FACULTY
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Selected Topics in AdS/CFT Correspondence

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Abstract

This is a study text for the "Selected topics in AdS/CFT correspondence" course taught at Charles University in 2022/23. The text is based on a number of sources stated below, as well as builds on similar courses delivered by Andrei Starinets and Veronika Hubeny.

- Sources
  - Gauge/gravity duality: Foundations and applications, M. Ammon and J. Erdmenger, Cambridge University Press, 2015.
  - Holographic entanglement entropy: An overview, T. Nishioka, S. Ryu, T. Takayanagi, J. Phys. A 42 (2009) 504008; ArXiv:0905.0932.
  - The entropy of Hawking radiation, A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, A. Tajdini, Rev. Mod. Phys. 93 (2021) 3, 035002; ArXiv:2006.6872.

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Chapter 1: Prelude: Black Hole Thermodynamics

1.1 Black Hole Thermodynamics

Motivation

Let us study some properties of the Schwarzschild black hole:

$$\boxed{ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,,} \qquad \boxed{f = 1 - \frac{2m}{r}\,,} \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2\,. \tag{1.1}$$

- Asymptotic mass (conserved energy)

Noether's theorem : symmetry $\leftrightarrow$ conserved quantity

described by Killing fields

$$\boxed{\nabla_{(\mu} k_{\nu)} = \nabla_\mu k_\nu + \nabla_\nu k_\mu = 0}$$

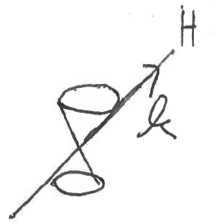
k^μ geometry "does not change"

Spec: $k = \partial t$ $\uparrow\uparrow^t$ spacetime is static $\Rightarrow$ energy is conserved

This leads to the following prescription for the *Komar mass*:

$$\boxed{M = -\frac{1}{8\pi}\int_{S^2_\infty} *dk = m\,.} \tag{1.2}$$

- Surface gravity. The black hole horizon($r = r_+ = 2M$) is a *Killing horizon*: a null surface generated by Killing field $k = \partial_t$.



$$k^\mu \nabla_\mu k^\nu |_H = \underbrace{\kappa}_{\text{surface gravity}} k^\nu |_H$$

It can be shown that κ reads

$$\kappa = \frac{f'(r_+)}{2} = \frac{1}{2}\frac{2M}{r_+^2} = \frac{1}{4M} = \frac{1}{2r_+}\,. \tag{1.3}$$

Note that this 'coincides' with Newtonian acceleration evaluated on the black hole horizon:

$$\kappa = \frac{M}{r_+^2} = \frac{M}{(2M)^2} = \frac{1}{4M}\,. \tag{1.4}$$

- Horizon area. Taking the $dt = 0 = dr$, the induced spatial metric 'on the horizon' is $d\gamma^2 = r_+^2 d\Omega^2$. The area then reads

$$A = \int \sqrt{\det \gamma} d\theta d\varphi = \int r_+^2 \sin\theta d\theta d\varphi = 4\pi r_+^2\,. \tag{1.5}$$

- Observation: Calculating the following differentials:

$$dM = \frac{dr_+}{2}\,, \quad dA = 8\pi r_+ dr_+\,, \tag{1.6}$$

we find that

$$\boxed{dM = \frac{\kappa}{2\pi}\frac{dA}{4}\,.} \tag{1.7}$$

Laws of Black Hole Mechanics

Bardeen, Carter and Hawking (1973) proved the following 4 laws of black hole mechanics. For a stationary, charged, and rotating black hole with mass M, angular momentum J, and charge Q, we have:

- **Zeroth law:** The surface gravity κ is constant on the black hole horizon.
- **First law:**

$$\boxed{dM = \frac{\kappa}{2\pi}\frac{dA}{4} + \underbrace{\Omega dJ + \Phi dQ}_{\text{work terms}}\,.} \tag{1.8}$$

Here, Ω is the angular velocity of the black hole horizon, and Φ is its 'electrostatic potential'.

- **Second law:** Classically, the area of the horizon never decreases (provided the null energy condition holds).
$$\boxed{dA \geq 0\,.} \tag{1.9}$$
- **Third law:** It is impossible to reduce κ to zero in a finite number of steps.

We would like to compare these to the laws of thermodynamics. In particular, the first law to
$$dE = TdS + \text{work terms}\,. \tag{1.10}$$
However, there is a problem: Classical black holes act as ultimate sponges: no heat can flow out, they are at absolute zero temperature. So we cannot have $\kappa \propto T$.

Black Hole Thermodynamics

- Wheeler's cup of tea: "If you throw a cup of tea to the black hole, where did its entropy go?" Based on analyzing this question Bekenstein proposed
$$\boxed{S \propto A\,.} \tag{1.11}$$
- Hawking 1974. When quantum effects are taken into account, black holes radiate away as black body with (upon restoring the fundamental units)
$$\boxed{T = \frac{\kappa}{2\pi}\frac{\hbar c^3}{k_B}\,, \quad S = \frac{A}{4}\frac{c^3 k_B}{\hbar G_N}\,.} \tag{1.12}$$
Note that the latter can also be written as
$$S = \frac{k_B}{4}\frac{A}{l_P^2}\,, \quad l_P = \sqrt{\frac{G\hbar}{c^3}}\,, \tag{1.13}$$
that is, it is given by measuring the area of the black hole horizon in Planck units – a huge entropy.

  Hawking's derivation was based on using the QFT in curved space (approximation of fixed background metric). Hawking basically showed "stimulated emission". The problem with his derivation is that due to the bluehift near the horizon, the test field approximation breaks down and we cannot really trust the result. However, since then the same result has been reproduced by many other approaches, e.g: Euclidean path integral, tunneling, string theory, LQG.

Euclidean Trick

We shall use the following two facts:

- Thermal Green functions have periodicity in imaginary Euclidean time $\boxed{\tau = it}$:

$$\boxed{G(\tau) = G(\tau + \beta)\,, \quad \beta = 1/T\,.} \tag{1.14}$$

Conversely, periodicity of G defines a thermal state. (Of course, other independent definitions of thermality can be given.) Green functions of quantum fields in the vicinity of black holes have this property (as seen by a static observer). What about gravitational field itself?

- Partition function. One can calculate the gravitational partition function at temperature T in the WKB (semiclassical) approximation as

$$\boxed{Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}\,,} \tag{1.15}$$

where g_c stands for the metric(s) describing the classical solution(s), and the integral is over all metrics periodic in imaginary time with period $\beta = 1/T$. Note that the Euclidean action S_E consists of three kinds of terms: the Einstein–Hilbert action, the York–Gibbons–Hawking term, and counter terms:

$$S_E = \int_\Omega \frac{d^4x\sqrt{g}R}{16\pi G} + \int_{\partial\Omega} \frac{d^3x\epsilon\sqrt{h}\mathcal{K}}{8\pi G} + \text{counter terms}\,, \tag{1.16}$$

where $\epsilon = -1$ for spacelike and $\epsilon = 1$ for timelike boundary. The second (York–Gibbons–Hawking) term is needed to ensure well-posed variational principle (it kills the unwanted boundary terms in the case of a compact manifold), while the third one is used to 'tune the value' of the action to make it finite and vanishing for the flat space. Here $\mathcal{K}$ stands for the extrinsic curvature and the second and third integrals are boundary integrals (with h_{ab} a boundary metric). Once the partition function is determined, we can calculate the free energy

$$F = -\frac{1}{\beta}\log Z \approx \frac{S_E}{\beta}\,, \tag{1.17}$$

which knows everything about thermodynamics. In particular, the entropy is given by

$$S = -\frac{\partial F}{\partial T}\,. \tag{1.18}$$

Let us explicitly demonstrate this procedure for the Schwarzschild black hole.

Schwarzschild black hole

- The Euclideanized Schwarzschild solution ($\tau = it$) is

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,. \tag{1.19}$$

Near the horizon we may expand

$$f = \underbrace{f(r_+)}_{0} + \underbrace{(r - r_+)}_{\Delta r} \underbrace{f'(r_+)}_{2\kappa} + \cdots = 2\kappa \Delta r\,. \tag{1.20}$$

Therefore, the near horizon limit of the 'Euclidean Schwarzschild solution' takes the following form:

$$ds^2 = 2\kappa \Delta r d\tau^2 + \frac{dr^2}{2\kappa \Delta r} + r_+^2 d\Omega^2\,. \tag{1.21}$$

We can now introduce a new coordinate ρ by

$$d\rho^2 = \frac{dr^2}{2\kappa\Delta r} \quad \Leftrightarrow \quad d\rho = \frac{dr}{\sqrt{2\kappa \Delta r}} \quad \Leftrightarrow \quad \Delta r = \frac{\kappa}{2}\rho^2\,, \tag{1.22}$$

getting

$$ds^2 = \kappa^2 \rho^2 d\tau^2 + d\rho^2 + r_+^2 d\Omega^2 = \rho^2 d\varphi^2 + d\rho^2 + \dots\,. \tag{1.23}$$

upon introducing a new angle coordinate, $\varphi = \kappa\tau$. This looks like a flat space written in polar coordinates, provided the angle φ has a period 2π, otherwise there is a conical singularity at $\rho = 0$, which corresponds to the original black hole horizon. The reasoning now goes as follows: since the black hole horizon was originally non-singular, we expect it to be non-singular again (otherwise we no longer solve vacuum Einstein equations). This is achieved by setting (we want to avoid conical singularity)

$$\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \tau \sim \tau + \underbrace{2\pi/\kappa}_{\beta} \quad \Leftrightarrow \quad \boxed{T = \frac{\kappa}{2\pi}\,,} \tag{1.24}$$

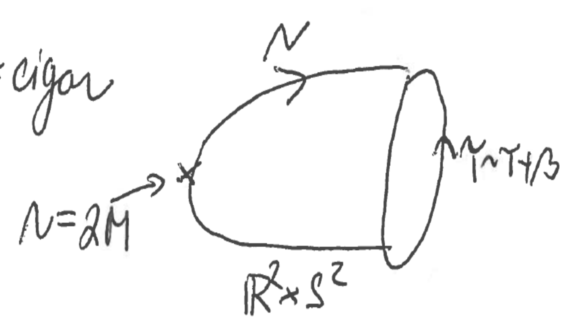
which is the Hawking temperature. In particular,

$$\boxed{T = \frac{1}{8\pi M}} \tag{1.25}$$

for the Schwarzschild solution.

