FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

AdS/CFT T3

David Kubiznak

Conformal and Weyl invariance

1 Flat space: conformal transformations

A conformal field theory in 'flat space' is invariant under a transformation of coordinates $x^\mu \to x'^\mu(x)$ so that

$$\eta_{\mu\nu} \;\to\; \Omega^2(x)\eta_{\mu\nu}\,. \tag{1}$$

Let us derive which transformations this includes.

a) Consider infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu\,. \tag{2}$$

Writing $\Omega = 1+\omega$, show that (1) implies the conformal Killing vector equation:

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \alpha \eta_{\mu\nu}\,. \tag{3}$$

What is α?

b) Verify that a solution to the above equation can be written as

$$\xi^\mu = \underbrace{a^\mu}_{\text{translations}} + \underbrace{\omega^\mu{}_\nu x^\nu}_{\text{LT}} + \underbrace{\lambda x^\mu}_{\text{scaling t}} + \underbrace{b^\mu x^2 - 2x^\mu b\cdot x}_{\text{special CT}}\,. \tag{4}$$

c) How many generators do we have in d dimensions? Is this what you would expect from the $SO(d,2)$ symmetry?

2 Curved space: Weyl invariance

a) In GR we may define the energy momentum tensor by the variation of the matter Lagrangian w.r.t. the (curved) metric:

$$\delta S_m[\phi, g_{\mu\nu}] \equiv -\frac{1}{2}\int d^d x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} + \int d^d x\sqrt{-g}\frac{\delta S_m}{\delta\phi}\delta\phi\,. \tag{5}$$

Show that when the action S_m is diffeomorphism invariant and the equation of motion for the matter are satisfied, $\delta S_m/\delta\phi = 0$, such $T_{\mu\nu}$ is inevitably conserved:

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{6}$$

Hint: recall that variations of the metric corresponding to a diffeomorphism generated by vector field ξ^μ ($x^\mu \to x^\mu - \xi^\mu$) are given by $\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$.

b) Consider next the (locally) Weyl invariant theory, that is a theory whose action remains invariant under (note that we only change the metric but not coordinates):

$$g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu} \quad \Rightarrow \quad \delta g_{\mu\nu} = \omega(x)g_{\mu\nu}\,. \tag{7}$$

Show that such a theory has to have traceless energy momentum tensor:

$$T^\mu{}_\mu = 0\,. \tag{8}$$

3 Massless scalar: Weyl vs. conformal

Consider the following (non-minimally coupled) scalar field action:

$$S = -\frac{1}{2}\int d^d x\sqrt{-g}\Big(g^{ab}(\nabla_a\phi)(\nabla_b\phi) + \eta R\phi^2\Big)\,. \tag{9}$$

a) Derive the corresponding EOM for ϕ. What happens in an 'Einstein space', that is in a vacuum with Λ spacetime where we have $R \propto \Lambda$, Λ =const?

b) Using the following identities:

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}\,,\\ \delta(\sqrt{-g}R) &= \sqrt{-g}(G_{ab}\delta g^{ab} + \nabla_a v^a)\,, \quad v^a = g_{cd}\nabla^a\delta g^{cd} - \nabla_b\delta g^{ab}\,,\end{aligned} \tag{10}$$

show that the associated energy-momentum tensor is

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}(\nabla\phi)^2 + \eta\Big(G_{ab} + g_{ab}\nabla^2 - \nabla_a\nabla_b\Big)\phi^2\,. \qquad (11)$$

c) By calculating $T^a{}_a$, show that for the following value of η:

$$\eta = \frac{d-2}{4(d-1)} \qquad (12)$$

the energy momentum tensor is (on-shell) traceless. (The fact that is has a Weyl symmetry can also be shown.) What happens in $d = 2$ dimensions?

d) Think about what happens in the flat space limit. Namely, does a massless scalar in flat space have conformal symmetry? Is its canonical energy momentum tensor traceless? What about the flat space limit of the above energy momentum tensor? What have we learned about the connection between flat space conformal symmetry and the Weyl symmetry in curved space?