FACULTY
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Selected Topics in AdS/CFT Correspondence

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Abstract

This is a study text for the "Selected topics in AdS/CFT correspondence" course taught at Charles University in 2024/25. The text is based on a number of sources stated below, as well as builds on similar courses delivered by Andrei Starinets, Veronika Hubeny, and myself in previous years. I would like to thank Petr Lukeš for useful comments on previous versions of this text.

- Sources
  - Gauge/gravity duality: Foundations and applications, M. Ammon and J. Erdmenger, Cambridge University Press, 2015.
  - AdS/CFT duality user guide, Makoto Natsuume, Lecture Notes in Physics Vol. 903, Springer, 2015.
  - Holographic entanglement entropy: An overview, T. Nishioka, S. Ryu, T. Takayanagi, J. Phys. A 42 (2009) 504008; ArXiv:0905.0932.
  - The entropy of Hawking radiation, A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, A. Tajdini, Rev. Mod. Phys. 93 (2021) 3, 035002; ArXiv:2006.6872.

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Chapter 1: Prelude: Lessons from black hole thermodynamics

1.1 Black hole thermodynamics

Characteristics of a BH

Let us study some properties of the Schwarzschild black hole:

$$\boxed{ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,,} \qquad \boxed{f = 1 - \frac{2m}{r}\,,} \qquad d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2\,. \tag{1.1}$$

- Asymptotic mass. It corresponds to a conserved energy associated with the asymptotic time translation symmetry, encoded in the following Killing vector field:

$$k = \partial_t\,. \tag{1.2}$$

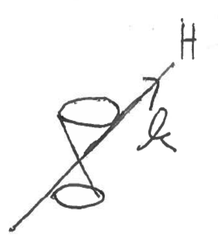
  In asymptotically flat space, the "Gaussian type integral" for the mass is given by the *Komar integral*:

$$M = -\frac{1}{8\pi}\int_{S^2_\infty} *dk = m\,. \tag{1.3}$$

- Surface gravity. The black hole horizon is located at

$$f(r_+) = 0 \quad \Rightarrow \quad r = r_+ = 2M\,. \tag{1.4}$$

  It is the so called *Killing horizon*: a null surface generated by Killing field $k = \partial_t$.



$$k^\mu \nabla_\mu k^\nu \big|_H = \kappa\, k^\nu \big|_H$$

(κ: surface gravity)

  It can be shown that κ reads

$$\kappa = \frac{f'(r_+)}{2} = \frac{1}{2}\frac{2M}{r_+^2} = \frac{1}{4M} = \frac{1}{2r_+}\,. \tag{1.5}$$

Note that this 'coincides' with Newtonian acceleration evaluated on the black hole horizon:

$$\kappa = \frac{M}{r_+^2} = \frac{M}{(2M)^2} = \frac{1}{4M}\,. \tag{1.6}$$

- Horizon area. Taking $dt = 0 = dr$, the induced spatial metric 'on the horizon' is $d\gamma^2 = r_+^2 d\Omega^2$. The horizon area then reads

$$A = \int \sqrt{\det\gamma}\, d\theta d\varphi = \int r_+^2 \sin\theta\, d\theta d\varphi = 4\pi r_+^2\,. \tag{1.7}$$

- Observation: Calculating the following differentials:

$$dM = \frac{dr_+}{2}\,, \quad dA = 8\pi r_+ dr_+\,, \tag{1.8}$$

we find that

$$\boxed{dM = \frac{\kappa}{2\pi}\frac{dA}{4}\,.} \tag{1.9}$$

Laws of Black Hole Mechanics

Bardeen, Carter and Hawking (1973) proved the following 4 laws of black hole mechanics. For a stationary, charged, and rotating black hole with mass M, angular momentum J, and charge Q, we have:

- **Zeroth law:** The surface gravity κ is constant on the black hole horizon.
- **First law:**

$$\boxed{dM = \frac{\kappa}{2\pi}\frac{dA}{4} + \underbrace{\Omega dJ + \Phi dQ}_{\text{work terms}}\,.} \tag{1.10}$$

  Here, Ω is the angular velocity of the hole, and Φ is its 'electrostatic potential'.

- **Second law:** Classically, the area of the horizon never decreases (provided the null energy condition holds).

$$\boxed{dA \geq 0\,.} \tag{1.11}$$

- **Third law:** It is impossible to reduce κ to zero in a finite number of steps.

We would like to compare these to the laws of thermodynamics. In particular, the first law to

$$dE = TdS + \text{work terms}\,. \tag{1.12}$$

However, there is a problem: Classical black holes act as ultimate sponges: no heat can flow out, they are at absolute zero temperature. So we cannot have $\kappa \propto T$.

Black hole thermodynamics

- Wheeler's cup of tea: "If you throw a cup of tea to the black hole, where did its entropy go?" Based on analyzing this question Bekenstein proposed

$$\boxed{S \propto A\,.} \tag{1.13}$$

- Hawking 1974. When quantum effects are taken into account, black holes radiate away as black body with (upon restoring the fundamental units)

$$\boxed{T = \frac{\kappa}{2\pi}\frac{\hbar c^3}{k_B} \quad \Rightarrow \quad S = \frac{A}{4}\frac{c^3 k_B}{\hbar G_N}\,,} \tag{1.14}$$

  where the formula for entropy simply follows from the 1st law.

  Hawking's derivation was based on using the QFT in curved space (approximation of fixed background metric). Hawking basically showed "stimulated emission". The problem with his derivation is that due to the bluehift near the horizon, the test field approximation breaks down and we cannot really trust the result. However, since then the same result has been reproduced by many other approaches, e.g: Euclidean path integral, tunneling, string theory, LQG.

Euclidean Trick

The simplest derivation of the black hole temperature is via the Euclidean trick. Let us repeat it here. It employs the fact that *thermal Green functions* have periodicity in imaginary Euclidean time $\boxed{\tau = it}$:

$$\boxed{G(\tau) = G(\tau + \beta)\,, \quad \beta = 1/T\,.} \tag{1.15}$$

Conversely, periodicity of G defines a thermal state. (It can be shown that a thermometer will register such temperature when interacting with the given field for a long time.) Green functions of quantum fields in the vicinity of black holes have this property (as seen by static observers).

What about gravitational field itself? Let us again focus on the Schwarzschild black hole. First, consider the Euclideanized Schwarzschild ($\tau = it$):

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,. \tag{1.16}$$

Near the horizon we may expand

$$f = \underbrace{f(r_+)}_{0} + \underbrace{(r - r_+)}_{\Delta r}\underbrace{f'(r_+)}_{2\kappa} + \cdots = 2\kappa\Delta r\,. \tag{1.17}$$

Therefore, the near horizon limit of the 'Euclidean Schwarzschild solution' takes the following form:

$$ds^2 = 2\kappa\Delta r d\tau^2 + \frac{dr^2}{2\kappa\Delta r} + r_+^2 d\Omega^2 \,. \tag{1.18}$$

We can now introduce a new coordinate ρ by

$$d\rho^2 = \frac{dr^2}{2\kappa\Delta r} \quad \Leftrightarrow \quad d\rho = \frac{dr}{\sqrt{2\kappa\Delta r}} \quad \Leftrightarrow \quad \Delta r = \frac{\kappa}{2}\rho^2 \,, \tag{1.19}$$

getting

$$ds^2 = \kappa^2\rho^2 d\tau^2 + d\rho^2 + r_+^2 d\Omega^2 = \rho^2 d\varphi^2 + d\rho^2 + \ldots \,. \tag{1.20}$$

upon introducing a new angle coordinate, $\varphi = \kappa\tau$. This looks like a flat space written in polar coordinates, provided the angle φ has a period 2π, otherwise there is a conical singularity at $\rho = 0$, which corresponds to the original black hole horizon. The reasoning now goes as follows: since the black hole horizon was originally non-singular, we expect it to be non-singular again (otherwise we no longer solve vacuum Einstein equations there). This is achieved by setting (we want to avoid conical singularity)

$$\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \tau \sim \tau + \underbrace{2\pi/\kappa}_{\beta} \quad \Leftrightarrow \quad \boxed{T = \frac{\kappa}{2\pi}\,,} \tag{1.21}$$

which is the Hawking temperature. In particular,

$$\boxed{T = \frac{1}{8\pi M}} \tag{1.22}$$

for the Schwarzschild solution.

Rindler space

As you will see in your tutorial, accelerated observers see a thermal bath at a temperature proportional to their acceleration,

$$\boxed{T = \frac{a}{2\pi}\,,} \tag{1.23}$$

which is the famous Unruh temperature. If you accelerate really fast, you can cook a chicken. Using the Euclidean action, one can also calculate the entropy of the associated Rindler horizon, yielding the Bekenstein result for this case as well.

Hawking evaporation

- The Hawking temperature for a Schwarzschild black hole reads

$$T = \frac{\hbar c^3}{8\pi k_B G M} \propto \frac{1}{M} \,, \tag{1.24}$$

smaller the black hole is the hotter it is. For a stellar mass black hole we get about $6 \times 10^{-8} K$, which is much smaller than the CMB temperature – the effect is not important for astrophysics.

We would need a black hole smaller than 4.5×10^{22}kg (size of the Moon) to reach at least the CMB temperature $T \approx 2.7K$.