- Let us now calculate the action (1.16). To this purpose we notice that the Euclidean Schwarzschild solution (1.19) looks like a surface of a *cigar*, where each point corresponds to a suppressed sphere:

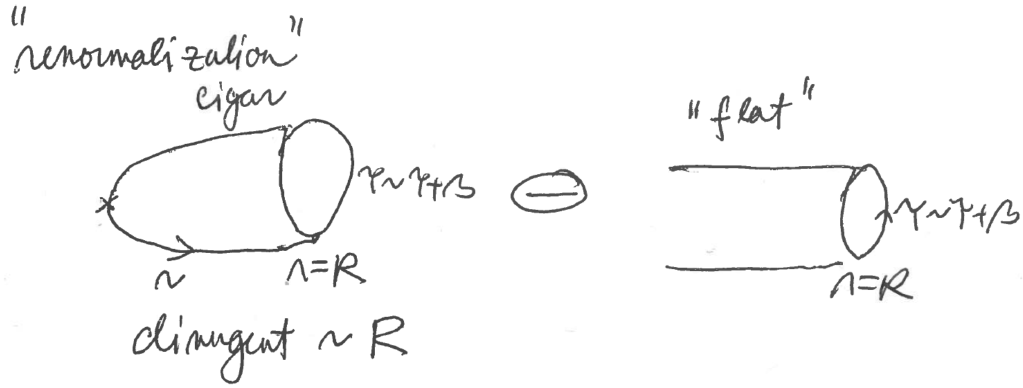
Euclidean Schwarzschild $\simeq$ cigar



Returning to the action (1.16), we note that $R = 0$ and the first term vanishes. Turning to the second term, we introduce a boundary at $r = R_0$, calculate the contribution of the second term and then let $R_0 \to \infty$. Unfortunately, this diverges and we have to renormalize by the corresponding counter term. In this case, this is given by

$$\text{counter term} = -\int_{\partial\Omega} \frac{d^3x\sqrt{h}\mathcal{K}_0}{8\pi G}\,, \tag{1.26}$$

the Gibbons–Hawking counter term for the flat space with the same boundary metric h_{ab} and the same periodicity of β, see figure:



We then get

$$S_E = S_E(\text{cigar}) - S_E(\text{flat}) = \frac{\beta M}{2}\,, \tag{1.27}$$

and the free energy becomes

$$F = -\frac{1}{\beta}\log Z = M/2\,, \tag{1.28}$$

giving the following entropy:

$$S = -\frac{\partial F}{\partial T} = \left| T = \frac{1}{8\pi M} \right| = \frac{1}{16\pi T^2} = \pi r_+^2 = \frac{A}{4}\,, \tag{1.29}$$

confirming Bekenstein's result.

Rindler space

As you will see in your tutorial, accelerated observers see a thermal bath at a temperature proportional to their acceleration,

$$\boxed{T = \frac{a}{2\pi}\,,} \tag{1.30}$$

which is the famous Unruh temperature. If you accelerate really fast, you can cook a chicken. Using the Euclidean action, one can also calculate the entropy of the associated Rindler horizon, yielding the Bekenstein result for this case as well.

1.2 A few remarks on Hawking evaporation

- The Hawking temperature for a Schwarzschild black hole reads

$$T = \frac{\hbar c^3}{8\pi k_B G M} \propto \frac{1}{M}\,, \tag{1.31}$$

smaller the black hole is the hotter it is. For a stellar mass black hole we get about $6 \times 10^{-8} K$, which is much smaller than the CMB temperature – the effect is not important for astrophysics.

We would need a black hole smaller than 4.5×10^{22}kg (size of the Moon) to reach at least the CMB temperature $T \approx 2.7K$.

Obviously, the evaporation accelerates and towards the end we can observe 'black hole explosions' (CERN?)

- Black holes do not radiate as true black body as some waves 'scatter back' to the horizon. For this reason the corresponding distribution reads

$$\langle n_\omega \rangle = \frac{\Gamma_\omega}{e^{\omega/T} - 1}\,, \tag{1.32}$$

where Γ_ω is the so called *greybody factor*.

Correspondingly, the black hole looses mass according to the 'effective' Stefan–Boltzmann law

$$\frac{dM}{dt} \propto -\sigma T^4 A \propto -\frac{1}{M^2}\,, \tag{1.33}$$

so that it would completely evaporate in (M_S denoting the mass of the Sun)

$$t_{\text{evap}} \approx \left(\frac{M}{M_S}\right)^3 \times 10^{71}\, s\,. \tag{1.34}$$

- Since $T \propto 1/M$, the Schwarzschild black hole has a *negative specific heat*:

$$C = T\frac{\partial S}{\partial T} = -\frac{1}{8\pi T^2}\,. \tag{1.35}$$

(It gets colder when mass is absorbed by a black hole and vice versa.)

This is very strange for ordinary matter, but it is quite typical for self-gravitating systems. For example, a satellite as it falls it increases its kinetic energy; a gravothermal catastrophe described by Lynden Bell.

It also means that the *canonical ensemble* is not well defined for Schwarschild black hole (as no stable thermal equilibrium exists). One way to 'stabilize the black hole system' is to place the black hole in a confining box. A natural such box is provided by the AdS space.

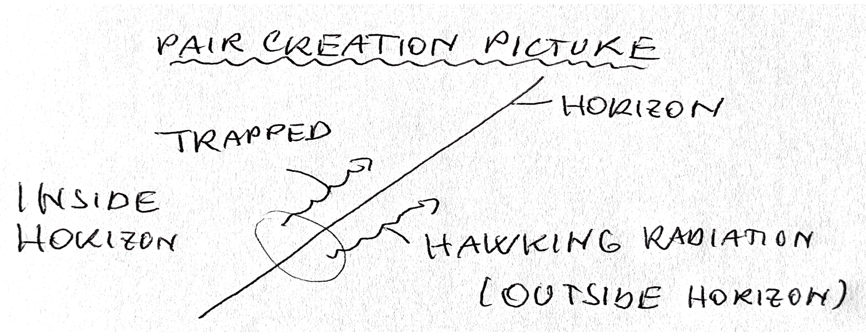
- Generalized second law. During the black hole evaporation, the black hole entropy (area) decreases – Hawking radiation violates the null energy condition. However, the total entropy of the black hole and of the outside Universe should never decrease:
$$\boxed{S_{\rm TOT} = S_{\rm BH} + S_{\rm outside} \geq 0\,.} \tag{1.36}$$
This is the content of the generalized second law of thermodynamics.

- Bekenstein bound provides an upper limit on the thermodynamic entropy of classical and quantum systems:
$$\boxed{S \leq \frac{2\pi k_B R E}{\hbar c}\,,} \tag{1.37}$$
where E is the total energy of the system, and R is the radius of a sphere enclosing it.

  Let us qualitatively outline proof of this statement for small E. Consider a system with (small) energy E and (arbitrary) entropy S, contained in a box of radius R. We then consider a black hole with (large) mass M and the same radius R. Its entropy is $S_{\rm BH} \propto M^2$. Let's next lower the system to the black hole, obtaining a black hole of mass $M + E$. Since the total entropy cannot decrease by the generalized second law, we must have (omitting all the pre-factors)
$$S + M^2 \leq (M+E)^2 \approx M^2 + 2ME + O(E^2)\,. \tag{1.38}$$
Hence we have,
$$S \leq 2ME \approx RE\,, \tag{1.39}$$
using the fact that the original black hole had a size $R \propto M$.

  Note that the above bound is saturated for the black hole entropy, $S = S_{\rm BH} = \pi r_+^2$. We can thus understand this as a universal bound on the amount of information in a given spatial region with a boundary of area A:
$$\boxed{S \leq \frac{A}{4}\,,} \tag{1.40}$$
as measured in Planck units. This means that the upper bound is bounded holographically – by the area of the region instead of its volume.

- Pair creation picture. Hawking radiation can be understood as originating from quantum pair creation of particles in the vicinity of the horizon. One of these remains trapped behind the horizon while the other one escapes to infinity as Hawking radiation:

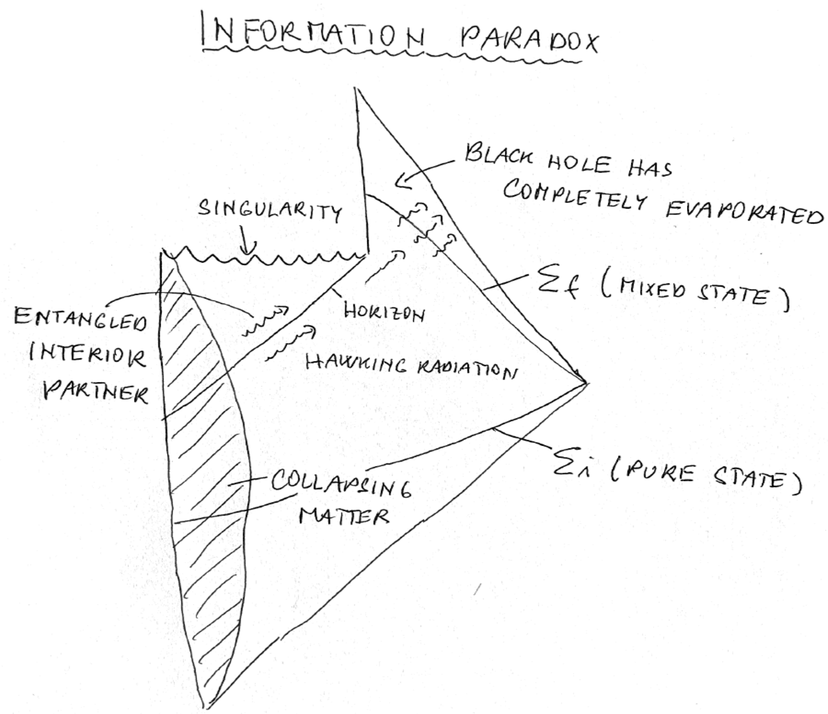


- Hawking radiation is a kinematic effect. (One needs equivalence principle, vacuum fluctuations, but the Einstein equations are not required.) This opens a possibility for observing this effect in '*analogue systems*', e.g. surface water waves (see Unruh's talk about fishes with ears).

