horizon r_H which in the case of a static black hole is the event horizon.
- The redshift factor serves to shift the application of this force to infinity so that the interpretation of κ is the *gravitational force (acceleration) that must be applied in order to hold a particle in place near the horizon (i.e. the surface gravity), where this force is not locally-applied but applied at infinity.*
- This ensures that the surface gravity κ is regular when evaluated on the horizon while also demonstrating the non-local nature intrinsic to this definition, Refs. [Poisson, Nielsen, Faraoni].

3 Surface gravity on a dynamical horizon

3.1 Hayward-Kodama surface gravity

As dynamical spacetimes don't allow for timelike Killing vectors, we need an alternative approach. One such approach is the Hayward-Kodama prescription which is applicable to spherically symmetric, dynamical spacetimes and, in particular, FLRW spacetimes ,

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{10}$$

which we write as $ds^2 = \gamma_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_2^2$, where $\tilde{r} = a(t)r$ is the *areal radius* with indices $a, b \in \{r, t\}$.

<u>Methodology:</u>

- First, define the Kodama vector

$$k^a \equiv \epsilon^{ab}\nabla_b\tilde{r}, \tag{11}$$

  where ϵ^{ab} is the $(1+1)-$dimensional Levi-Civita tensor with $\epsilon^{00} = \epsilon^{11} = 0$ and $\epsilon^{01} = 1 = -\epsilon^{10}$.

- One can verify that the divergence $\nabla_a k^a$ vanishes. As this is nothing more than the expansion tensor $\theta = \nabla_a k^a$, we expect, from the point of view of a Kodama observer that the background will appear not to expand and the areal radius will be a constant, i.e.

$$r \stackrel{?}{=} \frac{K}{a} \tag{12}$$

  will be the radial coordinate for some constant K.

- Taking a lead from the Killing vector case, we write

$$k^c\nabla_a k_c = \kappa_{HK}k_a \tag{13}$$

  where k_a and the gradient $\nabla_a(k^c k_c)$ are both normal to some null surface (e.g. a trapping surface or apparent horizon) analogous to the null hypersurface in the stationary example. We have a + sign rather than a − sign due to the cosmological setting where the direction is reversed, i.e. moving away from the singularity.

- In place of the Killing equation, we have the amended form

$$k^a(\nabla_a k_b + \nabla_b k_a) = 8\pi G\tilde{r}\psi_b, \tag{14}$$

  where ψ_b is the *energy flux vector* which tracks the deviation of the Kod. vector from the Kill. vector.

- We can, however, set $\psi_b = 0$ which ensures that the Kod. vector conforms to the Killing equation. Indeed. we must do this to ensure κ_{HK} is uniquely defined. The consequence of this is that the Kod. trajectory is no longer geodesic and requires some acceleration. From Eq. 13, we can write

$$\begin{aligned} \frac{1}{2}k^c\left(\nabla_a k_c + \nabla_a k_c\right) &= \kappa_{HK} k_a \\ \frac{1}{2}k^c\left(\nabla_a k_c - \nabla_c k_a\right) &= \kappa_{HK} k_a \\ \frac{1}{2}g^{ab}k^c\left(\nabla_a k_c - \nabla_c k_a\right) &= \kappa_{HK} k^b \end{aligned} \tag{15}$$

- Again, we can express this as

$$\kappa_{HK} = V_k \cdot A, \quad \text{where} \quad V_k = \sqrt{|k_c k^c|} \tag{16}$$

  and $a^a = u^c \nabla_c u^a = V_k^{-2} k^c \nabla_c k^a$ meaning that our prior interpretation of surface gravity is retained. *Exercise*.

- By decomposing into $ds^2 = \gamma_{ab} dx^a dx^b + \tilde{r}^2 d\Omega_2^2$, we can write κ_{HK} in the covariant form (*Exercise*)

$$\kappa_{HK} = -\frac{1}{2}\Box_\gamma \tilde{r} = -\frac{1}{2}\frac{1}{\sqrt{-\gamma}}\partial_a\left(\sqrt{-\gamma}\gamma^{ab}\partial_b\tilde{r}\right) \tag{17}$$

- *The Kodama Miracle.* Due to the divergence-free nature of the Kod. vector, i.e. $\nabla_a k^a = 0$, one can define a current $J^a \equiv G^{ab}k_b$ which is *covariantly conserved*, i.e. $\nabla_a J^a = 0$. This allows us to define physical quantities such as the four acceleration or surface gravity in a meaningful way.

- In an FLRW spacetime, we compute the surface gravity to be

$$\kappa_{HK} = \tilde{r}\left(H^2 + \frac{1}{2}\dot{H}\right), \tag{18}$$

  on some surface $\tilde{r}$. Question: without an event horizon, upon which surface should we evaluate this expression?

