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AdS/CFT T2
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Tutorial 2: Solutions

1 Nambu–Goto vs. Polyakov

In the lecture we have seen the Polyakov action for massless relativistic strings is given by:

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-h}\Gamma\,, \quad \Gamma = h^{AB}\gamma_{AB}\,. \tag{1}$$

Let us now integrate out the auxiliary metric h_{AB}. This is done as follows. First, we want to write down the EOM for h^{AB} obtained by varying the action. Using an amazing formula (derived in GR courses)

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,, \tag{2}$$

valid for any metric, we have

$$\begin{aligned} -4\pi\alpha'\delta S_P &= \int d^2\xi\delta(\sqrt{-h}h^{AB})\gamma_{AB} = \int d^2\xi\Big(-\frac{1}{2}\sqrt{-h}h_{AB}\delta h^{AB}\Gamma + \sqrt{-h}\gamma_{AB}\delta h^{AB}\Big) \\ &= \int d^2\xi\sqrt{-h}\Big(-\frac{1}{2}h_{AB}\Gamma + \gamma_{AB}\Big)\delta h^{AB}\,. \end{aligned} \tag{3}$$

The corresponding equations of motion are thus:

$$T_{AB} \equiv \gamma_{AB} - \frac{1}{2}h_{AB}\Gamma = 0\,. \tag{4}$$

Taking the determinant of this equation, and using the fact that we are in two dimensions, we get

$$\gamma = \frac{1}{4}\Gamma^2 h \quad \Rightarrow \quad \sqrt{-h} = \frac{2\sqrt{-\gamma}}{\Gamma}\,. \tag{5}$$

Plugging this back in the Polyakov action, we thus recovered the Nambu–Goto action:

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\xi\sqrt{-\det(\gamma_{AB})}\,. \tag{6}$$

The two actions are thus (at least classically) equivalent.

2 Extremal black holes & near horizon limit

There exists an interesting class of black hole solutions of the Einstein–Maxwell theory, given by the so called *Majumdar–Papapetrou solution*:

$$ds^2 = -\frac{1}{H^2}dt^2 + H^2 dx^i dx^i\,, \quad A_\mu dx^\mu = -\frac{1}{H}dt\,, \tag{7}$$

where H obeys $\Delta H = 0$, with Δ being the 3-dimensional Laplace operator in the flat space spanned by x^i, i.e., H is any harmonic function.

a) Obviously, by introducing spherical coordinates on the flat space, $dx^i dx^i = dr^2 + r^2 d\Omega^2$, we can have the following *monopole solution*:

$$ds^2 = -\frac{1}{H^2}dt^2 + H^2(dr^2 + r^2 d\Omega^2)\,, \quad A_\mu dx^\mu = -\frac{1}{H}dt\,, \quad H = 1 + \frac{M}{r}\,. \tag{8}$$

(Recall that the potential $1/r$ solves the Laplace equation in flat space.)

Upon redefining $R = r + M$, we recover the following metric:

$$ds^2 = -f dt^2 + \frac{dR^2}{f} + R^2 d\Omega^2\,, \quad f = \left(1 - \frac{M}{R}\right)^2\,, \tag{9}$$

which is the extremal Reissner–Nordström black hole with $M = Q$.

The horizon is located at $f = 0$, that is

$$R = M \quad \Leftrightarrow \quad r = 0\,, \tag{10}$$

or $H \to \infty$. The asymptotic region then corresponds to $r \to \infty$ or $R \to \infty$.

More generally, and restricting to asymptotically flat solutions ($H \to 1$ as $r \to \infty$), we have

$$H(r) = 1 + \sum_i \frac{M_i}{|\vec{r} - \vec{r}_i|}\,. \tag{11}$$

This corresponds to a stable configuration of charged black holes for which the gravitational pull is balanced by electrostatic repulsion.

b) Consider now a near horizon limit of (8), characterized by

$$r \ll M\,, \tag{12}$$

for which $H \approx M/r$. Here we are deeply in an (infinite) throat of the extremal black hole, and the metric reads

$$ds^2 = -\frac{r^2}{M^2}dt^2 + \frac{M^2}{r^2}dr^2 + M^2 d\Omega^2\,. \tag{13}$$

Introducing a new coordinate $z = M^2/r$, this then yields (up to an overall scale M^2) the direct product of $AdS_2 \times S_2$:

$$\frac{ds^2}{M^2} = \underbrace{\frac{1}{z^2}\left(-dt^2 + dz^2\right)}_{AdS_2} + \underbrace{d\Omega^2}_{S_2}\,. \tag{14}$$

This is precisely how the $AdS_5 \times S_5$ emerges in the AdS/CFT correspondence, upon replacing the 'point-like' extremal Reissner–Nordstrom solution by an extended (extremal) $D3$-brane.