1.3 Black hole information paradox

- Information loss. Classically, black holes absorb interesting stuff (for example an elephant, a star, and so on) but in response only get bigger – the only parameter that changes is the black hole mass – information disappears inside the black hole. (Only finite amount of information is radiated away during the in-fall.) Even quantum mechanically, if Hawking radiation is perfectly thermal, information must be lost inside a black hole.

- More precisely, following Hawking, let us draw the Penrose diagram of a completely evaporating black hole:

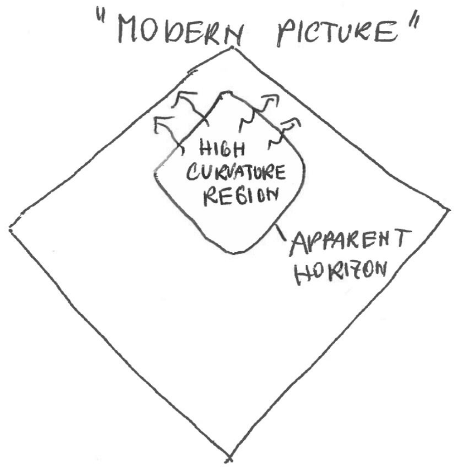


In this picture, Hawking radiation can be understood as originating from quantum pair creation of particles. These are entangled and together form a pure state. One of these remains trapped behind the horizon while the other one escapes to infinity as Hawking radiation. From outside, we only see one of them and thence a mixed (thermal) state. Thus, as the black hole has evaporated we evolved from a pure state on Σ_i to a mixed one on Σ_f, violating unitarity of quantum mechanics.

- Information paradox (Hawking 76). The above violates the following hypothesis: *As seen from outside, black holes can be described as ordinary quantum systems with $S = A/4$ dof; they should be described by a unitary time evolution.*

  (Or in other words: black holes are now fundamentally different from a piece of coal.)

- Various proposals as to what might happen as black hole evaporates were invented:
  - *Black hole remnants:* information is lost there
  - *Firewalls:* equivalence principle is violated in the vicinity of black holes
  - *Final bursts:* inconsistent with entropy bound
  - *Black holes do not exist:* e.g. 'leaking horizons', fuzzballs, or the following "QG picture":



Some of these are in favour of information loss (e.g. black hole remnants) some are in favour of restoring unitarity, e.g. AdS/CFT correspondence (black hole evaporation is dual to the evolution of the CFT on the boundary which is manifestly unitary).

If the unitarity is to be restored – information has to start coming out of the BH in order to purify Hawking's radiation. This has to happen quite early – around the Page time. We shall return to this problem in later chapters.

1.4 AdS black holes and their thermodynamics

- Let us now turn to the asymptotically AdS black holes, solutions of the Einstein equations with negative cosmological constant:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \quad \Lambda = -\frac{3}{\ell^2}\,, \tag{1.41}$$

  where ℓ is called the *AdS radius*.

- AdS action. In the AdS case one has a well defined local Euclidean action with unique counter terms (c.f. vague background subtraction of Gibbons and Hawking (1.26)). Namely, we have the following action:

$$\begin{aligned} S_E &= \frac{1}{16\pi G}\int_M d^4x\sqrt{g}\big(R+\frac{6}{\ell^2}\big) \\ &\quad +\frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}\left[\mathcal{K}-\frac{2}{\ell}-\frac{\ell}{2}\mathcal{R}\,(h)\right]\,, \end{aligned} \tag{1.42}$$

  where $\mathcal{K}$ and $\mathcal{R}\,(h)$ are respectively the extrinsic curvature and Ricci scalar of the boundary. In this expression we have included, apart from the Einstein–Hilbert and York–Gibbons–Hawking pieces, also the standard AdS counter-terms – constructed from the invariants on the boundary [1].

Varying this action yields the above Einstein equations, together with the following boundary term:

$$\delta S_E = -\frac{1}{2}\int_{\partial\Omega} \sqrt{-h} d^3x \tau_{ab} \delta h^{ab} \,, \tag{1.43}$$

where

$$8\pi\tau_{ab} = \mathcal{K}h_{ab} - \mathcal{K}_{ab} + \ell G_{ab}(h) - \frac{2}{\ell}h_{ab} \tag{1.44}$$

is (up to trivial scaling) the holographic stress energy tensor. This gives the expectation value of the energy momentum tensor of the dual CFT_3. Here, the first two terms come from varying the York–Gibbons–Hawking term and you may recognize them as left hand side of the Israel junction conditions. The latter two terms are innate to AdS and come from the corresponding counter-terms.

Obviously, when δh^{ab} vanishes, $\delta S_E = 0$, and we have a well defined Dirichlet principle (fixed boundary metric).

Alternatively, instead of considering S_E, (1.42), we may consider

$$\tilde{S}_E = S_E + \frac{1}{2}\int_{\partial\Omega} d^3x \sqrt{-h} h^{ab} \tau_{ab} \,. \tag{1.45}$$

This then yields a 'Neumann-type' variational principle [2], where $\delta(\sqrt{-h}\tau_{ab})$ is to be held fixed on the boundary, instead of δh^{ab}. However, as we shall see later, the holographic stress tensor is necessarily traceless, and so really $S_E = \tilde{S}_E$. (In flat space, on the other hand, we return back to the Einstein–Hilbert action.)

- Hawking–Page transition. As shown first by Hawking and Page in 1983, [3], thermodynamics of black holes in AdS space is not only (contrary to that of their asymptotically flat cousins) *well defined* (can have positive specific heat) but it is also rather interesting – for example it features a number of rather intriguing *phase transitions*. Such phase transitions then correspond to the associated phase transitions of the dual CFT.

  Let us show this explicitly for the Schwarzschild-AdS metric:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \,, \quad f = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2} \,. \tag{1.46}$$

  Using the Euclidan trick, one finds the black hole temperature

$$T = \frac{f'(r_+)}{4\pi} = \frac{\ell^2 + 3r_+^2}{4\pi\ell^2 r_+} \,. \tag{1.47}$$

  Note that, for fixed ℓ, there exists a minimal temperature

$$T_0 = \frac{\sqrt{3}}{2\pi}\frac{1}{\ell} \,, \tag{1.48}$$

below which AdS black holes cannot exist: For $T < T_0$ the 'thermal radiation' of AdS is stable against collapse to form a black hole. On the other hand, for $T > T_0$ two branches of black holes exist: large black holes ($r_+ > \ell/\sqrt{3}$) have positive specific heat and small black holes ($r_+ < \ell/\sqrt{3}$) have negative specific heat and are unstable.

To uncover the possible phase behavior, one has to study the behavior of the free energy, derived from the action (1.42). After some calculation this yields

$$F = -\frac{1}{\beta}\log Z \approx \frac{S_E}{\beta} = \frac{r_+(\ell^2 - r_+^2)}{4\ell^2}\,. \tag{1.49}$$

With this we can find

$$\begin{aligned} S &= -\frac{\partial F}{\partial T} = \pi r_+^2 = \frac{A}{4}\,, \\ E &= \frac{\partial(\beta F)}{\partial \beta} = \frac{\partial S_E}{\partial r_+}\frac{\partial r_+}{\partial \beta} = \frac{r_+}{2}\Big(1 + \frac{r_+^2}{\ell^2}\Big) = M\,. \end{aligned} \tag{1.50}$$

With this we can check that $F = M - TS$.

Finally, we can plot $F = F(T, l)$ and compare it to the free energy of thermal AdS F_{AdS} (AdS filled with thermal radiation). The global minimum then corresponds to the stable phase. Since $F_{\text{AdS}} \approx 0$, when F becomes negative, the black hole phase dominates the thermal AdS. This approximately happens at

$$T = T_{\text{HP}} = \frac{1}{\pi\ell}\,, \tag{1.51}$$

the so called *Hawking–Page temperature*. At this temperature there is a first-order phase transition from thermal AdS (which is stable for $T < T_c$) to large black hole phase (which dominates above T_c). The situation is displayed in figure 1.1:

Via the AdS/CFT correspondence, this phase transition has an interpretation of the confinement/deconfinement phase transition of the dual quark–gluon plasma [4]. Even more interesting phase transitions occur upon adding rotation and charges to the black hole, see e.g. [5].

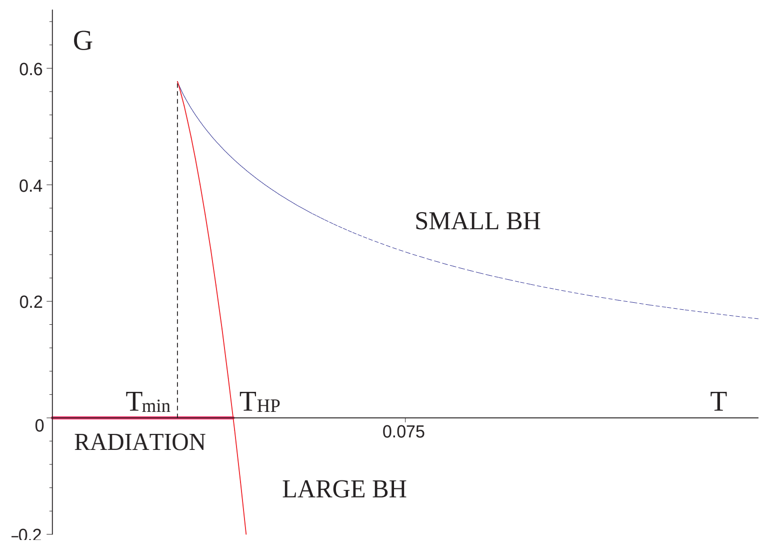


Figure 1.1: *Hawking–Page phase transition.* We display the free energy $G = F = F(T)$. There is no black hole configuration possible for $T < T_0 = T_{\min}$, while two possible branches of black holes are possible for $T > T_0$. The upper branch corresponds to unstable small AdS black holes, whereas the lower branch denotes large thermodynamically preferred black holes. Such black holes become globally stable when G becomes negative, i.e., for $T > T_{HP}$. Hence there is a thermal radiation/black hole transition in the system, called the *Hawking–Page transition.* [Units were chosen such that $G = 1$ and $l = 1$.]

Chapter 2: Motivating AdS/CFT

The best understood example of the holographic duality is the $AdS_5 \times S_5/CFT_4$ correspondence. In this chapter we will see how this 'emerges' from the results of string theory.

2.1 Elements of string theory

String theory is a quantum theory of interacting relativistic strings and higher-dimensional objects.