4 Cosmological horizons

For the purposes here, we work within a geometrically-flat FLRW universe with line element

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\Omega^2\right), \tag{19}$$

and define the *expansion* via the divergence

$$\theta \equiv \nabla_\mu n^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}n^\mu\right), \tag{20}$$

for some null ray n^μ. By appealing to the geodesic equation for null tangent vectors, see Appendix B. we find the ingoing null ray n^μ, and its associated outgoing ray l^μ, to be given by

$$n^\mu = \left(\frac{1}{a}, -\frac{1}{a^2}, 0, 0\right), \quad l^\mu = \left(\frac{1}{a}, \frac{1}{a^2}, 0, 0\right), \tag{21}$$

where we have used the fact that the determinant of metric is given by $g = \det g_{\mu\nu} = -a^6 r^4 \sin^2\phi$. The ingoing and outgoing expansions are then

$$\theta_{IN} = \frac{2}{a}\left(H - \frac{1}{\tilde{r}}\right), \quad \theta_{OUT} = \frac{2}{a}\left(H + \frac{1}{\tilde{r}}\right), \tag{22}$$

where $\tilde{r} \equiv ar$ is the areal radius.

Some terminology.

- An *apparent horizon* is defined by the locus of vanishing expansion of a null geodesic congruence, Ref. [Faraoni].
- Here, we consider the horizon which is formed when the *ingoing* expansion vanishes while the outgoing expansion remains positive. This is the *past-inner trapping horizon* of an *expanding cosmology* which we call the *cosmological apparent horizon* and it forms the boundary of the minimally anti-trapped surface, i.e. the anti-trapped surface of minimal size.
- This is not to be confused with the particle horizon, which is the maximum distance a particle can travel along a geodesic in proper conformal time, i.e.

  $$r_{PH} = \int_0^t \frac{dt'}{a(t')}, \tag{23}$$

  which (as we will see in Lecture 2) is related to conformal time η. As such, setting the ingoing expansion θ_{IN} to zero yields an apparent horizon with areal radius

  $$\tilde{r}_{AH} = H^{-1} \quad \text{where} \quad \tilde{r}_{AH} \equiv a r_{AH}. \tag{24}$$

- From Eq. (22), we can observe that when $\tilde{r} > \tilde{r}_{AH}$ both the ingoing and outgoing expansions are positive, i.e. $\theta_{IN,OUT} > 0$. The surfaces described by the expansion in this region are called *anti-trapped*, while surfaces in the region $0 \leq \tilde{r} < \tilde{r}_{AH}$, with $\theta_{OUT} > 0$ and $\theta_{IN} < 0$, are called *normal surfaces. Trapped surfaces* occur when $\theta_{IN,OUT} < 0$ and (by the Hawking-Penrose singularity theorems) lead to the formation of a singularity.
- In simple terms, outgoing geodesics in the normal region trace out a surface of larger area while ingoing geodesics trace out a shrinking surface with this being the familiar behaviour in flat space. We visualise the cosmological apparent horizon which forms the border between the normal and anti-trapped regions by considering an observer centred on a sphere, which we have positioned at $r = 0$. Events beyond the sphere are causally disconnected from our observer, meaning information is obscured, see Ref. [Faraoni] for a more detailed discussion on cosmological horizons.

Evaluating the surface gravity on the cosmological apparent horizon then leads to

$$\kappa_{HK} = \tilde{r}\left(H^2 + \frac{1}{2}\dot{H}\right) \quad \Longrightarrow \quad \kappa_{HK}|_{\tilde{r}=1/H} = \frac{1}{H}\left(H^2 + \frac{1}{2}\dot{H}\right), \tag{25}$$

which is sometime written in terms of the apparent horizon $\tilde{r}_{AH} = 1/H$ and its derivative $\dot{\tilde{r}}_{AH} = -\tilde{r}^3_{AH} H\dot{H}$ like so

$$\kappa_{HK}|_{\tilde{r}=\tilde{r}_{AH}} = \frac{1}{\tilde{r}}\left(H - \frac{1}{2}\frac{\dot{\tilde{r}}_{AH}}{\tilde{r}_{AH}}\right) \quad \text{with} \quad T = \frac{\kappa_{HK}}{2\pi}. \tag{26}$$

We now have a working definition of temperature on the cosmological apparent horizon.