Obviously, the evaporation accelerates and towards the end we can observe 'black hole explosions' (CERN?)

The black hole looses mass according to the 'effective' Stefan–Boltzmann law

$$\frac{dM}{dt} \propto -\sigma T^4 A \propto -\frac{1}{M^2}\,, \tag{1.25}$$

so that it would completely evaporate in (M_S denoting the mass of the Sun)

$$t_{\text{evap}} \approx \left(\frac{M}{M_\odot}\right)^3 \times 10^{71}\, s\,. \tag{1.26}$$

- Since $T \propto 1/M$, the Schwarzschild black hole has a *negative specific heat*:

  $$C = T\frac{\partial S}{\partial T} = -\frac{1}{8\pi T^2}\,. \tag{1.27}$$

  (It gets colder when mass is absorbed by a black hole and vice versa.)

  This is very strange for ordinary matter, but it is quite typical for self-gravitating systems. For example, a satellite as it falls it increases its kinetic energy; a gravothermal catastrophe described by Lynden Bell.

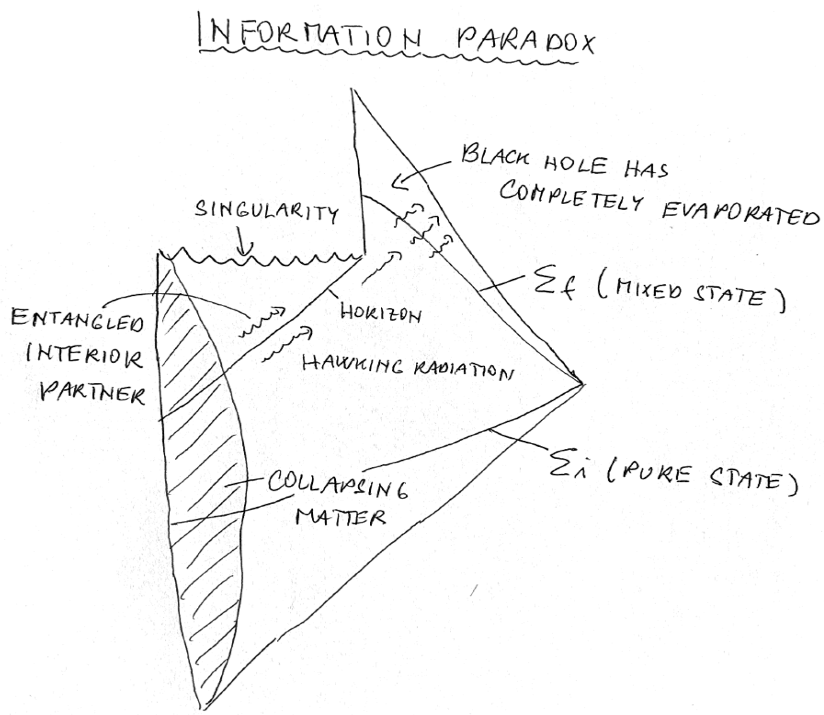
  It also means that the *canonical ensemble* is not well defined for Schwarschild black hole (as no stable thermal equilibrium exists). One way to 'stabilize the black hole system' is to place the black hole in a confining box. A natural such box is provided by the AdS space.

- Hawking radiation is a kinematic effect. (One needs equivalence principle, vacuum fluctuations, but the Einstein equations are not required.) This opens a possibility for observing this effect in '*analogue systems'*, e.g. surface water waves (see Unruh's talk about fishes with ears).

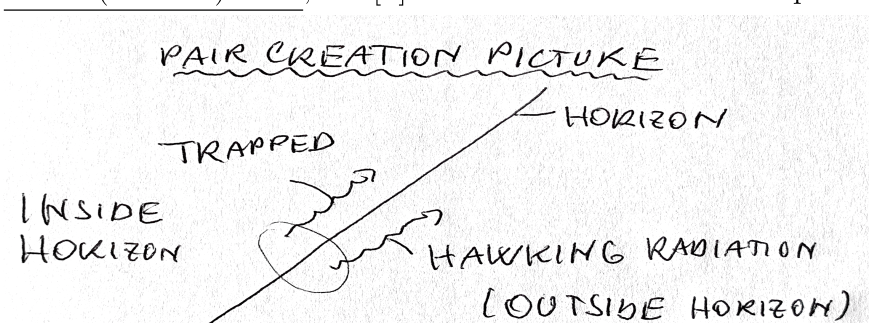
1.2 Black hole information paradox

- Information loss. Classically, black holes absorb interesting stuff (for example an elephant, a star, and so on) but in response only get bigger – the only parameter that changes is the black hole mass – information disappears inside the black hole. (Only finite amount of information is radiated away during the in-fall.) Even quantum mechanically, if Hawking radiation is perfectly thermal, information must be lost inside a black hole.

- More precisely, following Hawking, let us draw the Penrose diagram of a completely evaporating black hole:



In this picture, Hawking radiation can be understood as originating from quantum pair creation of particles. These are entangled and together form a pure state. One of these remains trapped behind the horizon while the other one escapes to infinity as Hawking radiation. From outside, we only see one of them and thence a mixed (thermal) state, see [?] for recent advances on this picture:



Thus, as the black hole has evaporated we evolved from a pure state on Σ_i to a mixed one on Σ_f, violating unitarity of quantum mechanics. This is the famous *black hole information paradox (Hawking 76)*.

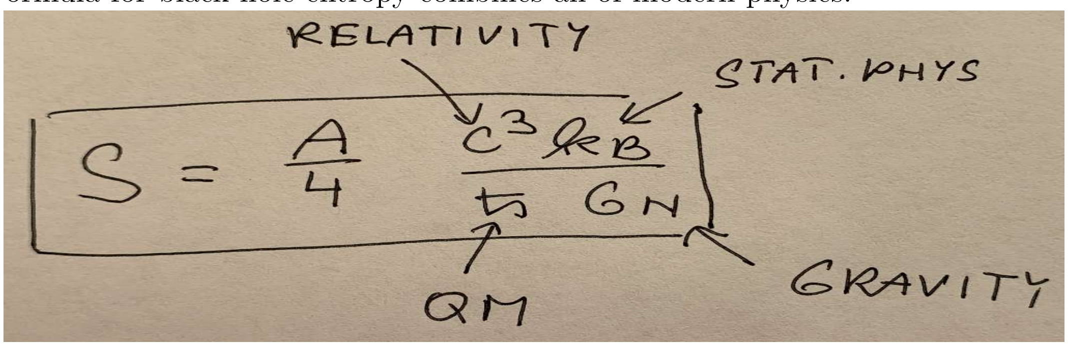
- Various proposals as to what might happen as black hole evaporates were invented: black hole remnants, firewalls, final bursts, black holes do not exist, 'leaking horizons', fuzzballs,... Some of these are in favour of information loss (e.g.

black hole remnants) some are in favour of restoring unitarity, e.g. AdS/CFT correspondence (black hole evaporation is dual to the evolution of the CFT on the boundary which is manifestly unitary).

If the unitarity is to be restored – information has to start coming out of the BH in order to purify Hawking's radiation. This has to happen quite early – around the Page time. We shall return to this problem in later chapters.

1.3 Black hole entropy

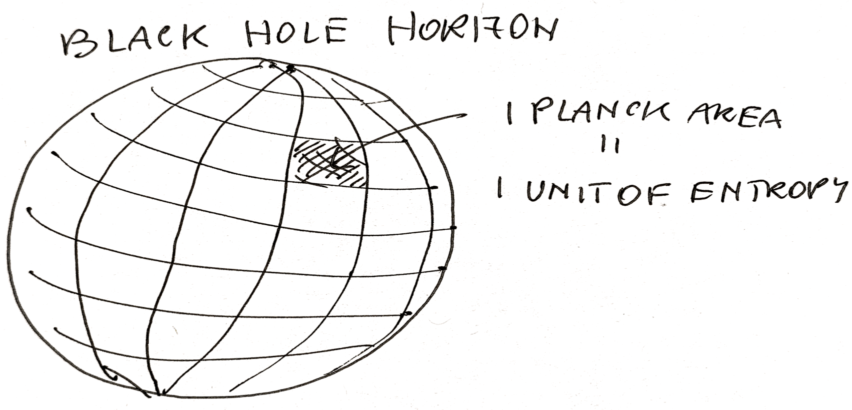
- Formula for black hole entropy combines all of modern physics:



- It can also be written as

$$S = \frac{k_B}{4} \frac{A}{l_P^2}\,, \quad l_P = \sqrt{\frac{G\hbar}{c^3}}\,, \tag{1.28}$$

that is, it is given by measuring the area of the black hole horizon in Planck units – a huge entropy:



- It is holographic, hinting on a holographic duality with a lower-dimensional quantum theory.

- It motivated the Bekenstein 's universal bound on the amount of information contained in a given spatial region with a boundary of area A:

$$\boxed{S \leq \frac{A}{4}\,,} \tag{1.29}$$

as measured in Planck units (see tutorial for a simple derivation of the existence of this bound).

- Generalized second law. During the black hole evaporation, the black hole entropy (area) decreases – Hawking radiation violates the null energy condition. However, the total entropy of the black hole and of the outside Universe should never decrease:

$$\boxed{S_{\text{TOT}} = S_{\text{BH}} + S_{\text{outside}} \geq 0\,.} \tag{1.30}$$

This is the content of the generalized second law of thermodynamics.