Classical p-branes

- Let us start with a motion of a *free particle*. It is governed by the following action:

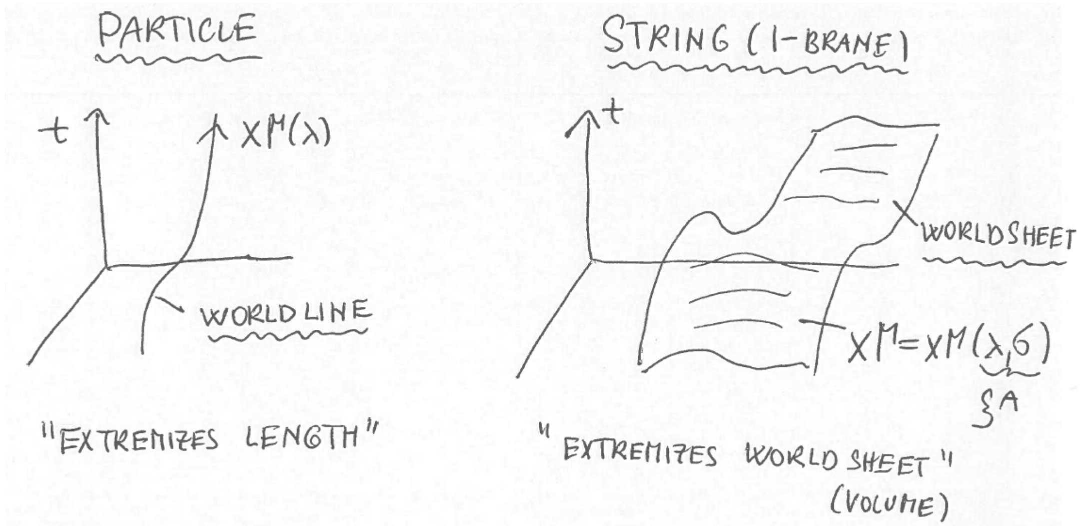
$$S[x^\mu] = -m_0 \int d\tau = -m_0 \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda = \int \sqrt{-\det(\gamma_{\lambda\lambda})} d\lambda \,, \qquad (2.1)$$

  where $\gamma_{\lambda\lambda} = \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$ is the 'induced metric' on the worldline, m_0 is particle's rest mass, and τ is its proper time. Upon varying w.r.t. δx^μ, this yields the (flat space) geodesic equation:

$$\frac{du^\alpha}{d\tau} = 0 \,, \quad u^\mu = \frac{dx^\alpha}{d\tau} \,, \qquad (2.2)$$

  or, upon using $\frac{d}{d\tau} = \frac{dx^\nu}{d\tau} \frac{\partial}{\partial x^\nu} = u^\nu \frac{\partial}{\partial x^\nu}$, we recover $u^\nu \partial_\nu u^\alpha = 0$.

- Motion of p-branes. A higher-dimensional object which sweeps a (p+1)-dimensional trajectory in the spacetime is called a p-brane. An example is a string, which is a 1-brane and sweeps a 2-dimensional worldsheet (or more generally p-dimensional worldvolume), see figure.



A *world-volume* of a p-brane is described by

$$x^\mu = x^\mu(\xi^A), \quad A = 0, 1, \ldots, p\,, \tag{2.3}$$

where ξ^A are the internal coordinates. One can easily write down the action for a free p-brane. It comes from the following observation. The above standard particle action (2.1) has a very intuitive meaning: "*the motion of a particle is such that it extremizes its proper time*". It is then natural to expect that *the motion of a free p-brane is such that it maximizes the p-brane's worldvolume.* Actions like this are called <u>Nambu–Goto-type actions</u>, generally they are of the form

$$\boxed{S_{\text{NG}}[x^\mu] = -T_p \int \sqrt{-\det\big(\gamma_{AB}(\eta)\big)}\, d^{p+1}\xi\,.} \tag{2.4}$$

Here, $\gamma_{AB}(\eta)$ is the so called <u>induced metric</u> (metric on the worldvolume induced from the metric of the Minkowski space):

$$\boxed{\gamma_{AB}(\eta) = \frac{\partial x^\mu}{\partial \xi^A}\frac{\partial x^\nu}{\partial \xi^B}\eta_{\mu\nu}\,,} \tag{2.5}$$

and T_p is the p-brane tension. (Note that (2.4) reduces to (2.1) for $p = 0$.)

- <u>Polyakov action.</u> Looking again at the action (2.1), we find that it is pretty complicated: it contains square root (and thence is difficult to quantize) and it does not work for massless particles. To avoid the square root and to make it work even for massless particles, let us consider instead a <u>Polyakov-type action</u>:

$$S[x^\mu, h] = \frac{1}{2}\int \Big(\frac{1}{h}\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - m_0^2 h\Big) d\lambda\,. \tag{2.6}$$

Here $h = h(\lambda)$ is an 'extra field' (or more precisely a Lagrange multiplier). Variation w.r.t. h yields

$$h = \frac{1}{m_0}\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,. \tag{2.7}$$

(Which is a constant provided we identify $\lambda = \tau$.) Plugging this back, we recover the previous Nambu–Goto action – classically, and for $m_0 \neq 0$, the actions are equivalent. At the same time we can take the limit $m_0 \to 0$ and recover the action for a massless point particle (definning $d\tilde{\lambda} = h d\lambda$).

Polyakov-type actions can also be written for p-branes. In particular, for a massless string we have (see tutorial)

$$\boxed{S[x^\mu, h^{AB}] = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-h}h^{AB}\gamma_{AB}\,,} \tag{2.8}$$

where α' is related to the string tension T and the fundamental string length l_s (the only dimensionful parameter of string theory) as follows:

$$\boxed{T = \frac{1}{2\pi\alpha'}\,, \quad \alpha' = l_s^2\,.} \tag{2.9}$$

Again, γ_{AB} is the induced metric (2.5) and h_{AB} is an auxiliary worldsheet metric (an analogue of e above). Note that this is nothing else than massless scalar field theory (with scalars x^μ) living in 2-dimensions. This has (in 2-dimensions only) conformal symmetry which is the cornerstone of string theory.

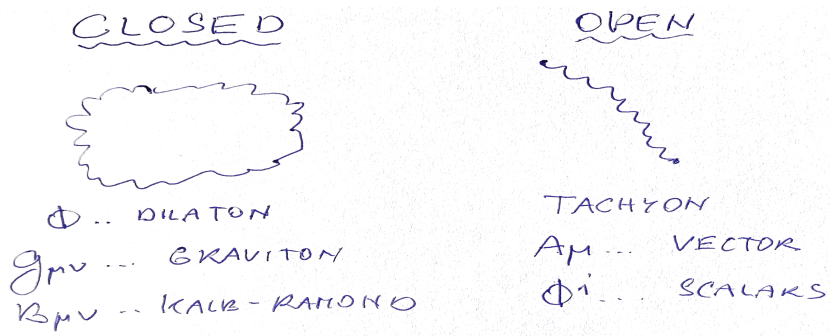
Quantum superstrings

By quantizing the above string action (2.8) (supported by fermionic degrees of freedom in a supersymmetric way) one finds:

- Depending on BC, 5 self-consistent theories (type IIA, IIB, type I, and 2 heterotic ones). These are related by a web of various dualities.
- Absence of negative norm states fixes the number of spacetime dimensions to

$$\boxed{d = 10\,.} \tag{2.10}$$

- Finite number of massless modes (including graviton for closed strings) and infinite tower of massive modes with $m \propto 1/l_s$. This means that string theory is a theory of quantum gravity. (Note the decomposition of a rank-2 tensor into its symmetric traceless part $g_{\mu\nu}$ with $d(d+1)/2-1$ dof, antisymmetric $B_{\mu\nu}$ with $d(d-1)/2$ dof, and the trace ϕ with 1 dof.)



- The fundamental strings can be coupled to non-trivial background of the massless closed string excitations (such as graviton $g_{\mu\nu}$, dilaton ϕ, Kalb–Ramond 2-form $B_{\mu\nu}$, ...). This is done by replacing in Eq. (2.8):[1]

$$h^{AB}\gamma_{AB}(\eta_{\mu\nu}) \;\to\; h^{AB}\gamma_{AB}(g_{\mu\nu}) + \epsilon^{AB}\gamma_{AB}(B_{\mu\nu}) + \alpha' R_h \phi + \dots \,. \tag{2.11}$$

  Moreover, by requiring the conformal symmetry to hold at the quantum level, the corresponding β-functions have to vanish (e.g. $\beta^g_{\mu\nu} \propto \mu\frac{\partial g_{\mu\nu}(x,\mu)}{\partial\mu} = 0$ at 1-loop). Consequently, the background fields have to satisfy a generalization of Einstein equations. This is how Einstein equations emerge in string theory (in the leading order of small α').

  For example, for type IIB, the vanishing of the β functions can be (secondarily) derived from the following Lagrangian (for fields $g_{\mu\nu}, \phi, A_0, B_{\mu\nu}, A_{\mu\nu}, A_{\mu\nu\kappa\lambda}$):

$$\boxed{\mathcal{L} = *\Big(e^{-2\phi}\Big[R + 4(\partial_\mu\phi)^2 - \frac{1}{2}H_3^2\Big] - \frac{1}{2}F_1^2 - \frac{1}{2}F_3^2 - \frac{1}{4}F_5^2\Big) - \frac{1}{2}A_4 \wedge H_3 \wedge F_3\,,} \tag{2.12}$$

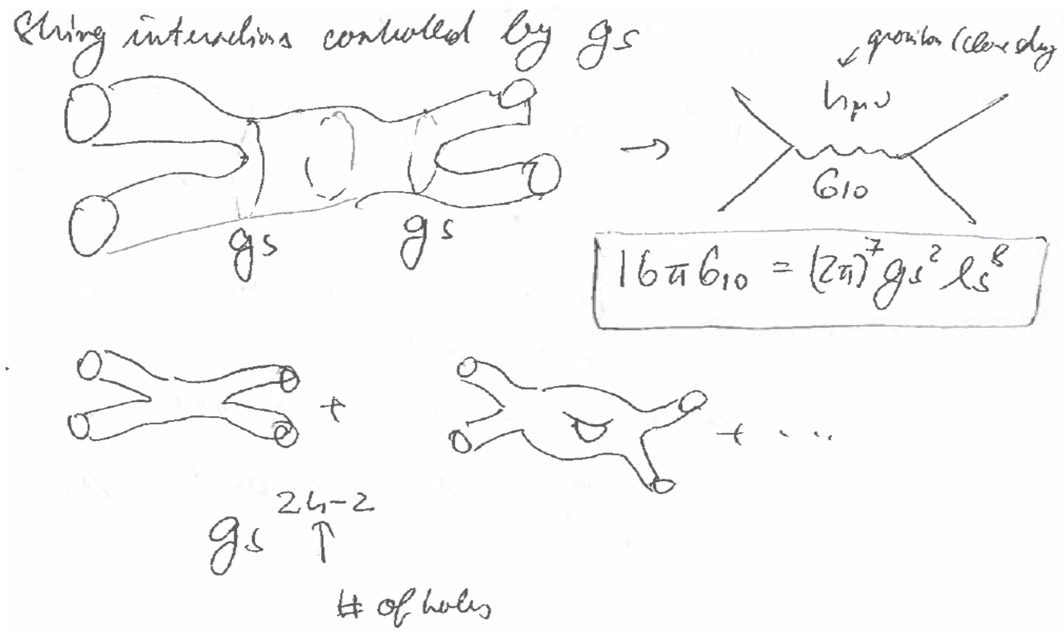
  plus fermions. This is the action of the type IIB, $D = 10$, $\mathcal{N} = 2$ SUGRA.

