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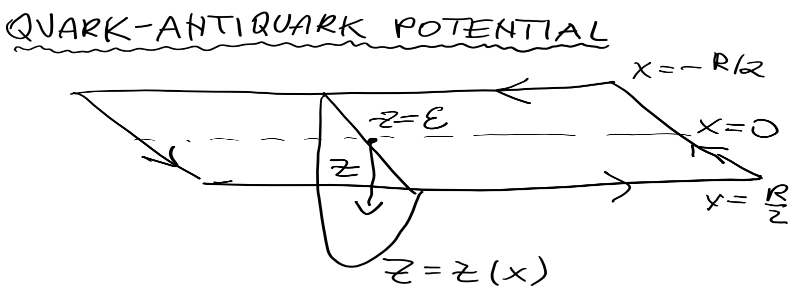
Tutorial 4: Solutions

1 Wilson loops: quark-antiquark potential



Let us consider the above quark-antiquark contour, where we have chosen the separation R in the x-direction, and parametrized $z = z(x)$ in AdS_5 (while fixing the string in S^5).

a) Since the string is fixed on S^5 and so is the direction y in AdS_5, and we choose the coordinates $\sigma^A = (\tau, x)$ to parametrize the strings, that is $z = z(x)$, we have the following induced metric on the string worldsheet:

$$\gamma = \frac{\ell^2}{z^2}\Big(d\tau^2 + (1 + z'^2)dx^2\Big)\,, \tag{1}$$

where $z' = dz/dx$. The corresponding determinant is thus

$$\sqrt{\det \gamma_{AB}} = \frac{\ell^2}{z^2}\sqrt{1+z'^2}\,. \tag{2}$$

Hence, the Nambu–Goto action reads:

$$S_{\rm NG} = \frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma \sqrt{\det \gamma_{AB}} = \frac{T\ell^2}{2\pi\alpha'}\int dx \frac{\sqrt{1+z'^2}}{z^2}\,, \tag{3}$$

where we integrated over τ to yield T.

b) One can think about the above action as a mechanical problem, with the following Lagrangian:

$$L = \frac{\sqrt{1+z'^2}}{z^2}\,, \tag{4}$$

where x plays the role of time. Since L is independent of x, there is a corresponding conserved 'energy':

$$E = \frac{\partial L}{\partial z'}z' - L = -\frac{1}{z^2\sqrt{1+z'^2}}\,. \tag{5}$$

We can invert this, and consider instead the following conserved quantity:

$$z_*^2 = z^2\sqrt{1+z'^2} = \text{const.} \tag{6}$$

Of course, z_* has a physical meaning of the 'largest z' the string can reach in the bulk (where $z' = 0$).

c) The expression for z_* yields

$$\frac{dz}{dx} = \pm\sqrt{(z_*/z)^4 - 1}\,, \tag{7}$$

or

$$\int dx = \pm \int \frac{dz}{\sqrt{(z_*/z)^4-1}}\,, \tag{8}$$

where $\pm$ corresponds to two branches of the string. Using the fact that

$$z(R/2) = z(-R/2) = 0\,, \quad z(0) = z_*\,, \tag{9}$$

we have for example

$$\int_0^{R/2} dx = -\int_{z_*}^{0} \frac{dz}{\sqrt{(z_*/z)^4 - 1}} \,. \tag{10}$$

Finally, introducing the dimensionless quantity $q = z/z_*$, we recover

$$z_* = kR\,, \quad k = \Big(2\int_0^1 \frac{q^2 dq}{\sqrt{1-q^4}}\Big)^{-1} \approx 0.83. \tag{11}$$

d) Let us now plug back the first integral to the action. We have

$$S_{\rm NG} = \frac{T\ell^2 z_*^2}{2\pi\alpha'}\int dx \frac{1}{z^4} = \frac{2T\ell^2 z_*^2}{2\pi\alpha'}\int^{z_*} dz \frac{1}{z^4\sqrt{(z_*/z)^4-1}}\,, \tag{12}$$

Choosing again $z/z_* = q$, we thus have

$$S_{\rm NG} = \frac{2}{z_*}\frac{T\ell^2}{2\pi\alpha'}k_\epsilon\,, \quad \tilde{k}_\epsilon \sim \int_\epsilon^1 \frac{dq}{q^2\sqrt{1-q^4}} \approx \frac{1}{\epsilon} - k_0 + O(\epsilon)\,, \quad k_0 \approx 0.599\,. \tag{13}$$

e) Obviously, the previous result is infinite (contains infinite self-energy of the quarks) and has to be renormalized. This is, for example, done by subtracting $S^0_{\rm NG}$ of two parallel strings hanging between $z = \epsilon z_*$ and $z = \infty$. The corresponding action is given by

$$S^0_{\rm NG} = \frac{2T\ell^2}{2\pi\alpha' z_*}\int_\epsilon^\infty \frac{dq}{q^2} = \frac{2T\ell^2}{2\pi\alpha' z_* \epsilon}\,. \tag{14}$$

Thus we have

$$S_{\rm NG} - S^0_{\rm NG} = -\frac{2}{z_*}\frac{T\ell^2}{2\pi\alpha'}k_0\,, \tag{15}$$

which, when compared to $TV(R)$, yields:

$$V = -\frac{\ell^2}{\alpha'}\frac{k_0}{k\pi}\frac{1}{R} \approx -0.23\frac{\sqrt{2\lambda}}{R}\,. \tag{16}$$

So we arrived at the Coulomb law :)

2 CFT at finite temperature: dimensional analysis

a) Consider a d-dimensional CFT at finite temperature T. Since T is the only dimensionfull parameter, and we have $[T] = 1/\text{length}$, we must have

$$s = \frac{S}{V_{d-1}} = g(\lambda) T^{d-1} \tag{17}$$

for the entropy density s, where $g(\lambda)$ is some function of the 't Hooft coupling λ.

b) To calculate the internal energy, we can use the $d\epsilon = T ds$. Namely, we have

$$\frac{d\epsilon}{dT} = T\frac{ds}{dT} = g(d-1)T^{d-1} \quad \Rightarrow \quad \epsilon = \frac{d-1}{d} g T^d\,. \tag{18}$$

c) In the rest frame we have $\langle T_{\mu\nu}\rangle = \mathrm{diag}(\epsilon, P, P, P)$. Since this must be traceless for any CFT, we have

$$\langle T^\mu{}_\mu \rangle = -\epsilon + (d-1)P = 0 \quad \Rightarrow \quad P = \frac{1}{d-1}\epsilon = \frac{1}{d} g T^d \tag{19}$$

for the CFT pressure P. Similarly, we have the following prediction for the speed of sound:

$$v_s^2 = \frac{\partial P}{\partial \epsilon} = \frac{1}{d-1}\,. \tag{20}$$

d) Finally, we can calculate the free energy density f from the standard relation $f = \epsilon - Ts$:

$$f = \epsilon - Ts = \frac{d-1}{d} g T^d - g T^d = -\frac{1}{d} g T^d = -P\,, \tag{21}$$

which is the standard Euler relation:

$$\epsilon = Ts - P\,. \tag{22}$$