- Partition function. One can calculate the gravitational partition function at temperature T in the WKB (semiclassical) approximation as

$$\boxed{Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}\,,} \tag{1.31}$$

where g_c stands for the metric(s) describing the classical solution(s), and the integral is over all metrics periodic in imaginary time with period $\beta = 1/T$. Note that the Euclidean action S_E consists of three kinds of terms: the Einstein–Hilbert action, the York–Gibbons–Hawking term, and counter terms:

$$S_E = \int_\Omega \frac{d^4x\sqrt{g}R}{16\pi G} + \int_{\partial\Omega} \frac{d^3x\epsilon\sqrt{h}\mathcal{K}}{8\pi G} + \text{counter terms}\,, \tag{1.32}$$

where $\epsilon = -1$ for spacelike and $\epsilon = 1$ for timelike boundary. The second (York–Gibbons–Hawking) term is needed to ensure well-posed variational principle with Dirichlet boundary conditions (it kills the unwanted boundary terms in the case of a compact manifold), while the third one is used to 'tune the value' of the action to make it finite and vanishing for the flat space. Here $\mathcal{K}$ stands for the extrinsic curvature and the second and third integrals are boundary integrals (with h_{ab} a boundary metric). Once the partition function is determined for a given solution, we can calculate the corresponding free energy

$$F = -\frac{1}{\beta}\log Z \approx \frac{S_E}{\beta}\,, \tag{1.33}$$

which knows everything about thermodynamics. In particular, the entropy is given by

$$S = -\frac{\partial F}{\partial T}\,. \tag{1.34}$$

One can check that for black holes in Einstein gravity this confirms the area law.

1.4 AdS black holes and their thermodynamics

- Let us now turn to the asymptotically AdS black holes, solutions of the Einstein equations with negative cosmological constant:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \quad \Lambda = -\frac{3}{\ell^2}\,, \tag{1.35}$$

  where ℓ is called the *AdS radius.*

- AdS action. In the AdS case one has a well defined local Euclidean action with unique counter terms (c.f. vague background subtraction of Gibbons and Hawking). Namely, we have the following action:

$$\begin{aligned} S_E &= \frac{1}{16\pi G}\int_M d^4x\sqrt{g}\big(R+\frac{6}{\ell^2}\big) \\ &+\frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}\left[\mathcal{K}-\frac{2}{\ell}-\frac{\ell}{2}\mathcal{R}\,(h)\right]\,, \end{aligned} \tag{1.36}$$

  where $\mathcal{K}$ and $\mathcal{R}\,(h)$ are respectively the extrinsic curvature and Ricci scalar of the boundary. In this expression we have included, apart from the Einstein–Hilbert and York–Gibbons–Hawking pieces, also the standard AdS counter-terms – constructed from the invariants on the boundary [?].

  Varying this action yields the above Einstein equations, together with the following boundary term:

$$\delta S_E = -\frac{1}{2}\int_{\partial\Omega} d^3x\sqrt{-h}\tau_{ab}\delta h^{ab}\,, \tag{1.37}$$

  where

$$8\pi\tau_{ab} = \mathcal{K}h_{ab} - \mathcal{K}_{ab} + \ell G_{ab}(h) - \frac{2}{\ell}h_{ab} \tag{1.38}$$

  is (up to trivial infinite scaling) the holographic stress energy tensor. This gives the expectation value for the energy momentum tensor of the dual CFT_3. Here, the first two terms come from varying the York–Gibbons–Hawking term and you may recognize them as left hand side of the Israel junction conditions. The latter two terms are innate to AdS and come from the corresponding counter-terms.

  Obviously, when δh^{ab} vanishes, $\delta S_E = 0$, and we have a well defined Dirichlet principle (fixed boundary metric).[1]

[1] Alternatively, instead of considering S_E, (1.36), we may consider

$$\tilde{S}_E = S_E + \frac{1}{2}\int_{\partial\Omega} d^3x\sqrt{-h}h^{ab}\tau_{ab}\,. \tag{1.39}$$

This then yields a 'Neumann-type' variational principle [?], where $\delta(\sqrt{-h}\tau_{ab})$ is to be held fixed on the boundary, instead of δh^{ab}. However, as we shall see later, the holographic stress tensor is necessarily traceless, and so really $S_E = \tilde{S}_E$. (In flat space, on the other hand, we return back to the Einstein–Hilbert action – please check!)

- Hawking–Page transition. As shown first by Hawking and Page in 1983, [?], thermodynamics of black holes in AdS space is not only (contrary to that of their asymptotically flat cousins) *well defined* (can have positive specific heat) but it is also rather interesting – for example it features a number of rather intriguing *phase transitions*. Such phase transitions then correspond to the associated phase transitions of the dual CFT.

  Let us show this explicitly for the Schwarzschild-AdS metric (please see Homework 1 for more details):

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2}\,. \tag{1.40}$$

  Using the Euclidan trick, one finds the following black hole temperature and free energy

$$T = \frac{1}{\beta} = \frac{\ell^2 + 3r_+^2}{4\pi\ell^2 r_+}\,, \tag{1.41}$$

$$G = -\frac{1}{\beta}\log Z \approx \frac{S_E}{\beta} = \frac{r_+(\ell^2 - r_+^2)}{4\ell^2}\,. \tag{1.42}$$

  Plotting this parametrically and comparing to the free energy of thermal AdS G_{AdS} (AdS filled with thermal radiation), we get the pictured displayed in Fig. 1.1. The global minimum corresponds to the stable phase. Since $G_{\mathrm{AdS}} \approx 0$, when G becomes negative, the black hole phase dominates the thermal AdS. This approximately happens at

$$T = T_{\mathrm{HP}} = \frac{1}{\pi\ell}\,, \tag{1.43}$$

  the so called *Hawking–Page temperature*. At this temperature there is a first-order phase transition from thermal AdS (which is stable for $T < T_{\mathrm{HP}}$) to large black hole phase (which dominates above T_{HP}), as displayed in figure 1.1:

  Via the AdS/CFT correspondence, this phase transition has an interpretation of the confinement/deconfinement phase transition of the dual quark–gluon plasma [?]. Even more interesting phase transitions occur upon adding rotation and charges to the black hole, see e.g. [?].

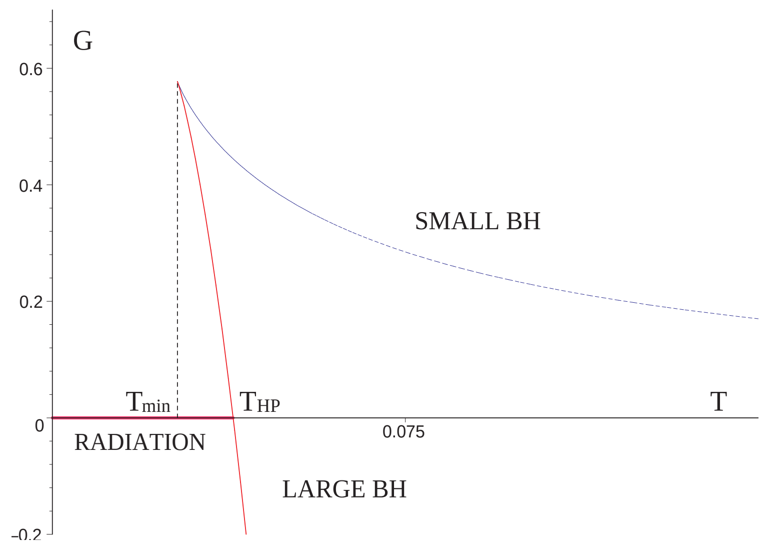


Figure 1.1: *Hawking–Page phase transition.* We display the free energy $G = G(T)$. There is no black hole configuration possible for $T < T_{\min} = \sqrt{3}/(2\pi\ell)$, while two possible branches of black holes are possible for $T > T_{\min}$. The upper branch corresponds to unstable small AdS black holes (with negative heat capacity), whereas the lower branch denotes large thermodynamically preferred black holes (with positive heat capacity). Such black holes become globally stable when G becomes negative, i.e., for $T > T_{HP}$. Hence there is a thermal radiation/black hole transition in the system, called the *Hawking–Page transition.* [Units were chosen such that $G_N = 1$ and $\ell = 1$.]

Chapter 2: Motivating AdS/CFT

The best understood example of the holographic duality is the $AdS_5 \times S_5/CFT_4$ correspondence. In this chapter we will see how this 'emerges' from the results of string theory.

2.1 String theory cartoons

String theory is a quantum theory of interacting relativistic strings and higher-dimensional objects.