- String interactions are controlled by g_s, related to the 10-dimensional gravitational constant as:

$$\boxed{16\pi G_{10} = (2\pi)^7 g_s^2 l_s^8 \propto l_P^8\,,} \tag{2.13}$$

  see the following figure:

[1] Please see D. Tong's lectures, Chapter 7, for the explanation as to how the quantum modes of individual strings give rise to the 'classical background' fields.



g_s is not really a free parameter, but rather an expectation value of the dilatonic massless mode, $g_s = e^{\phi}$. However, effectively, string theory has 2 parameters:

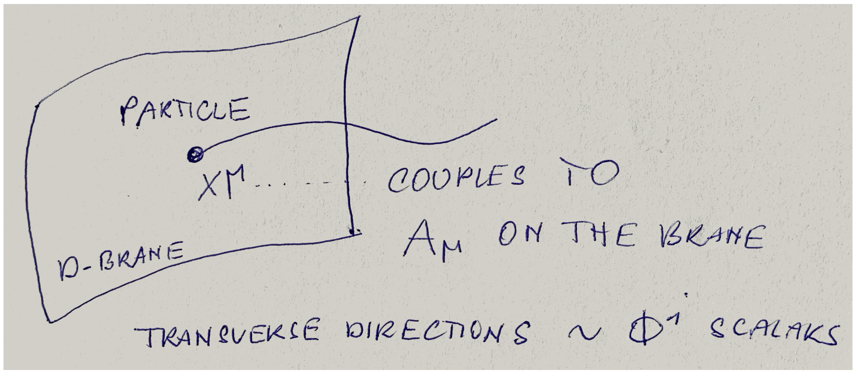
$$\boxed{l_s\,,\, g_s\,.} \tag{2.14}$$

D-branes

- Dp-branes are 'topological defects' (non-perturbative in g_s) with $(p+1)$-dimensional world-volume on which open strings can end (and move freely along the world volume).

  Fluctuations of a Dp-brane are determined by the quantum spectrum of open strings attached to it. At low energy, only the massless modes are of interest. For a single Dp-brane we have a $(p+1)$-dimensional $U(1)$ <u>gauge field</u> $A_\mu(x)$ living on the world-volume of the brane (in the Neumann BC directions), and $\phi^i(x)$ <u>scalars</u> (corresponding to transverse directions to the brane – corresponding to the Dirichlet BC):

$$A_\mu(x): \;\; \mu = 0, \ldots, p\,, \qquad \phi^i(x): \;\; i = 1, \ldots, 9-p\,. \tag{2.15}$$



These are goldstone modes associated with the spontaneous symmetry breaking by the branes.

The motion of a single brane is governed by the Dirac–Born–Infeld action (a generalization of the Nambu–Goto action):

$$S_{\text{DBI}} = -T_{Dp} \int d^{p+1}\xi e^{-\phi} \sqrt{-\det(\gamma_{AB}(g) + \gamma_{AB}(B) + 2\pi\alpha' F_{AB})}\,, \tag{2.16}$$

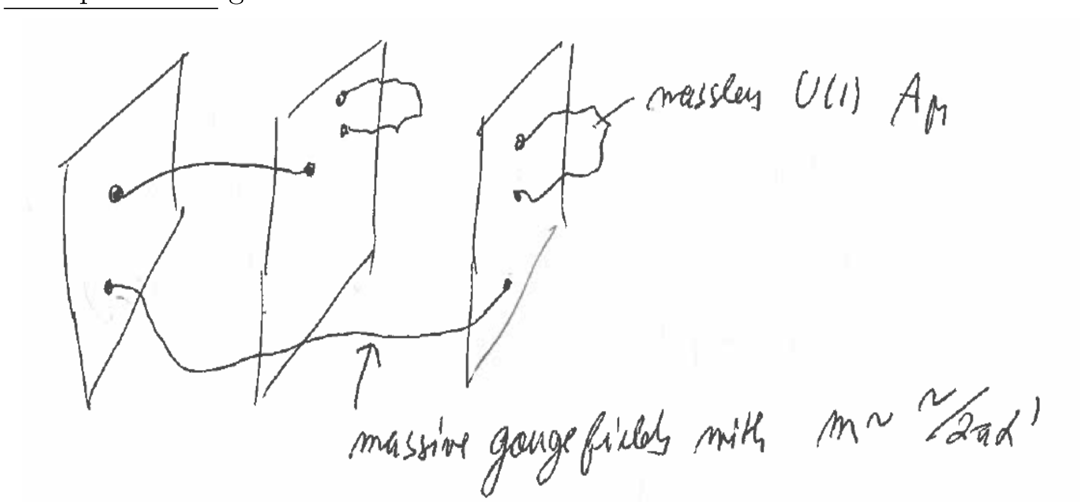
where the D-brane tension is $T_{Dp} = \frac{1}{(2\pi)^p l_s^{p+1}}$. For weak EM fields in flat space, setting $B = 0$ and $e^{\phi} = g_s$, we can then approximate

$$S_{\text{DBI}} \approx -(2\pi\alpha')^2 \frac{T_{Dp}}{4g_s} \int d^{p+1}\xi F_{AB}F^{AB} + O(F^4)\,. \tag{2.17}$$

Thence we can read off the Yang–Mills coupling constant as

$$g_{YM}^2 = \frac{g_s}{T_{Dp}(2\pi\alpha')^2} = (2\pi)^{p-2} g_s l_s^{p-3}\,. \tag{2.18}$$

- Multiple branes give rise to:



In the coalescence limit, $r \to 0$, non-Abelian gauge theory arises, $(A_\mu)^a_b$. (Naively, we have added another internal (geometrical) index indicating where the string ends), see Zweibach [6] and Witten [7].

- N_c coalescent $D3$-branes in type IIB then give rise to the $\mathcal{N}=4$ $U(N_c)$ SYM in $d=4$, with the following field content: A_μ and ϕ^i $(i=1,\dots,6)$ and 4 Weyl fermions in adjoint representation of $U(N_c)$, governed by the following action:

$$\boxed{\mathcal{L}=-\frac{1}{g_{YM}^2}\mathrm{Tr}\Big(\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}D_\mu\phi^i D^\mu\phi^i+[\phi^i,\phi^j]^2\Big)+\text{fermions}\,,}\quad \boxed{g_{YM}^2=2\pi g_s}\,. \tag{2.19}$$

- Alternatively we may view D-branes as BPS (extremal) massive solitons of $d=10$ type IIB SUGRA. Let all fields except $g_{\mu\nu}$, F_5, and ϕ =const be zero. This is a self-consistent truncation of the EOMs derived from (2.12), which now become:

$$R_{\mu\nu}=\frac{1}{96}F_{\mu\alpha\beta\gamma\delta}F_\nu{}^{\alpha\beta\gamma\delta}\,,\quad F_5=*F_5\,. \tag{2.20}$$

(Note that $d*F_5=0$ is satisfied as a consequence of Bianchi identity, $dF_5=0$ and self duality condition.)

In particular, we find the following *near-extremal* black brane solution:

$$\begin{aligned} ds_{10}^2 &= H^{-1/2}\Big[-f dt^2+dx^2+dy^2+dz^2\Big]+H^{1/2}\Big(\frac{dr^2}{f}+r^2d\Omega_5^2\Big)\,,\\ F_5 &= -\frac{4L^2}{H^2r^5}\sqrt{r_0^4+L^4}(1+*)dt\wedge dx\wedge dy\wedge dz\wedge dr\,,\\ H &= 1+\frac{L^4}{r^4}\,,\quad f=1-\frac{r_0^4}{r^4}\,. \end{aligned} \tag{2.21}$$

Here f is the so called blackening factor, and $r=r_0$ is the horizon. Setting

$$\boxed{f=1\,,} \tag{2.22}$$

we recover the BPS solution describing the $D3$-brane. The length L can be determined from the flux of the F_5 (which is quantized and 'counts number of $D3$-branes):

$$Q=\frac{1}{2\kappa_{10}^2}\int_{S^5}*F_5=N_c\mu_3\propto M\,, \tag{2.23}$$

as we have N_c branes and $\mu_3=T_{D3}/g_s$, yielding

$$\boxed{\frac{L^4}{l_s^4}=4\pi g_sN_c=2g_{YM}^2N_c=2\lambda\,,\quad \lambda=g_{YM}^2N_c\,,} \tag{2.24}$$

the latter know as the 't Hooft coupling.[2] Obviously, it is this effective coupling which 'decides' about the strength of gravitational interaction – whether gravity

[2] That λ is the coupling to consider can easily be seen by looking at the strength of gravitational potential, which for a p-brane goes like

$$\phi\sim\frac{G_{10}M_{\mathrm{Tot}}}{r^{d-p-3}}\sim\frac{G_{10}N_c\mu_3}{r^4}\sim g_sN_c\frac{l_s^4}{r^4}\sim\lambda\frac{l_s^4}{r^4}\,, \tag{2.25}$$