4.1 Kodama trajectory

Here we compute the Kodama trajectory. First, decompose the metric like so

$$ds^2 = \gamma_{ab}dx^a dx^b + \tilde{r}^2 d\Omega^2, \tag{27}$$

so that the Kod. vector develops like so

$$\begin{aligned} k^a &= \epsilon^{ab}_{\perp}\nabla_b \tilde{r} \\ &= \epsilon^{ab}_{\perp}\left(\delta^t_b \dot{a}r + \delta^r_b a\right) \\ &= \sqrt{-\gamma}(\gamma^{at}\gamma^{br} - \gamma^{ar}\gamma^{bt})\left(\delta^t_b \dot{a}r + \delta^r_b a\right) \\ &= \sqrt{-\gamma}\left(\gamma^{at}\gamma^{tr}\dot{a}r + \gamma^{at}\gamma^{rr}a - \gamma^{ar}\gamma^{tt}\dot{a}r - \gamma^{ar}\gamma^{rt}a\right). \end{aligned} \tag{28}$$

If we further restrict our metric to be isotropic, i.e. where off-diagonal terms vanish, we find the Kodama vector to be given by

$$\begin{aligned} k^a &= \sqrt{-\gamma}\left(a\gamma^{at}\gamma^{rr} - \dot{a}r\gamma^{ar}\gamma^{tt}\right). \\ &= \sqrt{-\gamma}\left(a\delta^a_t\gamma^{tt}\gamma^{rr} - \dot{a}r\delta^a_r\gamma^{rr}\gamma^{tt}\right) \end{aligned} \tag{29}$$

For an FLRW metric, $ds^2_\gamma = -dt^2 + a^2(t)dr^2$ we have $\sqrt{-\gamma} = a$ and

$$k^a = -\delta^a_t + \delta^a_r Hr = (-1, Hr). \tag{30}$$

That is, $k^0 = -1$, $k^1 = Hr$ and so $k^c k_c = \gamma_{00}k^0k^0 + g_{11}k^1k^1 = -1 + a^2H^2r^2 = -1 + H^2\tilde{r}^2$, i.e. in terms of the apparent horizon $\tilde{r}_{AH} = 1/H$ we can write

$$k^c k_c = -1 + (r/r_{AH})^2. \tag{31}$$

In this form, it is clear to see that the Kodama vector does indeed mimic the Killing vector in that it becomes null on the surface of the apparent horizon r_{AH} and is timelike in the region $r < r_{AH}$. In the region where it is timelike, the Kodama vector evokes a class of preferred observers with four-velocity $u^a \equiv k^a/V_k$, given by

$$u^a = \frac{1}{\sqrt{1-\dot{a}^2r^2}}(-1, Hr). \tag{32}$$

Let's now compute the trajectories starting with the radial trajectory which is related to the time trajectory like so

$$\frac{dr}{d\tau} = -Hr\frac{dt}{d\tau}. \tag{33}$$

Next note that $H = \frac{1}{a}\frac{da}{dt} = \frac{1}{a}\frac{d\tau}{dt}\frac{da}{d\tau} = \frac{d\tau}{dt}H(\tau)$ so that we can write

$$\frac{r'(\tau)}{r(\tau)} = -\frac{a'(\tau)}{a(\tau)} \qquad \Longrightarrow \qquad \int d\tau \frac{r'(\tau)}{r(\tau)} = -\int d\tau \frac{a'(\tau)}{a(\tau)}, \tag{34}$$

which we solve to find

$$\ln(r/K) = -\ln(a) \qquad \Longrightarrow \qquad r(\tau) = \frac{K}{a} \tag{35}$$

for some constant K. This agrees with our earlier intuition of a constant areal radius for a Kod. observer. Thus

$$\frac{dt}{d\tau} = -\frac{1}{\sqrt{1-\dot{a}^2r^2}} \qquad \Longrightarrow \qquad \left(\frac{dt}{d\tau}\right)^2\left[1 - H^2K^2\right] = 1 \tag{36}$$

i.e.

$$\left(\frac{dt}{d\tau}\right)^2 = 1 + H^2(\tau)K^2 \tag{37}$$

Defining $V(\tau) = 1 + H^2(\tau)K^2$, we arrive at the trajectories

$$t(\tau) = \int \sqrt{V(\tau)}d\tau, \quad r(\tau) = \frac{K}{a(\tau)}. \tag{38}$$

We will return to these trajectories later in the course when considering an Unruh-DeWitt particle detector traveling through a cosmological spacetime.

4.2 Unified first law of thermodynamics

Claim: The Hayward-Kodama prescription is consistent with a *unified first law of thermodynamics:*

$$dE = TdS + WdV, \tag{39}$$

where

- E =total energy, temperature $T = \kappa/2\pi$, entropy $S = \text{Area}/4G$, work density $W = \frac{1}{2}(\rho - p)$,and V is the volume of the apparent horizon.