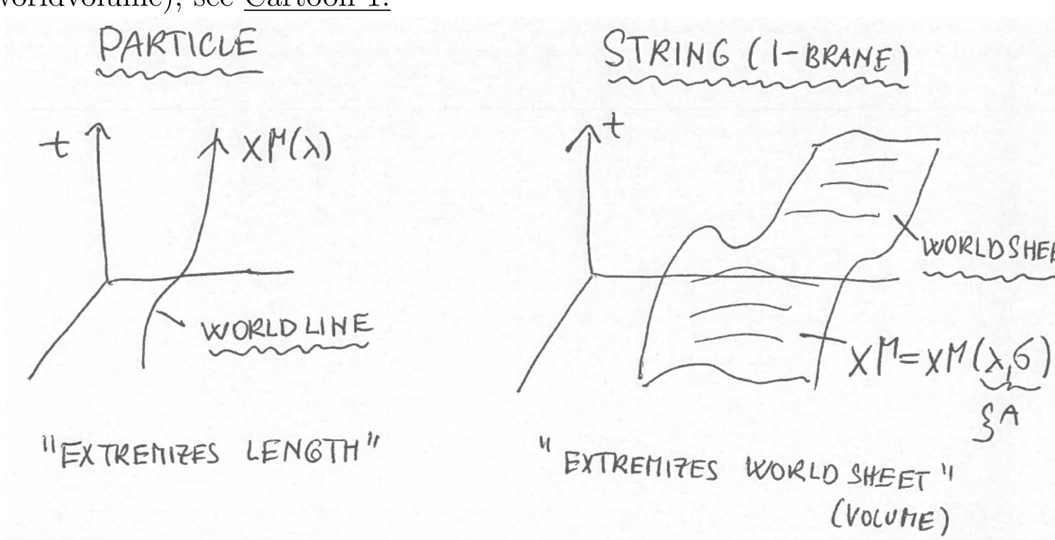
Classical p-branes

- Let us start with a motion of a *free particle*. It is governed by the following action:

$$S[x^\mu] = -m_0 \int d\tau = -m_0 \int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda = \int \sqrt{-\det(\gamma_{\lambda\lambda})}d\lambda\,, \tag{2.1}$$

  where $\gamma_{\lambda\lambda} = \eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ is the 'induced metric' on the worldline, m_0 is particle's rest mass, and τ is its proper time. Upon varying w.r.t. δx^μ, this yields the (flat space) geodesic equation.

- Motion of p-branes. A higher-dimensional object which sweeps a (p+1)-dimensional trajectory in the spacetime is called a p-brane. An example is a string, which is a 1-brane and sweeps a 2-dimensional worldsheet (or more generally p-dimensional worldvolume), see Cartoon 1:



A *world-volume* of a p-brane is described by

$$x^\mu = x^\mu(\xi^A), \quad A = 0, 1, \ldots, p\,, \tag{2.2}$$

where ξ^A are the internal coordinates. One can easily write down the action for a free p-brane. It comes from the following observation. The above standard particle action (2.1) has a very intuitive meaning: "*the motion of a particle is such that it extremizes its proper time*". It is then natural to expect that *the motion of a free p-brane is such that it maximizes the p-brane's worldvolume.* Actions like this are called Nambu–Goto-type actions, generally they are of the form

$$\boxed{S_{\text{NG}}[x^\mu] = -T_p \int \sqrt{-\det\bigl(\gamma_{AB}(\eta)\bigr)}\, d^{p+1}\xi\,.} \tag{2.3}$$

Here, $\gamma_{AB}(\eta)$ is the so called induced metric (metric on the worldvolume induced from the metric of the Minkowski space):

$$\boxed{\gamma_{AB}(\eta) = \frac{\partial x^\mu}{\partial \xi^A}\frac{\partial x^\nu}{\partial \xi^B}\eta_{\mu\nu}\,,} \tag{2.4}$$

and T_p is the p-brane tension. (Note that (2.3) reduces to (2.1) for $p = 0$.)

- Polyakov action. Looking again at the action (2.1), we find that it is pretty complicated: it contains square root (and thence is difficult to quantize) and it does not work for massless particles. To avoid the square root and to make it work even for massless particles, let us consider instead a Polyakov-type action:

$$S[x^\mu, h] = \frac{1}{2}\int (\frac{1}{h}\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - m_0^2 h)d\lambda\,. \tag{2.5}$$

Here $h = h(\lambda)$ is an 'extra field' (or more precisely a Lagrange multiplier). Variation w.r.t. h yields

$$h = \frac{1}{m_0}\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,. \tag{2.6}$$

(Which is a constant provided we identify $\lambda = \tau$.) Plugging this back, we recover the previous Nambu–Goto action – classically, and for $m_0 \neq 0$, the actions are equivalent. At the same time we can take the limit $m_0 \to 0$ and recover the action for a massless point particle (definning $d\tilde{\lambda} = hd\lambda$).

Polyakov-type actions can also be written for p-branes. In particular, for a massless string we have (see tutorial)

$$\boxed{S[x^\mu, h^{AB}] = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-h}h^{AB}\gamma_{AB}\,,} \tag{2.7}$$

where α' is related to the string tension T and the fundamental string length l_s (the only dimensionful parameter of string theory) as follows:

$$\boxed{T = \frac{1}{2\pi\alpha'}\,, \quad \alpha' = l_s^2\,.} \tag{2.8}$$

Again, γ_{AB} is the induced metric (2.4) and h_{AB} is an auxiliary worldsheet metric (an analogue of h above).

Comparing this to the action of the massless scalar field φ coupled to gravitational field h_{AB} : $\mathcal{L} = \sqrt{-h}h^{AB}\nabla_A\varphi\nabla_B\varphi$, we see that the Polyakov action is nothing else than massless scalar field theory (with several scalars x^μ) living in 2-dimensions. This has (in 2-dimensions only) Weyl symmetry which is the cornerstone of string theory.

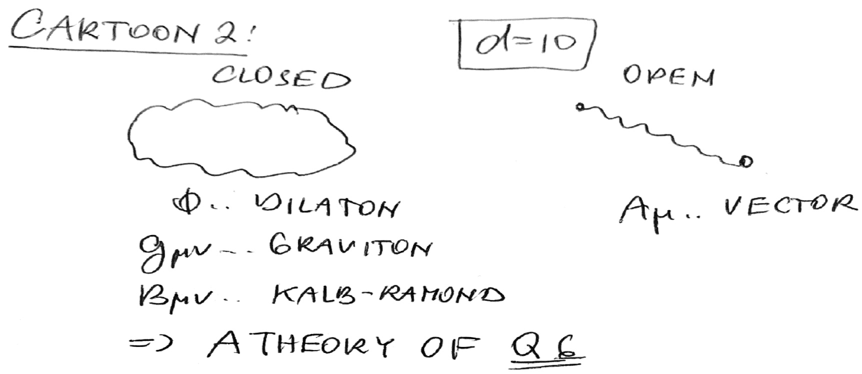
Quantum superstrings

By quantizing the above string action (2.7) (supported by fermionic degrees of freedom in a supersymmetric way) one finds:

- Depending on BC, 5 self-consistent theories (type IIA, IIB, type I, and 2 heterotic ones). These are related by a web of various dualities.
- By requiring the fields with massless polarizations to have vanishing mass, fixes the number of spacetime dimensions to

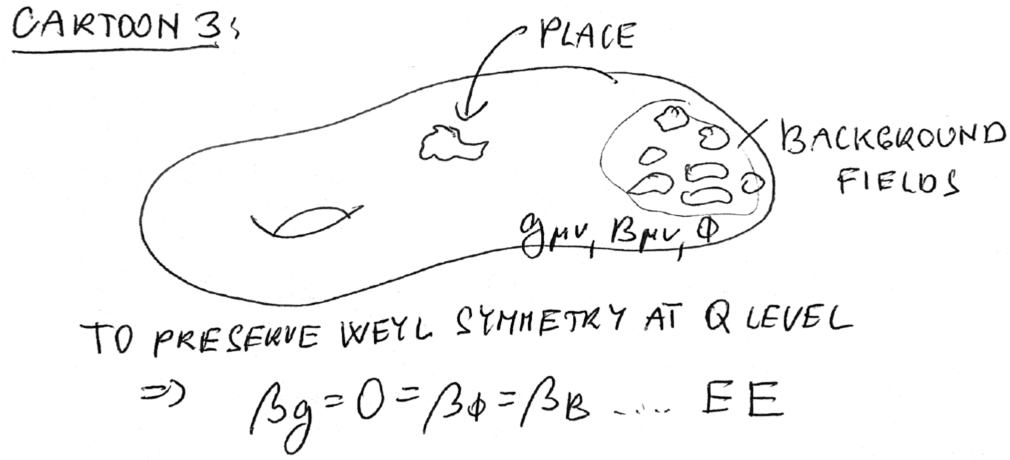
$$\boxed{d = 10\,.} \tag{2.9}$$

- Finite number of massless modes (including graviton for closed strings) and infinite tower of massive modes with $m \propto 1/l_s$. This means that string theory is a theory of quantum gravity:



Note the decomposition of a rank-2 tensor into its symmetric traceless part $g_{\mu\nu}$ with $d(d+1)/2-1$ dof, antisymmetric $B_{\mu\nu}$ with $d(d-1)/2$ dof, and the trace ϕ with 1 dof.

- The fundamental strings can be coupled to non-trivial background of the massless closed string excitations (such as graviton $g_{\mu\nu}$, dilaton ϕ, Kalb–Ramond 2-form $B_{\mu\nu}$, ...):



This is done by replacing in Eq. (2.7):[1]

$$h^{AB}\gamma_{AB}(\eta_{\mu\nu}) \;\to\; h^{AB}\gamma_{AB}(g_{\mu\nu}) + \epsilon^{AB}\gamma_{AB}(B_{\mu\nu}) + \alpha' R_h \phi + \dots \,. \tag{2.10}$$

Moreover, by requiring the conformal symmetry to hold at the quantum level, the corresponding β-functions have to vanish (e.g. $\beta^g_{\mu\nu} \propto \mu\frac{\partial g_{\mu\nu}(x,\mu)}{\partial\mu} = 0$ at 1-loop). Consequently, the background fields have to satisfy a generalization of Einstein equations. This is how Einstein equations emerge in string theory (in the leading order of small α').

