Tutorial 3: Solutions

1 Flat space: conformal transformations

A conformal field theory in flat space is invariant under a transformation of coordinates $x^\mu \to x'^\mu(x)$ so that

$$\eta_{\mu\nu} \;\to\; \Omega^2(x)\eta_{\mu\nu}\,. \tag{1}$$

Let us derive which transformations this includes.

a) Consider infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu\,, \tag{2}$$

or equivalently

$$x^\mu = x'^\mu + \xi^\mu\,. \tag{3}$$

Let us also write $\Omega = 1+\omega$. The metric then transforms as

$$\begin{aligned}\Omega^2\eta_{\mu\nu} = (1+2\omega+O(\omega^2))\eta_{\mu\nu} = g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}\\ &= (\delta^\alpha_\mu + \xi^\alpha_{,\mu})(\delta^\beta_\nu + \xi^\beta_{,\nu})\eta_{\alpha\beta}\\ &= \eta_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} + O(\xi^2)\,.\end{aligned} \tag{4}$$

Thus we found a conformal Killing vector equation:

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = 2\omega\eta_{\mu\nu}\,. \tag{5}$$

Note that, we can contract this equation to get

$$2\partial \cdot \xi = 2\omega\delta^\mu_\mu = 2d\omega \quad \Rightarrow \quad \omega = \frac{1}{d}\partial \cdot \xi \tag{6}$$

in d number of spacetime dimensions.

b) The most general solution to the above equation can be written as

$$\xi^\mu = \underbrace{a^\mu}_{\text{translations}} + \underbrace{\omega^\mu{}_\nu x^\nu}_{\text{LT}} + \underbrace{\lambda x^\mu}_{\text{scaling t}} + \underbrace{b^\mu x^2 - 2x^\mu b \cdot x}_{\text{special CT}}, \tag{7}$$

where $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ are the parameters of the Lorentz transformation. Obviously, the first two leave the Minkowski metric invariant and thence solve Eq. (5). The most interesting are the scaling transformations, for which

$$\partial_\mu \xi_\nu = \lambda \eta_{\mu\nu}\,, \quad \partial \cdot \xi = d\lambda\,, \tag{8}$$

and so (5) is also satisfied. I leave up to you to show that the special transformations also solve (5).

c) So how many generators do we have in d dimensions? We have the following parameters:

$$a^\mu\,,\ \omega_{\mu\nu}\,,\ \lambda\,,\ b^\mu\,. \tag{9}$$

This gives

$$d + \binom{d}{2} + 1 + d = \frac{(d+2)(d+1)}{2} = \binom{d+2}{2}\,, \tag{10}$$

which is the same as the number of parameters of the $SO(d,2)$ symmetry.

2 Curved space: Weyl invariance

a) In GR we define the energy momentum tensor by varying the matter Lagrangian w.r.t. the (curved) metric:

$$\delta S_m[\phi, g_{\mu\nu}] \equiv -\frac{1}{2}\int d^dx\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} + \int d^dx\sqrt{-g}\frac{\delta S_m}{\delta\phi}\delta\phi\,. \tag{11}$$

Such tensor is automatically symmetric. Moreover, the prescription can be also used in flat space, upon promoting the metric to a curved one, calculating $T^{\mu\nu}$, and plugging back the Minkowski metric.

For a diffeomorphism invariant matter action we have

$$\delta S_m = 0\,. \tag{12}$$

Moreover, in that case

$$\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}\,. \tag{13}$$

At the same time there is some $\delta\phi$. When the equations of motion for the matter are satisfied, the second term vanishes. Thus, integrating by parts, we have

$$\begin{aligned} 0 &= \delta S_m \propto \int d^d x\sqrt{-g}T^{\mu\nu}\nabla_{(\mu}\xi_{\nu)} = \int d^d x\sqrt{-g}T^{\mu\nu}\nabla_\mu\xi_\nu \\ &= -\int d^d x\sqrt{-g}\nabla_\mu T^{\mu\nu}\xi_\nu\,, \end{aligned} \tag{14}$$

for any diffeomorphism ξ that vanishes on the boundary (so that the corresponding boundary term vanishes) or for $T_{\mu\nu}$'s of compact support. In other words, we have to have

$$\nabla_\mu T^{\mu\nu} = 0 \quad \text{or} \quad \partial_\mu T^{\mu\nu} = 0\,, \tag{15}$$

with the latter valid in flat space.