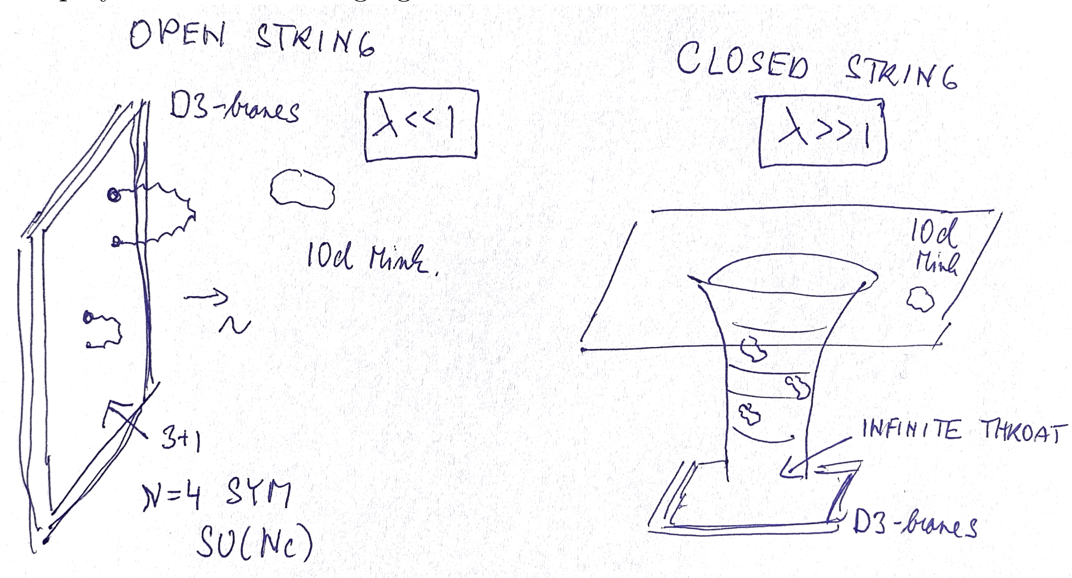
with the last two valid for the stack of N_c $D3$ branes. Here we have used that $G_{10}\sim g_s^2l_s^8$, $\mu_3\sim 1/(g_sl_s^4)$.

(closed strings) are important or not. We thus have two opposite limits: that of opens strings (for $\lambda \ll 1$) and that of closed strings (for $\lambda \gg 1$).

2.2 AdS/CFT conjecture

Two D-brane pictures

Let us consider a stack of N_c $D3$-branes. We can look at them from two perspectives: the 'open string picture' (reliable for $\lambda \ll 1$) and the 'closed string picture' (for $\lambda \gg 1$), as displayed in the following figure:



- *open string picture.* In this picture the strings are treated as perturbations – we have $\lambda \ll 1$. For low energies $E \ll 1/l_s$ only the massless excitations are relevant – this is how we recover the $\mathcal{N} = 4$ SYM on the brane (plus decoupled SUGRA modes in $d = 10$ Minkowski space). The two parameters describing SYM are:

$$N_c\,, \quad \lambda = g_{YM}^2 N_c\,. \tag{2.26}$$

- *closed string picture.* This picture is valid for strong coupling $\lambda \gg 1$, the D-branes are very massive and described by the SUGRA solution (2.21), (2.22) above:

$$\begin{aligned} ds_{10}^2 &= H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}\Big(dr^2 + r^2 d\Omega_5^2\Big)\,, \\ H &= 1 + \frac{L^4}{r^4}\,, \quad \frac{L^2}{l_s^2} = 2\lambda\,. \end{aligned} \tag{2.27}$$

We have two different low energy modes: SUGRA modes propagating in 10d Minkowski background ($r \gg L$) and full stringy modes propagating deep in the throat of the D-brane $r \ll L$ (whose energy is infinitely red-shifted as seen by the asymptotic observer) – these two kinds of modes are completely decoupled. Concentrating on the near horizon region

$$r \ll L \tag{2.28}$$

we recover the $AdS_5 \times S_5$ near horizon geometry:

$$ds^2 = \underbrace{\frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2}_{AdS_5 \text{ of size } L} + \underbrace{L^2 d\Omega_5^2}_{S^5 \text{ of size } L} , \tag{2.29}$$

or, upon setting $z = L^2/r$:

$$\boxed{ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) + L^2 d\Omega_5^2 .} \tag{2.30}$$

Note that the radius of both AdS_5 and S^5 is L; the metric is supported by the F_5 flux through both. Sometimes we can 'smear out' over the S^5 and consider only the geometry of AdS_5. On this side we have two dimensionless parameters:

$$g_s, \quad L/l_s , \tag{2.31}$$

related to the two parameters in the open string picture as

$$\boxed{2\pi g_s = g_{YM}^2 = \frac{\lambda}{N_c} , \quad \frac{L^4}{l_s^2} = 4\pi g_s N_c = 2\lambda .} \tag{2.32}$$

- Classical SUGRA is only valid when

  1. *Stringy corrections* are suppressed (strings are almost point like, classical geometry valid):
  $$L \gg l_s \quad \Rightarrow \quad \lambda \gg 1 . \tag{2.33}$$
  2. *Quantum gravity corrections* (loops) are suppressed
  $$1 \gg g_s \sim \frac{\lambda}{N_c} . \tag{2.34}$$

  Combining the two, we thus get

  $$\boxed{1 \ll \lambda \ll N_c ,} \tag{2.35}$$

  for validity of classical gravity in the bulk. (For finite λ and N_c we can do perturbations in $1/N$ and $1/\lambda$).

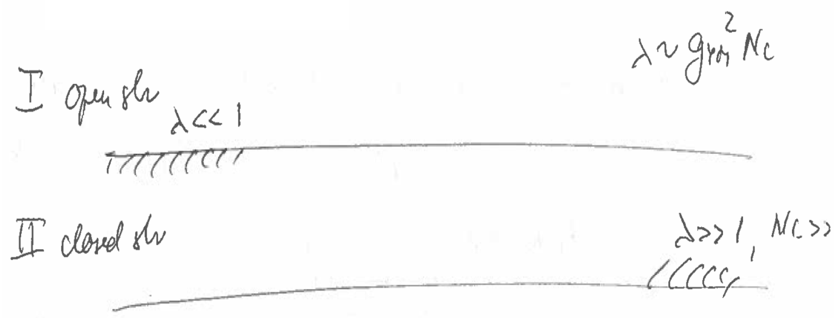
  Note also that N_c governs the ratio of L/l_P. Namely, since $G_{10} \sim g_s^2 l_s^8$ and $L_P^8 = G_{10}\hbar/c^3$, we have

  $$\frac{L^4}{l_P^4} \sim \frac{\lambda l_s^4}{G_{10}^{1/2}} \sim \frac{g_s N_c l_s^4}{g_s l_s^4} \sim N_c . \tag{2.36}$$

  Thus, requiring $L \gg l_P$ requires $N_c \gg 1$.

AdS/CFT conjecture

In principle the two descriptions are valid for all values of λ and N_c but we do not really know how to extend there:



We have a conjecture that I and II describe the same object in different languages:

> **Conjecture (Maldacena 1997).** Type IIB superstring theory on $AdS_5 \times S^5$ is *dual* to $\mathcal{N} = 4$ $SU(N_c)$ SYM in $d = (3 + 1)$ dimensions.

A few remarks:

1. **Duality** means exact equivalence at the full quantum level (Hilbert spaces and the dynamics of the two theories agree). For many dualities one can find a 'change of variables' in the partition function which directly maps dof of one theory to the dof of the dual theory. Such a map is currently not known for the AdS/CFT.

   Remarkably, this duality relates a theory of *quantum gravity* to a quantum field theory in flat space.

   Moreover, this duality is *holographic*: relates the dof of gravitational theory in AdS to dof of QFT on its conformal boundary ∂AdS.

   It is an example of *strong-weak* coupling duality: if the field theory is strongly coupled, the dual gravity theory is classical and weak. This provides a tool for studying strongly coupled QFTs.

   The conjecture is supported by case by case evidence and as Veronika says: "it is non-trivial and still holds water".

2. We have $SU(N_c)$ rather than $U(N_c) = SU(N_c) \times U(1)$ SYM. $U(1)$ can be decoupled – describes the motion of the center of mass of the system of N_c branes, which corresponds to *singleton* fields in the gravity theory (only located on the boundary and cannot propagate into the bulk of AdS_5)

3. Symmetries match. On the $AdS_5 \times S^5$ we have $SO(4, 2)$ and $SO(6)$. This corresponds to the conformal group in $d = 4$: $SO(4, 2)$ (with generators $P_\mu, L_{\mu\nu}, D, K_\mu$) and additional R-symmetry $SO(6)_R$.

   We also have $SL(2, \mathbb{Z})$ duality on both sides ($g_{YM} \to \frac{4\pi}{g_{YM}}$).

4. Various versions. For example *strong version* restricts to classical string theory: $g_s \ll 1$ and L/l_s =const., which implies λ finite and $N_c \to \infty$, known as *'t Hooft limit* (planar limit of the gauge theory). $1/N_c \sim g_s$ expansion then corresponds to genus expansion in string theory.

   The above *weak form* is then obtained by taking $\lambda \to \infty$ and point-like strings ($l_s \ll L$).

   More generally, we have *gauge/gravity duality.* Can go to AdS_{d+1}/CFT_d, or even go beyond asymptotically AdS.

2.3 Short CFT propaganda

- Conformal field theory in d dimensions is invariant under conformal transformations $SO(2,d)$:

$$\boxed{x^\mu \;\to\; x'^\mu(x): \quad \eta_{\mu\nu} \;\to\; \Omega^2(x)\eta_{\mu\nu}\,.} \tag{2.37}$$

  These include the Poincare group, the 'special conformal transformations', and the scaling transformation[3]

$$t \to \lambda t\,, \quad \vec{x} \to \lambda \vec{x}\,, \tag{2.38}$$

  under which the field transforms as

$$\phi(x) \;\to\; \phi'(x') = \lambda^{-\Delta}\phi(x)\,, \tag{2.39}$$

  where Δ is the scaling dimension.

conformal invariant equations

1) Maxwell equations: $\partial_\mu F^{\mu\nu} = 0$
2) Massless Dirac equation: $\gamma^\mu \partial_\mu \psi = 0 \qquad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$
3) classical $\lambda\phi^4$ in $D=4$: $\Box\phi = \frac{\lambda}{3!}\phi^3$
4) classical YM: $D_\mu F^{\mu\nu} = 0 \qquad \partial_\mu F^{\mu\nu} - ig[A_\mu, F^{\mu\nu}] = 0$ } not quantum mechanically!