For example, for type IIB (theory of closed strings), the vanishing of the β functions can be (secondarily) derived from the following Lagrangian (for fields $g_{\mu\nu}, \phi, A_0, B_{\mu\nu}, A_{\mu\nu}, A_{\mu\nu\kappa\lambda}$):

$$\boxed{\mathcal{L} = *\Big(e^{-2\phi}\Big[R + 4(\partial_\mu\phi)^2 - \frac{1}{2}H_3^2\Big] - \frac{1}{2}F_1^2 - \frac{1}{2}F_3^2 - \frac{1}{4}F_5^2\Big) - \frac{1}{2}A_4 \wedge H_3 \wedge F_3\,,} \tag{2.11}$$

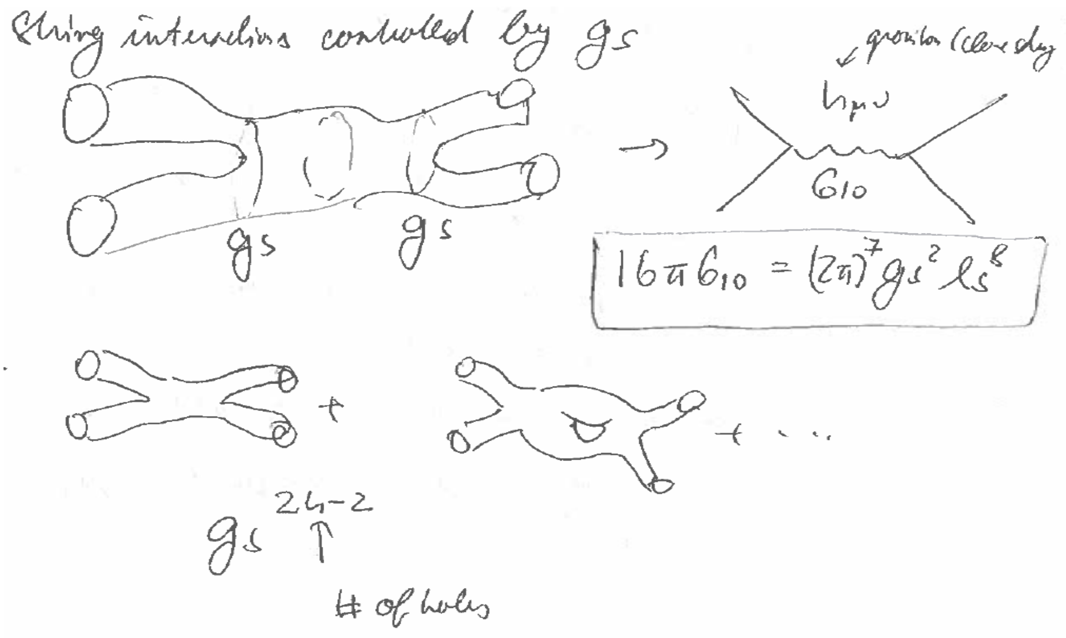
plus fermions. This is the action of the type IIB, $D = 10$, $\mathcal{N} = 2$ SUGRA.

- String interactions are controlled by g_s, related to the 10-dimensional gravitational constant as:

$$\boxed{16\pi G_{10} = (2\pi)^7 g_s^2 l_s^8 \propto l_P^8\,,} \tag{2.12}$$

see the following Cartoon 4:

[1] Please see D. Tong's lectures, Chapter 7, for the explanation as to how the quantum modes of individual strings give rise to the 'classical background' fields.



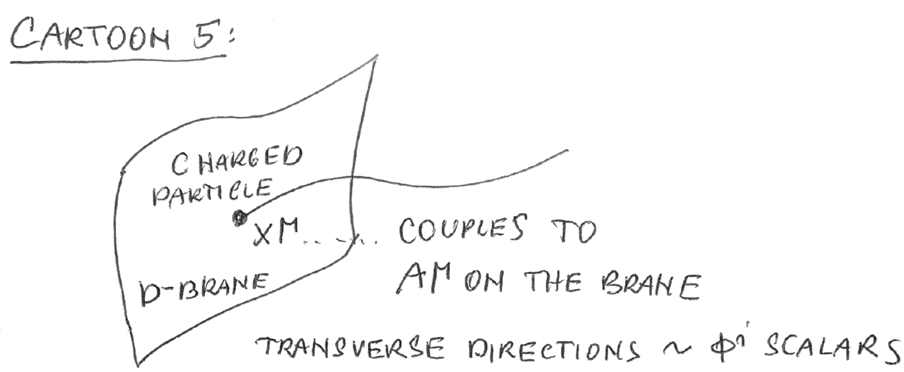
g_s is not really a free parameter, but rather an expectation value of the dilatonic massless mode, $g_s = e^{\phi}$. However, effectively, string theory has 2 parameters:

$$\boxed{l_s \, , g_s \, .} \tag{2.13}$$

D-branes

- Dp-branes are 'topological defects' (non-perturbative in g_s) with (p+1)-dimensional world-volume on which open strings can end (and move freely along the world volume).

  Fluctuations of a Dp-brane are determined by the quantum spectrum of open strings attached to it. At low energy, only the massless modes are of interest. For a single Dp-brane we have a $(p+1)$-dimensional $U(1)$ gauge field $A_\mu(x)$ living on the world-volume of the brane (in the Neumann BC directions), and $\phi^i(x)$ scalars (corresponding to transverse directions to the brane – corresponding to the Dirichlet BC):



$$A_\mu(x):\ \mu=0,\dots,p\,,\qquad \phi^i(x):\ i=1,\dots,9-p\,. \tag{2.14}$$

These are goldstone modes associated with the spontaneous symmetry breaking by the branes.

The motion of a single brane is governed by the Dirac–Born–Infeld action (a generalization of the Nambu–Goto action):

$$S_{\rm DBI}=-T_{Dp}\int d^{p+1}\xi e^{-\phi}\sqrt{-\det(\gamma_{AB}(g)+\gamma_{AB}(B)+2\pi\alpha' F_{AB})}\,, \tag{2.15}$$

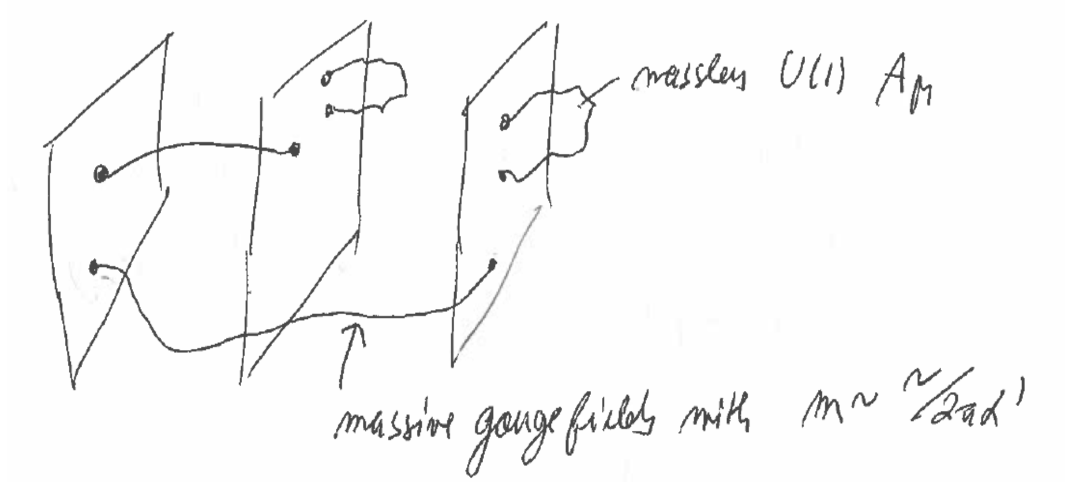
where the D-brane tension is $T_{Dp}=\frac{1}{(2\pi)^p l_s^{p+1}}$. For weak EM fields in flat space, setting $B=0$ and $e^\phi=g_s$, we can then approximate (using $\det(1+A)=1-\frac{1}{2}{\rm Tr}(A^2)+\dots$)

$$S_{\rm DBI}\approx-(2\pi\alpha')^2\frac{T_{Dp}}{4g_s}\int d^{p+1}\xi F_{AB}F^{AB}+O(F^4)\,. \tag{2.16}$$

Thence we can read off the Yang–Mills coupling constant as

$$g_{YM}^2=\frac{g_s}{T_{Dp}(2\pi\alpha')^2}=(2\pi)^{p-2}g_s l_s^{p-3}\,. \tag{2.17}$$

- Multiple branes give rise to Cartoon 6:



In the coalescence limit, $r\to 0$, non-Abelian gauge theory arises, $(A_\mu)^a_b$. (Naively, we have added another internal (geometrical) index indicating where the string ends), see Zweibach [?] and Witten [?].