3 Feelings for D-branes

In this problem we study transverse vibrations of classical strings, governed by the following action:

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} dt \int_0^a dx \mathcal{L}(\psi, \dot{\psi}, , \psi_x, t, x)\,, \quad \mathcal{L} = \frac{1}{2}\mu\dot{\psi}^2 - \frac{1}{2}T\psi_x^2\,, \tag{15}$$

where μ is the mass density of the string, and T is its tension. We also denoted by $\psi_x = \partial\psi/\partial x$ and $\dot{\psi} = \partial\psi/\partial t$.

a) Denoting by

$$p^t = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\,, \quad p^x = \frac{\partial \mathcal{L}}{\partial \psi_x}\,, \tag{16}$$

we have the following variation of the action:

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} dt \int_0^a \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\psi}}}_{p^t}\,\delta\dot{\psi} + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_x}}_{p^x}\,\delta\psi_x + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi}}_{0}\,\delta\psi\Big]dx \\ &= -\int_{t_1}^{t_2} dt \int_0^a dx\big[\dot{p}^t + \partial_x p^x\big]\delta\psi + \int_0^a \big[p^t\delta\psi\big]_{t_1}^{t_2}dx + \int_{t_1}^{t_2}\big[p^x\delta\psi\big]_0^a dt\,, \end{aligned}\tag{17}$$

where we have integrated by parts (over t in the first term and over x in the second one), using the fact that the variation δ commutes with the partial derivatives:

$$\delta\dot{\psi} = \frac{\partial}{\partial t}\delta\psi\,, \quad \delta\psi_x = \frac{\partial}{\partial x}\delta\psi\,. \tag{18}$$

In order to impose the action principle, $\delta S = 0$, the three terms that we got must vanish independently!

b) The first (bulk) term describes the motion of the string for $x \in (0, a)$ and $t \in (t_1, t_2)$. Since explicitly we have

$$p^t = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \mu\dot{\psi}\,, \quad p^x = \frac{\partial \mathcal{L}}{\partial \psi_x} = -T\psi_x\,, \tag{19}$$

this gives

$$\dot{p}^t + \partial_x p^x = 0 \quad \Leftrightarrow \quad \psi_{xx} - \frac{1}{v^2}\ddot{\psi} = 0\,, \quad v = \sqrt{T/\mu}\,, \tag{20}$$

which is the wave equation with the corresponding speed of propagation v.

c) The second (time boundary) term is determined by the the string configuration at t_1 and t_2 and corresponds to *initial data*. It is natural to set

$$\delta\psi(t_1, x) = 0\,, \quad \delta\psi(t_2, x) = 0\,, \tag{21}$$

fixing the ‘shape of the string’ at t_1 and t_2, upon which this term vanishes.

d) The last term is related to the evolution of endpoints $\psi(t,0)$ and $\psi(t,a)$ and represents the <u>*boundary conditions*</u>. Two types of boundary conditions are often prescribed:

i) <u>Dirichlet</u> boundary conditions describe *fixed endpoints*:

$$\delta\psi|_a = 0 = \delta\psi|_0 \quad \Leftrightarrow \quad \boxed{\dot\psi|_0 = 0 = \dot\psi|_a\,.} \tag{22}$$

ii) <u>Neumann</u> boundary conditions describe *free* to move endpoints (no friction). This means that $\delta\psi|_0$ and $\delta\psi|_a$ are unconstrained but

$$p^x|_a = 0 = p^x|_0 \quad \Leftrightarrow \quad \boxed{\psi_x|_0 = 0 = \psi_x|_a\,.} \tag{23}$$

e) Let us finally calculate the momentum/energy carried by the string:

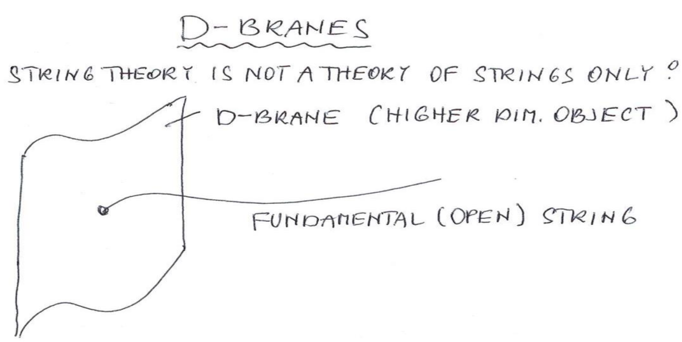
$$P = \int_0^a p^t dx = \int_0^a \mu\dot\psi dx\,. \tag{24}$$

For Neumann boundary conditions P is conserved, since

$$\dot P = \int_0^a \mu\ddot\psi dx = \mu v^2 \int_0^a \frac{d}{dx}\psi_x dx = \mu v^2[\psi_x]_0^a = 0\,. \tag{25}$$

On the other hand, with Dirichlet conditions momentum can exchange with the 'wall" holding the fixed endpoints.

In string theory, the endpoints of open strings may be attached to <u>*D-branes*</u> that can exchange momentum with the string, see the following picture:



and [1] for more details.

References

[1] B. Zwiebach, *A first course in string theory*. Cambridge university press, 2004.