Let us stress that this is truly true only when the equations of motion for the matter are satisfied! (Diffeomorphism induces some variations of the fields as well and in order the corresponding terms vanish we must ensure that the equations of the motion for the matter are satisfied. For his reason the conservation of $T_{\mu\nu}$ is more or less equivalent to the equations of motion for the matter!)

b) Similarly, for the Weyl invariant theory, we have the following symmetry:

$$g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu} \quad \Rightarrow \quad \delta g_{\mu\nu} = \omega(x)g_{\mu\nu}\,. \tag{16}$$

Thus we have

$$0 = \delta S_m \propto \int d^d x\sqrt{-g}T^{\mu\nu}2\omega g_{\mu\nu} = 2\int d^d x\sqrt{-g}\omega T^\mu{}_\mu\,. \tag{17}$$

Since this must be true for any ω, we must have

$$T^\mu{}_\mu = 0 \tag{18}$$

for energy-momentum tensors of Weyl invariant theories.

3 Massless scalar: Weyl vs. conformal

Consider the following (non-minimally coupled) scalar field action:

$$S = -\frac{1}{2}\int d^dx\sqrt{-g}\Big(g^{ab}(\nabla_a\phi)(\nabla_b\phi) + \eta R\phi^2\Big)\,. \tag{19}$$

a) Varying w.r.t ϕ we recover the following EOM:

$$(\nabla^2 - \eta R)\phi = 0\,. \tag{20}$$

In particular, when we have an Einstein space, we have

$$R \propto \Lambda = \text{const.} \equiv \frac{1}{\eta}m^2\,, \tag{21}$$

we recover the massive Klein–Gordon equation with mass given by the cosmological constant.

b) To calculate the energy momentum tensor, we use the formula (11), that is we have to vary the above Lagrangian w.r.t. the metric. For this we will employ the following 2 identities:

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}\,, \\ \delta(\sqrt{-g}R) &= \sqrt{-g}(G_{ab}\delta g^{ab} + \nabla_a v^a)\,, \quad v^a = g_{cd}\nabla^a\delta g^{cd} - \nabla_b\delta g^{ab}\,, \end{aligned} \tag{22}$$

Obviously, employing the first identity to the first term in Lagrangian, we recover the first two terms in T_{ab}. More interesting is the η term, for which we use the second identity. Namely, we have

$$\delta(\sqrt{-g}R)\phi^2 = \sqrt{-g}(G_{ab}\delta g^{ab} + \nabla_a v^a)\phi^2 \tag{23}$$

We now concentrate on the second term and integrate it by parts, shifting the derivatives onto ϕ^2:

$$\begin{aligned} \sqrt{-g}\nabla_a v^a\phi^2 &= -\sqrt{-g}v^a\nabla_a\phi^2 = -\sqrt{-g}(g_{cd}\nabla^a\delta g^{cd} - \nabla_b\delta g^{ab})\nabla_a\phi^2 \\ &= \sqrt{-g}(g_{ab}\nabla^2\phi^2 - \nabla_b\nabla_a\phi^2)\delta g^{ab}\,. \end{aligned} \tag{24}$$

Combining all terms together, we just recover the following expression for the energy momentum tensor:

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}(\nabla\phi)^2 + \eta\Big(G_{ab} + g_{ab}\nabla^2 - \nabla_a\nabla_b\Big)\phi^2\,. \tag{25}$$

c) Let us now calculate the trace of the above expression. In particular, we have $G^a{}_a = R - \frac{1}{2}Rd = (2-d)/2R$. Thus we have

$$\begin{aligned} T^a{}_a &= \frac{2-d}{2}(\nabla\phi)^2 + \eta\Big(\frac{2-d}{2}R + (d-1)\nabla^2\Big)\phi^2 \\ &= \Big(\frac{2-d}{2} + 2\eta(d-1)\Big)(\nabla\phi)^2 + 2(d-1)\eta\phi\Big(\nabla^2\phi + \frac{2-d}{4(d-1)}R\phi\Big) \,. \end{aligned} \tag{26}$$

Vanishing of the first term requires

$$\eta = \frac{d-2}{4(d-1)} \,. \tag{27}$$

The second term then vanishes on behalf of EOM for ϕ, Thence, the above scalar field enjoys conformal symmetry in curved space for this choice of η. In particular, when $d = 2$, the standard massless theory is Weyl invariant.

d) Let us now consider the flat space limit of the above. First, we recover the standard massless scalar theory in flat space. This is known to have conformal symmetry (check scalings if you are not convinced). One can easily check that the corresponding canonical energy-momentum is not traceless. Conformal symmetry, however, guarantees that one can find an improved energy momentum tensor that will be traceless. One such choice is given by the flat space limit of the above energy-momentum of the Weyl invariant scalar:

$$T_{ab} = \partial_a\phi\partial_b\phi - \frac{1}{2}\eta_{ab}(\partial\phi)^2 + \eta\Big(\eta_{ab}\partial^2 - \partial_a\partial_b\Big)\phi^2 \,. \tag{28}$$

In general, Weyl symmetry is very strong and leads to traceless energy-momentum tensor. In the flat space limit, it yields the conformal symmetry. Since in the former only the metric changes, whereas the latter included the change of coordinates, the fields have different 'conformal weights' in the two cases.