- N_c coalescent $D3$-branes in type IIB then give rise to the $\mathcal{N}=4$ $U(N_c)$ SYM in $d=4$, with the following field content: A_μ and ϕ^i $(i=1,\dots,6)$ and 4 Weyl fermions in adjoint representation of $U(N_c)$, governed by the following action:

$$\boxed{\mathcal{L}=-\frac{1}{g_{YM}^2}{\rm Tr}\Big(\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}D_\mu\phi^i D^\mu\phi^i+[\phi^i,\phi^j]^2\Big)+\text{fermions}\,,}\quad\boxed{g_{YM}^2=2\pi g_s}\,. \tag{2.18}$$

- Alternatively we may view D-branes as BPS (extremal) massive solitons of $d = 10$ type IIB SUGRA. Let all fields except $g_{\mu\nu}$, F_5, and ϕ =const be zero. This is a self-consistent truncation of the EOMs derived from (2.11), which now become:

$$R_{\mu\nu} = \frac{1}{96} F_{\mu\alpha\beta\gamma\delta} F_{\nu}{}^{\alpha\beta\gamma\delta} \,, \quad F_5 = *F_5 \,. \tag{2.19}$$

(Note that $d * F_5 = 0$ is satisfied as a consequence of Bianchi identity, $dF_5 = 0$ and self duality condition.)

In particular, we find the following *near-extremal* black brane solution:

$$\begin{aligned} ds^2_{10} &= H^{-1/2}\Big[-f dt^2 + dx^2 + dy^2 + dz^2\Big] + H^{1/2}\Big(\frac{dr^2}{f} + r^2 d\Omega_5^2\Big) \,, \\ F_5 &= -\frac{4\ell^2}{H^2 r^5}\sqrt{r_0^4 + \ell^4}(1 + *) dt \wedge dx \wedge dy \wedge dz \wedge dr \,, \\ H &= 1 + \frac{\ell^4}{r^4} \,, \quad f = 1 - \frac{r_0^4}{r^4} \,. \end{aligned} \tag{2.20}$$

Here f is the so called blackening factor, and $r = r_0$ is the horizon. Setting

$$\boxed{f = 1 \,,} \tag{2.21}$$

we recover the BPS solution describing the $D3$-brane. The length ℓ can be determined from the flux of the F_5 (which is quantized and 'counts' number of $D3$-branes):

$$Q = \frac{1}{2G_{10}} \int_{S^5} *F_5 = N_c \mu_3 \propto M \,, \tag{2.22}$$

as we have N_c branes and $\mu_3 = T_{D3}/g_s$, yielding

$$\boxed{\frac{\ell^4}{l_s^4} = 4\pi g_s N_c = 2 g_{YM}^2 N_c = 2\lambda \,, \quad \lambda = g_{YM}^2 N_c \,,} \tag{2.23}$$

the latter know as the 't Hooft coupling.[2] Obviously, it is this effective coupling which 'decides' about the strength of gravitational interaction – whether gravity (closed strings) are important or not. We thus have two opposite limits: that of opens strings (for $\lambda \ll 1$) and that of closed strings (for $\lambda \gg 1$).

[2] That λ is the coupling to consider can easily be seen by looking at the strength of gravitational potential, which for a p-brane goes like (check Schw for $d = 4$ and $p = 0$):

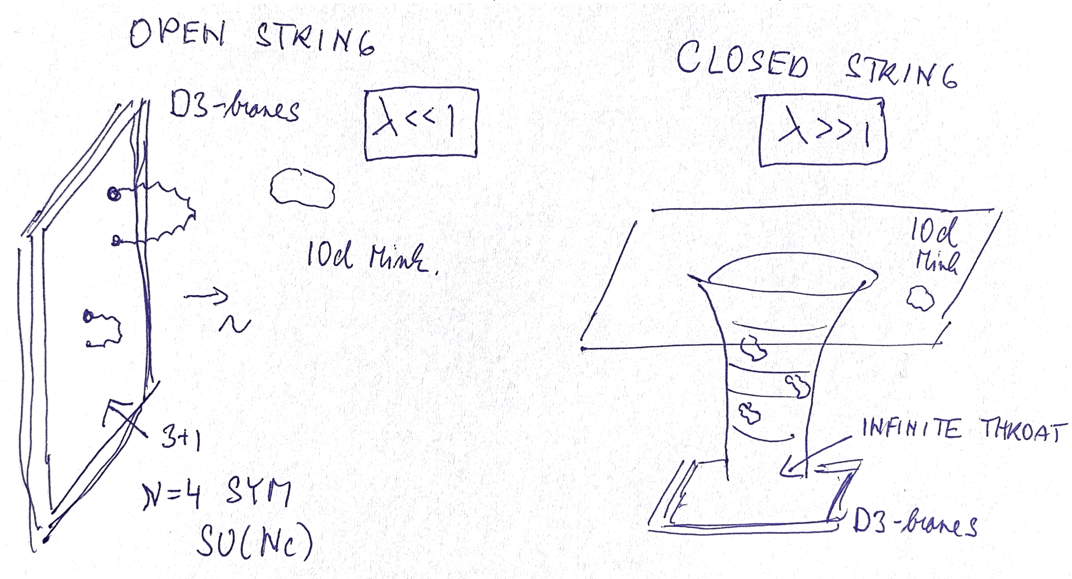
$$\phi \sim \frac{G_{10} M_{\rm Tot}}{r^{d-p-3}} \sim \frac{G_{10} N_c \mu_3}{r^4} \sim g_s N_c \frac{l_s^4}{r^4} \sim \lambda \frac{l_s^4}{r^4} \sim \frac{\ell^4}{r^4} \,, \tag{2.24}$$

with the last two valid for the stack of N_c $D3$ branes. Here we have used that $G_{10} \sim g_s^2 l_s^8$, $\mu_3 \sim 1/(g_s l_s^4)$.

2.2 AdS/CFT conjecture

Two D-brane pictures

Let us consider a stack of N_c $D3$-branes. We can look at them from two perspectives: the 'open string picture' (reliable for $\lambda \ll 1$) and the 'closed string picture' (for $\lambda \gg 1$), as displayed in the following picture (assuming low energy):



- *open string picture.* In this picture the strings are treated as perturbations – we have $\lambda \ll 1$. For low energies $E \ll 1/l_s$ only the massless excitations are relevant – this is how we recover the $\mathcal{N} = 4$ SYM on the brane (plus decoupled SUGRA modes in $d = 10$ Minkowski space). The two parameters describing SYM are:

$$N_c\,, \quad \lambda = g_{YM}^2 N_c\,. \tag{2.25}$$

- *closed string picture.* This picture is valid for strong coupling $\lambda \gg 1$, the D-branes are very massive and described by the SUGRA solution (2.20), (2.21) above:

$$\begin{aligned} ds_{10}^2 &= H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}\Big(dr^2 + r^2 d\Omega_5^2\Big)\,, \\ H &= 1 + \frac{\ell^4}{r^4}\,, \quad \frac{\ell^4}{l_s^4} = 2\lambda\,. \end{aligned} \tag{2.26}$$

We have two different low energy modes: SUGRA modes propagating in 10d Minkowski background ($r \gg \ell$) and full stringy modes propagating deep in the throat of the D-brane $r \ll \ell$ (whose energy is infinitely red-shifted as seen by the asymptotic observer) – these two kinds of modes are completely decoupled. Concentrating on the near horizon region

$$r \ll \ell \tag{2.27}$$

we recover the $AdS_5 \times S_5$ near horizon geometry:

$$ds^2 = \underbrace{\frac{r^2}{\ell^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{\ell^2}{r^2}dr^2}_{AdS_5 \text{ of size } \ell} + \underbrace{\ell^2 d\Omega_5^2}_{S^5 \text{ of size } \ell} \,, \tag{2.28}$$

or, upon setting $z = \ell^2/r$:

$$\boxed{ds^2 = \underbrace{\frac{\ell^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right)}_{AdS_5 \text{ of size } \ell} + \underbrace{\ell^2 d\Omega_5^2}_{S^5 \text{ of size } \ell} \,.} \tag{2.29}$$

Note that the radius of both AdS_5 and S^5 is ℓ; the metric is supported by the F_5 flux through both. Sometimes we can 'smear out' over the S^5 and consider only the geometry of AdS_5. On this side we have two dimensionless parameters:

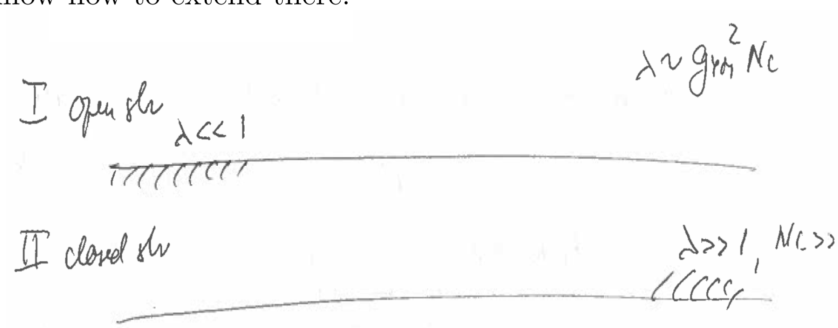
$$g_s, \quad \ell/l_s \,, \tag{2.30}$$

related to the two parameters in the open string picture as

$$\boxed{2\pi g_s = g_{YM}^2 = \frac{\lambda}{N_c} \,, \quad \frac{\ell^4}{l_s^4} = 4\pi g_s N_c = 2\lambda \,.} \tag{2.31}$$

AdS/CFT conjecture

In principle the two descriptions are valid for all values of λ and N_c but we do not really know how to extend there:



We have a conjecture that I and II describe the same object in different languages:

Conjecture (Maldacena 1997). Type IIB superstring theory on $AdS_5 \times S^5$ is *dual* to $\mathcal{N} = 4$ $SU(N_c)$ SYM in $d = (3+1)$ dimensions.