  Scale invariant theories have no dimensionful parameter. Usually theories that are scale invariant are also conformal invariant. These are theories where the energy momentum tensor is traceless (see Tutorial).

[3] More generally, one can consider $t \to \lambda^z t$ where z is called the dynamical exponent: $z = 1$ for relativistic theories; $z = 2$ for non-relativistic e.g. $E \sim p^2$.

- In QFT there are no coupling constants; they depend on energy scale: $g = g(E), \lambda = \lambda(E)$: we may define the corresponding β function:

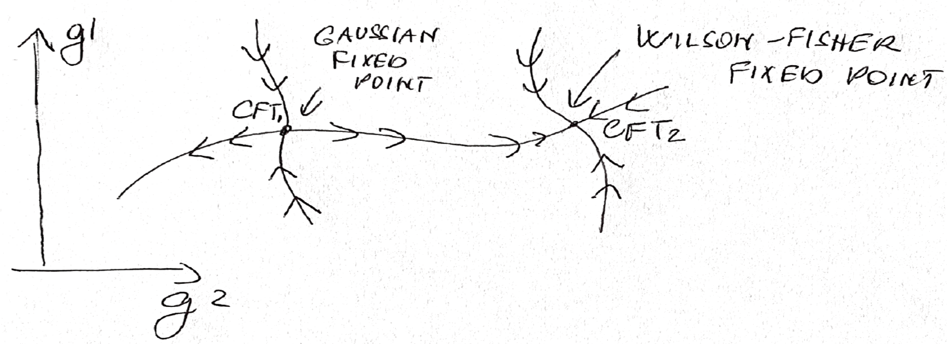
$$\boxed{\beta(g) = \frac{dg}{d\log E}\,.} \tag{2.40}$$

If the conformal symmetry is preserved even quantum mechanically we must have

$$\beta(g) = 0\,. \tag{2.41}$$

This is the case for the $d = 4 \mathcal{N} = 4$ SYM.

- In the space of QFT we have RG flow: $\beta_I(g^J)dg_I/d\log E$; fixed points are described by interacting CFTs:



Can classify operators according to their behavior near fixed points: relevant: flow away from fixed point $\Delta < d$ (there is only finite number of them), irrelevant ($\Delta > d$), or marginal (with $\Delta = d$).

Exercise: Show that considering a scalar field ϕ in $d = 4$ only ϕ^2, ϕ^3 and ϕ^4 are relevant.

- Conformal symmetry restricts significantly the form of correlation functions. For example, for a scalar conformal primary operator O of dimension Δ we have

$$\boxed{\langle O(x)\bar{O}(y)\rangle = \frac{1}{(x-y)^{2\Delta}}\,.} \tag{2.42}$$

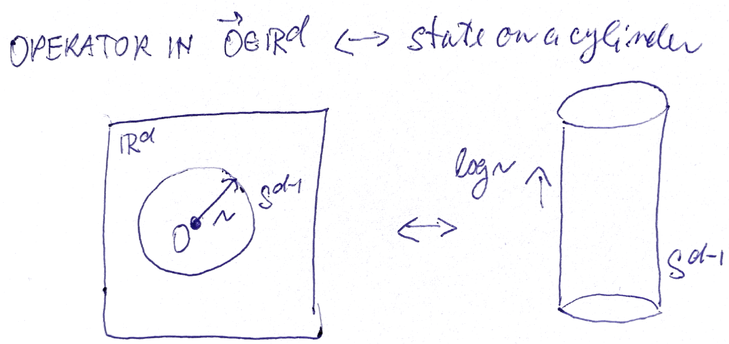
- Role of CFTs in nature:
  - Describe critical phenomena in statistical physics (driven by thermal fluctuations; critical exponents correspond to operator dimensions of fields in CFT)
  - Quantum critical phenomena (driven by quantum fluctuations at $T = 0$)

We also have applications to *String theory*: described by 2d CFT and *AdS/CFT*. It is useful to think about FT as a perturbation of CFT.

- Acting with an operator on field theory vacuum we create a state in the field theory. In general CFT we have a map between states on the cylinder $\mathbb{R} \times S^{d-1}$ and operators on the plave $\mathbb{R}$. The dimension of the operator is equal to the energy of the corresponding state. THis is because the Euclidean cylinder and the plane differ by Weyl transformation:

$$d(\log r)^2 + d\Omega^2 = \frac{1}{r^2}\left(dr^2 + r^2 d\Omega^2\right), \tag{2.43}$$

  and are thus equivalent in a CFT, see picture:



  Both states and operators can be 'classified' by their transformation laws under conformal group. These are characterized by the spin and its scaling dimensions Δ. E.g. $T_{\mu\nu}$ has $\Delta = d$; it creates a graviton in AdS. Single (multipe) trace operators correspond to single (multiple) particle states in the bulk.

2.4 AdS primer

AdS_d geometry is a maximally symmetric solution of Einstein equations with negative cosmological constant. It has an $O(d-1,2)$ symmetry.

Embedding perspective: AdS as maximally symmetric space

- One way to understand the AdS_d space is as a (maximally symmetric) hypersurface in higher-dimensional space. Namely, consider the following $(d+1)$-dimensional metric in $\mathbb{R}^{2,d-1}$ and a d-dimensional hyperboloid in it:

$$\begin{aligned} ds^2 &= -dY_{-1}^2 - dY_0^2 + dY_1^2 + dY_2^2 + \cdots + dY_{d-1}^2 \equiv \eta_{AB}^{2,d-1} dY^A dY^B, \\ -\ell^2 &= -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + \cdots + Y_{d-1}^2 = \eta_{AB}^{2,d-1} Y^A Y^B. \end{aligned} \tag{2.44}$$

  Solving the constraint for Y_{-1} and plugging back to the metric, we obtain a geometry of the AdS space in these coordinates:

$$\boxed{g_{ab} = \eta_{ab}^{1,d-1} - \frac{Y_a Y_b}{\ell^2 + Y^a Y_a}.} \tag{2.45}$$

- Note that the constraint equations (**??**) are invariant under

$$\tilde{x}^A = \Lambda^A{}_B x^B \quad \text{where} \quad \eta^{2,d-1}_{AB} = \eta^{2,d-1}_{CD} \Lambda^C{}_A \Lambda^D{}_B \,. \tag{2.46}$$

Group matrices $\Lambda^A{}_B$ form a representation of $O\big(d-1,2\big)$. Infinitesimally, we write

$$\Lambda^A{}_B = \delta^A_B + \lambda^A{}_B \quad \Rightarrow \quad \lambda_{AB} = \eta^{1,d-1}_{AC} \lambda^C{}_B = -\lambda_{BA} \ldots \binom{d+1}{2} \text{ generators}\,.$$

Since this is the maximum number of symmetries one can have in d number of dimensions, the spacetime is maximally symmetric.

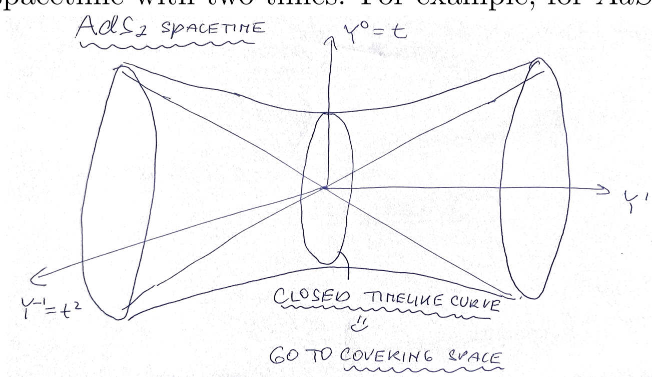
- One can show that

$$R_{abcd} = -\frac{1}{\ell^2}(g_{ac}g_{bd} - g_{ad}g_{bc})\,. \tag{2.47}$$

Thence we solve the Einstein equations $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$ with $T_{ab} = 0$ and

$$\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}\,. \tag{2.48}$$

- In this description the AdS_d can be understood as a hyperboloid embedded in a spacetime with two times. For example, for AdS_2 we have the following picture:



The conformal boundary corresponds to set of lines on the lightcone originating at $0 \in \mathbb{R}^{d-1,2}$: $\eta^{2,d-1}_{AB} Y^A Y^B = 0$ – it is a conformal compactification of $(d-1)$-dimensional Minkowski.

Global coordinates

- Let us now parametrize (solving automatically the constraint)

$$Y_{-1} = \ell \cosh\tilde{\rho} \cos\tilde{t}\,, \quad Y_0 = \ell \cosh\tilde{\rho} \sin\tilde{t}\,, \quad Y_i = \ell \sinh\tilde{\rho}\,\Omega_i\,, \tag{2.49}$$

where Ω_i are angular coordinates that parametrize the S^{d-2} sphere: $\Omega_i^2 = 1$. We then have the AdS in global coordinates:

$$ds^2 = \ell^2\Big(-\cosh^2\tilde{\rho} d\tilde{t}^2 + d\tilde{\rho}^2 + \sinh^2\tilde{\rho} d\Omega^2_{d-2}\Big)\,. \tag{2.50}$$

If we do identify $\tilde{t} \sim \tilde{t} + 2\pi$ and instead take $\tilde{t} \in (-\infty, \infty)$, we have covered the hyperboloid infinitely many times – obtaining so universal covering of AdS. This is well motivated as we do not want to have closed timelike curves.

- To investigate the conformal boundary, we set $\tan\theta = \sinh\tilde{\rho}$, upon which we recover

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(-d\tilde{t}^2 + d\theta^2 + \sin^2\theta d\Omega_{d-2}^2\right), \tag{2.51}$$

where $\theta \in [0, \pi/2)$. Stripping of the conformal factor, we get a (non-Euclidean) cylinder.

- Boundary at $\theta = \pi/2$ is identical to the Einstein static universe $\mathbb{R} \times S^{d-2}$. This is where the field theory lives. The isometries of AdS act on the boundary: send points on the boundary to points on the boundary. This action is simply that of the conformal group in $(d-1)$ dimensions: $SO(2, d-1)$. Thus the field theory is CFT. In particular, the rescaling symmetry (2.56) translates into a dilatation on the boundary. The boundary theory is thus scale invariant and has no dimension full parameter.