A Exercise 1

Exercise: Show that $a_\mu = u^\sigma \nabla_\sigma u_\mu = \nabla_\mu \ln V$. (tip: also show that $a_\mu u^\mu = 0$). Consider

$$\begin{aligned}
\nabla_\mu \ln V &= \frac{\nabla_\mu V}{V}, \\
&= V^{-1} \nabla_\mu \sqrt{\chi_\sigma \chi^\sigma} \\
&= -\frac{1}{2V^2} \nabla_\mu (\chi_\sigma \chi^\sigma) \\
&= -\frac{1}{V^2} \chi^\sigma \nabla_\mu \chi_\sigma \\
&= \frac{1}{V^2} \chi^\sigma \nabla_\sigma \chi_\mu \\
&= \frac{1}{V} u^\sigma \nabla_\sigma (V u_\mu) \\
&= \frac{1}{V} u^\sigma \nabla_\sigma V u_\mu + u^\sigma \nabla_\sigma u_\mu \\
&= u_\mu u^\sigma \nabla_\sigma \ln V + a_\mu
\end{aligned}$$

Contract with u^μ so that

$$u^\mu \nabla_\mu \ln V = -u^\sigma \nabla_\sigma \ln V \quad \Longrightarrow \quad u^\mu \nabla_\mu \ln V = 0$$

where

$$\begin{aligned}
a_\mu u^\mu &= u^\lambda \nabla_\lambda u_\mu u^\mu \\
&= \frac{1}{2} \left(u^\lambda \nabla_\lambda u_\mu u^\mu + u^\lambda \nabla_\lambda u_\mu u^\mu \right) \\
&= \frac{1}{2} \left(u^\lambda \nabla_\lambda (u_\mu u^\mu) - u^\lambda u_\mu \nabla_\lambda u^\mu + u^\lambda \nabla_\lambda u_\mu u^\mu \right) \\
&= 0
\end{aligned}$$

Thus

$$a_\mu = \nabla_\mu \ln V = u^\sigma \nabla_\sigma u_\mu$$

B Expansion tensor FLRW

Derive the expansion tensor for an FLRW metric with line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{40}$$

Without loss of generality, we cane restrict the trajectory to $x^\mu = (t(\lambda), x(\lambda), 0, 0)$ due to the isotropic nature of the spacetime. From the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0, \tag{41}$$

we read off

$$\frac{d^2t}{d\lambda^2} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \quad \text{and} \quad \frac{d^2x}{d\lambda^2} + \Gamma^1_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0. \tag{42}$$

The non-vanishing Christoffel symbols are

$$\Gamma^0_{ij} = g_{ij}\frac{\dot{a}}{a} \quad \text{and} \quad \Gamma^i_{0j} = \Gamma^i_{j0} = \delta^i_j\frac{\dot{a}}{a}. \tag{43}$$

Thus

$$\frac{d^2t}{d\lambda^2} + \dot{a}a\left(\frac{dx}{d\lambda}\right)^2 = 0 \quad \text{and} \quad \frac{d^2x}{d\lambda^2} + \Gamma^1_{01}\frac{dt}{d\lambda}\frac{dx}{d\lambda} = 0. \tag{44}$$

Restricting the line element to null rays with $ds^2|_{null} = 0$ implies $dt^2 = a^2(t)dx^2$,i.e.

$$\frac{dt}{d\lambda} = a(t)\frac{dx}{d\lambda} \qquad \Longrightarrow \qquad \frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0. \tag{45}$$

Next note that from the chain rule we have

$$\frac{\dot{a}}{a} = \frac{da}{dt}\frac{1}{a} = \frac{d\lambda}{dt}\frac{da}{d\lambda}\frac{1}{a} \tag{46}$$

so that

$$t''(\lambda) + \frac{a'(\lambda)}{a(\lambda)}t'(\lambda) = 0 \qquad \Longrightarrow \qquad \int \frac{t''(\lambda)}{t'(\lambda)}d\lambda = -\int \frac{a'(\lambda)}{a(\lambda)}d\lambda \tag{47}$$

which we solve to find

$$\ln(t'/C) = -\ln a = \ln(1/a) \quad \text{i.e.} \quad \frac{dt}{d\lambda} = \frac{C}{a}. \tag{48}$$

Thus

$$\left(\frac{dt}{d\lambda}, \frac{dx}{d\lambda}\right) = \left(\frac{C}{a}, \frac{C}{a^2}\right). \tag{49}$$

More generally, we note that $dt^2 = a^2(t)dx^2$ implies $\frac{dt}{d\lambda} = \pm a(t)\frac{dx}{d\lambda}$ so that for a general trajectory $k^\mu = (k^0, k^i)$ we can write

$$k^\mu = \left(\frac{1}{a}, \pm\frac{1}{a^2}\right) = (k^0, k^i), \tag{50}$$

where the sign attached to the spatial vectors indicates whether it is an ingoing $(-)$ or outgoing $(+)$ null tangent vector and we have set the integration constant $C = 1$.