A few remarks:

1. **Duality** means exact equivalence at the full quantum level (Hilbert spaces and the dynamics of the two theories agree). For many dualities one can find a 'change

of variables' in the partition function which directly maps dof of one theory to the dof of the dual theory. Such a map is currently not known for the AdS/CFT.

Remarkably, this duality relates a theory of *quantum gravity* to a quantum field theory in flat space.

Moreover, this duality is *holographic*: relates the dof of gravitational theory in AdS to dof of QFT on its conformal boundary ∂AdS.

2. Classical SUGRA is valid when

   a) *Stringy corrections* are suppressed (strings are almost point like, classical geometry valid):

   $$\ell \gg l_s \quad \Rightarrow \quad \lambda \gg 1\,. \tag{2.32}$$

   b) *Quantum gravity corrections* (loops) are suppressed

   $$1 \gg g_s \sim \frac{\lambda}{N_c}\,. \tag{2.33}$$

   Combining the two, we thus get

   $$\boxed{1 \ll \lambda \ll N_c\,,} \tag{2.34}$$

   for validity of classical gravity in the bulk. (For finite λ and N_c we can do perturbations in $1/N$ and $1/\lambda$).

   Note also that N_c governs the ratio of ℓ/l_P. Namely, since $G_{10} \sim g_s^2 l_s^8$ and $L_P^8 = G_{10}\hbar/c^3$, we have

   $$\frac{\ell^4}{l_P^4} \sim \frac{\lambda l_s^4}{G_{10}^{1/2}} \sim \frac{g_s N_c l_s^4}{g_s l_s^4} \sim N_c\,. \tag{2.35}$$

   Thus, requiring $\ell \gg l_P$ requires $N_c \gg 1$.

3. Strong-weak duality. Taking into account the above, we see that AdS/CFT is an example of *strong-weak* coupling duality: if the field theory is strongly coupled, the dual gravity theory is classical and weak. This provides a tool for studying strongly coupled QFTs.

   The conjecture is supported by case by case evidence and as Veronika says: "it is non-trivial and still holds water".

4. We have $SU(N_c)$ rather than $U(N_c) = SU(N_c) \times U(1)$ SYM. $U(1)$ can be decoupled – describes the motion of the center of mass of the system of N_c branes, which corresponds to *singleton* fields in the gravity theory (only located on the boundary and cannot propagate into the bulk of AdS_5)

5. Symmetries match. On the $AdS_5 \times S^5$ we have $SO(4,2)$ and $SO(6)$. This corresponds to the conformal group in $d = 4$: $SO(4,2)$ (with generators $P_\mu, L_{\mu\nu}, D, K_\mu$) and additional R-symmetry $SO(6)_R$.

   We also have $SL(2,\mathbb{Z})$ duality on both sides ($g_{YM} \to \frac{4\pi}{g_{YM}}$).

6. Various versions. For example *strong version* restricts to classical string theory: $g_s \ll 1$ and ℓ/l_s =const., which implies λ finite and $N_c \to \infty$, known as *'t Hooft limit* (planar limit of the gauge theory). $1/N_c \sim g_s$ expansion then corresponds to genus expansion in string theory.

   The above *weak form* is then obtained by taking $\lambda \to \infty$ and point-like strings ($l_s \ll \ell$).

   More generally, we have *gauge/gravity duality.* Can go to AdS_{d+1}/CFT_d, go beyond CFT, or even go beyond asymptotically AdS.

2.3 Short CFT propaganda

- Conformal field theory is a field theory with additional spacetime symmetries. In fact, we have the following theorem: *Conformal (superconformal) symmetry is the largest admissible symmetry of a non-trivial QFT.*

  CFT in d dimensions is invariant under conformal transformations $SO(2, d)$:

$$\boxed{x^\mu \;\to\; x'^\mu(x) : \quad \eta_{\mu\nu} \;\to\; \Omega^2(x)\eta_{\mu\nu}\,.} \tag{2.36}$$

  These include the Poincare group, the 'special conformal transformations', and the scaling transformation[3]

$$t \to \lambda t\,, \quad \vec{x} \to \lambda\vec{x}\,, \tag{2.37}$$

  under which the field transforms as

$$\phi(x) \;\to\; \phi'(x') = \lambda^{-\Delta}\phi(x)\,, \tag{2.38}$$

  where Δ is the scaling dimension (eigenvalue of the dilatation operator).

- We have the following CFTs in flat space:

  conformal invariant equations

  1) Maxwell equations: $\partial_\mu F^{\mu\nu} = 0$
  2) Massless Dirac equation: $\gamma^\mu \partial_\mu \psi = 0$ $\quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$
  3) classical $\lambda\phi^4$ in $D=4$: $\Box\phi = \frac{\lambda}{3!}\phi^3$
  4) classical YM: $D_\mu F^{\mu\nu} = 0$ $\quad \partial_\mu F^{\mu\nu} - ig[A_\mu, F^{\mu\nu}] = 0$ } not quantum mechanically!

[3] More generally, one can consider $t \to \lambda^z t$ where z is called the *dynamical critical exponent*: $z = 1$ for relativistic theories; $z = 2$ for non-relativistic, e.g. Schrodinger eq. $i\partial_t\psi = -\frac{1}{2}\nabla^2\psi$ (or $E \sim p^2$).

Scale invariant theories have no dimensionful parameter. Usually theories that are scale invariant are also conformal invariant. Conformal field theories have traceless (upgraded) energy momentum tensor:

$$\boxed{T^{\mu}{}_{\mu} = 0\,.} \tag{2.39}$$

- In QFT there are no coupling constants; they depend on energy scale, e.g. $g = g(E)$: we may define the corresponding β function:

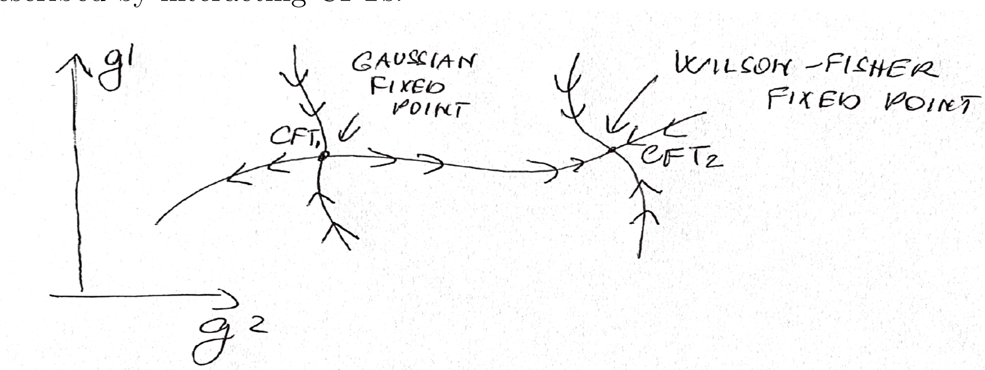
$$\boxed{\beta(g) = \frac{dg}{d\log E}\,.} \tag{2.40}$$

If the conformal symmetry is preserved even quantum mechanically we must have

$$\beta(g) = 0\,. \tag{2.41}$$

This is the case for the $d = 4$ $\mathcal{N} = 4$ SYM.

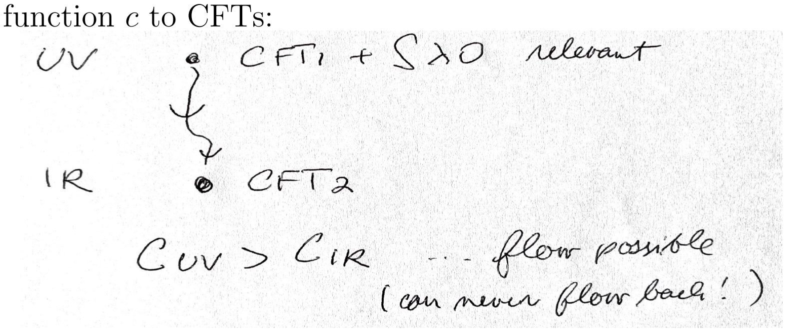
- In the space of QFT we have RG flow: $\beta_I(g^J) = dg_I/d\log E$; fixed points are described by interacting CFTs:



It is useful to think about QFT as a deformation of CFT (think of Higgs):

$$S = S_{\text{CFT}} + \int \lambda O d^d x\,. \tag{2.42}$$

Moreover, CFTs induce an ordering in the space of QFTs by assigning a 'height' function c to CFTs:



- Conformal symmetry restricts significantly the form of correlation functions. For example, for a scalar conformal primary operator O of dimension Δ we have[4]

$$\boxed{\langle O(x)O(y)\rangle \propto \frac{1}{(x-y)^{2\Delta}}\,.} \tag{2.43}$$

- Role of CFTs in nature:
  - Describe critical phenomena (2nd-order PTs) in statistical physics (driven by thermal fluctuations; critical exponents correspond to operator dimensions of fields in CFT: $\{\alpha,\beta,\dots\}\leftrightarrow\{\Delta\}$)
  - Quantum critical phenomena (driven by quantum fluctuations at $T=0$; gapless phase described by CFT)

  We also have applications to *String theory*: described by 2d CFT (determines Einstein equations, YM equations, ...) and *AdS/CFT*.