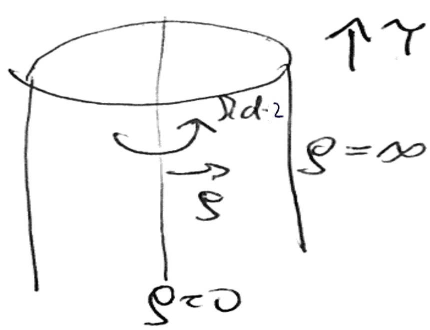
- Setting finally

$$\rho = \ell \sinh\tilde{\rho}\,, \quad \tau = \ell\tilde{t}\,, \tag{2.52}$$

we recover AdS_d in the usual global coordinates

$$\boxed{ds^2 = -f d\tau^2 + \frac{dr^2}{f} + \rho^2 d\Omega_{d-2}^2\,, \quad f = 1 + \frac{\rho^2}{\ell^2}\,.} \tag{2.53}$$

This is manifestly static and spherically symmetric, boundary is located at $\rho \to \infty$:



Poincare AdS

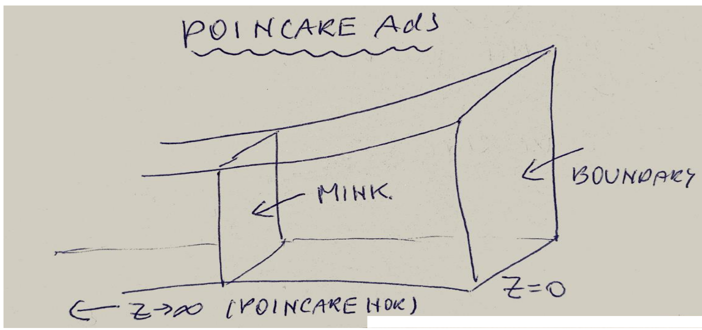
- Defining instead

$$Y_{-1} = \frac{1 + z^2 + \eta_{\mu\nu}x^\mu x^\nu}{2z}\,, \quad Y_\mu = \frac{x_\mu}{z}\,, \quad Y_{d-1} = \frac{1 - z^2 - \eta_{\mu\nu}x^\mu x^\nu}{2z}\,, \tag{2.54}$$

We recover the Poincare coordinates for AdS_d:

$$\boxed{ds^2 = \frac{\ell^2}{z^2}\Big(\underbrace{-dt^2 + d\vec{x}^2}_{\eta_{\mu\nu}dx^\mu dx^\nu} + dz^2\Big)\,.} \tag{2.55}$$

The boundary is now located at $z = 0$. Since $z > 0$, we only cover half of the original hyperboloid. Obviously, the slices of constant z have Poincare symmetry: Poincare AdS = volume filling slices of Minkowski:



Moreover, we clearly see an isometry

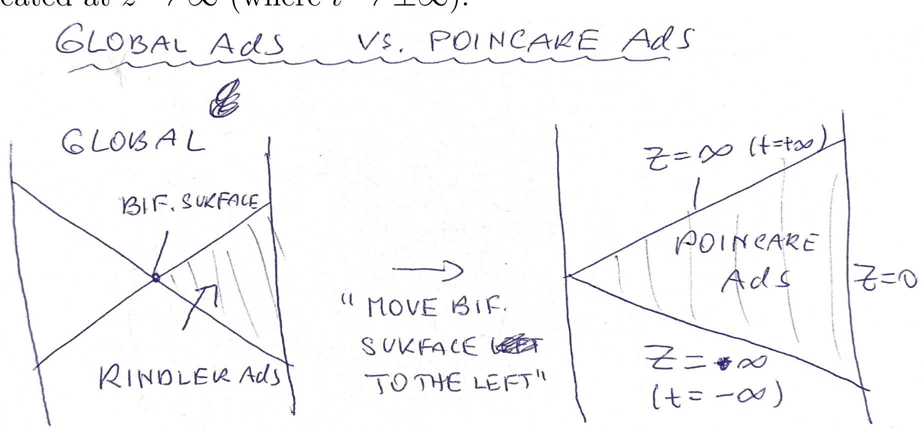
$$(t, \vec{x}, z) \;\to\; \lambda(t, \vec{x}, z)\,. \tag{2.56}$$

In order to continue the metric to the boundary we have to ensure finiteness by multiplying (2.55) by $\Omega^2(z, x^\mu)$. For example, we may choose

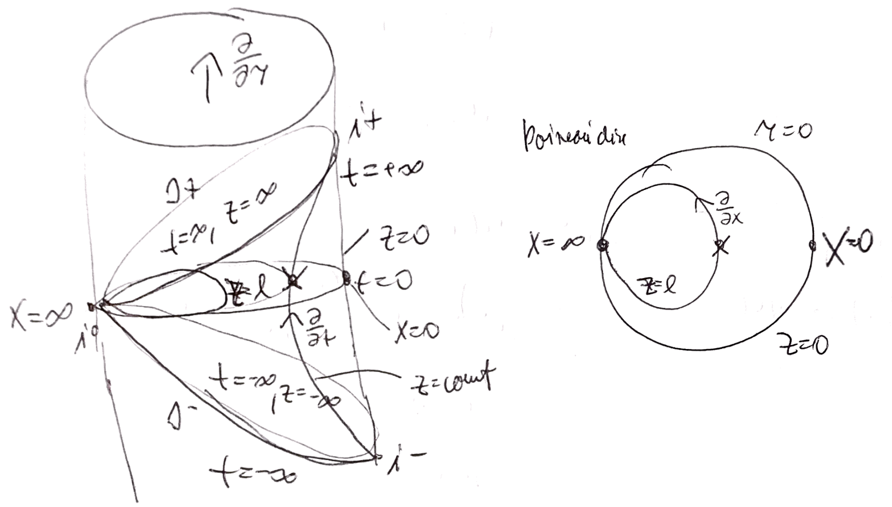
$$\Omega^2 = \frac{z^2}{\ell^2}\omega^2(x^\mu) \quad \Rightarrow \quad ds^2|_{\partial AdS} = \omega^2(x^\mu)\eta_{\mu\nu}dx^\mu dx^\nu\,, \tag{2.57}$$

which is a class of boundary metrics that are related by conformal transformations. This is why we say that the boundary is conformal.

Poincare coordinates are in many respects similar to Rindler coordinates, with ∂_t playing the role of the boost vector. In particular, we have a Poincare horizon located at $z \to \infty$ (where $t \to \pm\infty$):



Sometimes it is useful to consider Poincare disc obtained by setting τ=const. Together with the relation between global and Poincare coordinates this is displayed in the following figure:



Two more coordinate systems

- The following coordinate system is used in brane world scenarios:

$$ds^2 = dr^2 + \ell^2 e^{2r/\ell} \eta_{\mu\nu} dx^\mu dx^\nu \,, \tag{2.58}$$

  which is obtained from Poincare by $z = exp(-r/\ell)$. Here the conformal boundary is at $r \to \infty$ and the horizon at $r \to -\infty$.

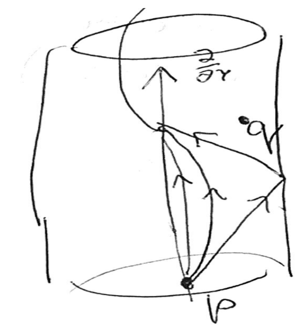
- Fefferman–Graham metric is obtained by setting $\rho = z^2$, to get

$$\boxed{ds^2 = \ell^2 \Big(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} \eta_{\mu\nu} dx^\mu dx^\nu \Big) \,.} \tag{2.59}$$

  This will play a role for the holographic renormalization, and calculation of the holographic stress tensor.

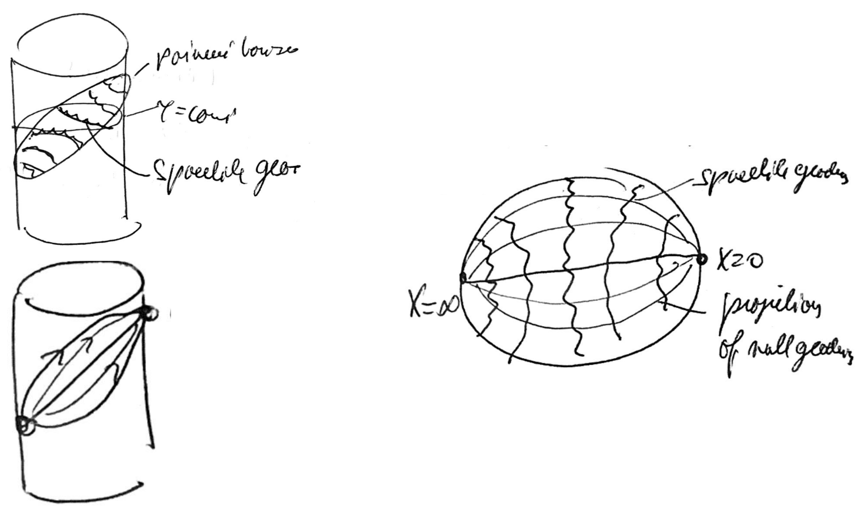
Geodesics

Due to the gravitational pull of AdS toweards its origin, the timelike geodesics oscillate with perior $\tau = 2\pi$ around the origin, as displayed in the following figure:



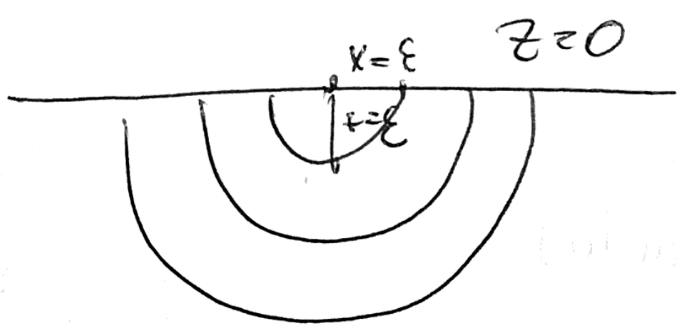
Note that despite p and q being timelike separated, no geodesic connects them (one has to accelerate to get to q).

We may also project the spacelike and null geodesics to the Poincare disc, to obtain:



Scale/radius duality: UV/IR correspondence

As obvious from the following picture: UV CFT (with huge energies able to probe small distances on the boundary) corresponds to IR gravity (large distances from the origin in AdS):



In other words UV cutoff in CFT corresponds to IR (large distance) cutoff in AdS (not sensitive to the interior of the bulk).

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