- A curved space "generalization" of the conformal symmetry is the Weyl symmetry. Contrary to the conformal symmetry, this is an infinite dimensional symmetry. In the flat space limit, the Weyl invariance implies conformal invariance (though not necessarily with the same conformal weight). See tutorial for more information on these two symmetries.

2.4 AdS primer

AdS_d geometry is a maximally symmetric solution of Einstein equations with negative cosmological constant. It has an $O(d-1,2)$ symmetry.

Embedding perspective: AdS as a maximally symmetric space

- One way to understand the AdS_d space is as a (maximally symmetric) hypersurface in higher-dimensional space. Namely, consider the following $(d+1)$-dimensional metric in $\mathbb{R}^{2,d-1}$ and a d-dimensional hyperboloid in it:

$$\begin{aligned} ds^2 &= -dY_{-1}^2 - dY_0^2 + dY_1^2 + dY_2^2 + \cdots + dY_{d-1}^2 \equiv \eta_{AB}^{2,d-1}dY^A dY^B\,, \\ -\ell^2 &= -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + \cdots + Y_{d-1}^2 = \eta_{AB}^{2,d-1}Y^A Y^B\,. \end{aligned} \tag{2.44}$$

  Solving the constraint for Y_{-1} and plugging back to the metric, we obtain a geometry of the AdS space in these coordinates $(a,b=0,\dots,d-1)$:

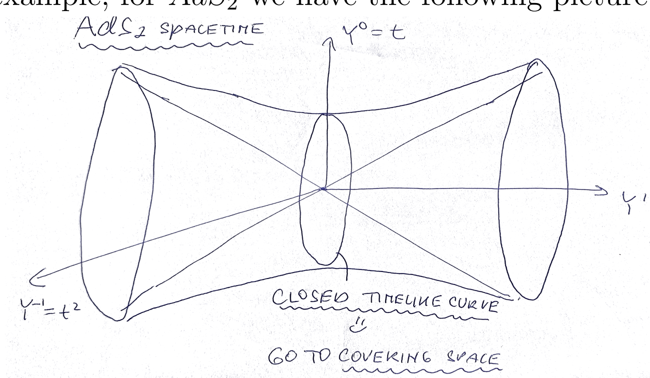
$$\boxed{g_{ab} = \eta_{ab}^{1,d-1} - \frac{Y_a Y_b}{\ell^2 + Y^a Y_a}\,.} \tag{2.45}$$

[4] In a given 'class', *primary operator* is the one with the lowest scaling dimension. *Conformal descendants* of this operator are obtained by taking its derivatives.

- Note that the constraint equations (2.44) are invariant under
$$\tilde{x}^A = \Lambda^A{}_B x^B \quad \text{where} \quad \eta^{2,d-1}_{AB} = \eta^{2,d-1}_{CD}\Lambda^C{}_A\Lambda^D{}_B\,. \tag{2.46}$$
Group matrices $\Lambda^A{}_B$ form a representation of $O\big(d-1,2\big)$. Infinitesimally, we write
$$\Lambda^A{}_B = \delta^A_B + \lambda^A{}_B \quad \Rightarrow \quad \lambda_{AB} = \eta^{1,d-1}_{AC}\lambda^C{}_B = -\lambda_{BA}\ldots \binom{d+1}{2} \text{ generators}\,.$$
Since this is the maximum number of symmetries one can have in d number of dimensions, the spacetime is maximally symmetric.

- One can show that
$$R_{abcd} = -\frac{1}{\ell^2}(g_{ac}g_{bd} - g_{ad}g_{bc})\,. \tag{2.47}$$
Thence we solve the Einstein equations $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$ with $T_{ab} = 0$ and
$$\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}\,. \tag{2.48}$$

- Embedding picture. In this description the AdS_d can be understood as a hyperboloid embedded in a spacetime with two perpendicular timelike directions. For example, for AdS_2 we have the following picture:



Global coordinates

- Let us now parametrize (solving automatically the constraint)
$$Y_{-1} = \ell\cosh\tilde{\rho}\cos\tilde{t}\,, \quad Y_0 = \ell\cosh\tilde{\rho}\sin\tilde{t}\,, \quad Y_i = \ell\sinh\tilde{\rho}\Omega_i\,, \tag{2.49}$$
where Ω_i are angular coordinates that parametrize the S^{d-2} sphere: $\Omega_i^2 = 1$ (constraint). We then have the AdS in global coordinates:
$$ds^2 = \ell^2\Big(-\cosh^2\tilde{\rho}d\tilde{t}^2 + d\tilde{\rho}^2 + \sinh^2\tilde{\rho}d\Omega^2_{d-2}\Big)\,. \tag{2.50}$$
If we do identify $\tilde{t} \sim \tilde{t} + 2\pi$ and instead take $\tilde{t} \in (-\infty, \infty)$, we have covered the hyperboloid infinitely many times – obtaining so universal covering of AdS. This is well motivated as we do not want to have closed timelike curves.

- We may further compactify $\tan\theta = \sinh\tilde{\rho}$, upon which we recover

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(-d\tilde{t}^2 + d\theta^2 + \sin^2\theta d\Omega_{d-2}^2\right), \tag{2.51}$$

  where $\theta \in [0, \pi/2)$. Stripping of the conformal factor, we get a (non-Euclidean) cylinder.

- Conformal boundary of the above cylinder is located at $\theta = \pi/2$ – it is identical to the Einstein static universe $\mathbb{R} \times S^{d-2}$:

$$\boxed{ds^2|_{\partial\Omega} = -d\tilde{t}^2 + d\Omega_{d-2}^2\,.} \tag{2.52}$$

  This is where the field theory lives. The isometries of AdS act on the boundary: send points on the boundary to points on the boundary. This action is simply that of the conformal group in $(d-1)$ dimensions: $SO(2, d-1)$. Thus the field theory is CFT. In particular, the rescaling symmetry (2.57) translates into a dilatation on the boundary. The boundary theory is thus scale invariant and has no dimensionful parameter.

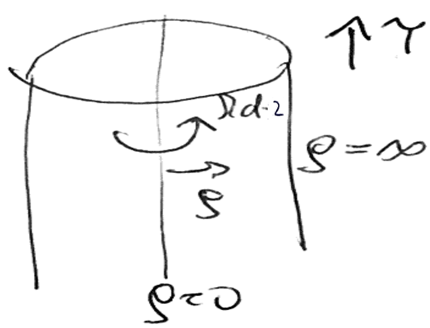
- Setting finally

$$\rho = \ell\sinh\tilde{\rho}\,, \quad \tau = \ell\tilde{t}\,, \tag{2.53}$$

  we recover AdS_d in the 'usual' global coordinates

$$\boxed{ds^2 = -fd\tau^2 + \frac{d\rho^2}{f} + \rho^2 d\Omega_{d-2}^2\,, \quad f = 1 + \frac{\rho^2}{\ell^2}\,.} \tag{2.54}$$

  This is manifestly static and spherically symmetric, boundary is located at $\rho \to \infty$:



Poincare AdS

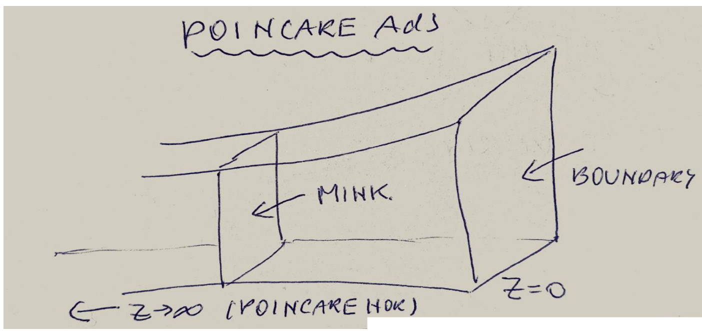
- Defining instead

$$Y_{-1} = \frac{1 + z^2 + \eta_{\mu\nu}x^\mu x^\nu}{2z}\,, \quad Y_\mu = \frac{x_\mu}{z}\,, \quad Y_{d-1} = \frac{1 - z^2 - \eta_{\mu\nu}x^\mu x^\nu}{2z}\,, \tag{2.55}$$

we recover the Poincare coordinates for AdS_d:

$$\boxed{ds^2 = \frac{\ell^2}{z^2}\Big(\underbrace{-dt^2 + d\vec{x}^2}_{\eta_{\mu\nu}dx^\mu dx^\nu} + dz^2\Big)\,.} \tag{2.56}$$

The boundary is now located at $z = 0$. Since $z > 0$, we only cover half of the original hyperboloid. Obviously, the slices of constant z have Poincare symmetry: Poincare AdS = volume filling slices of Minkowski:



Moreover, we clearly see an isometry

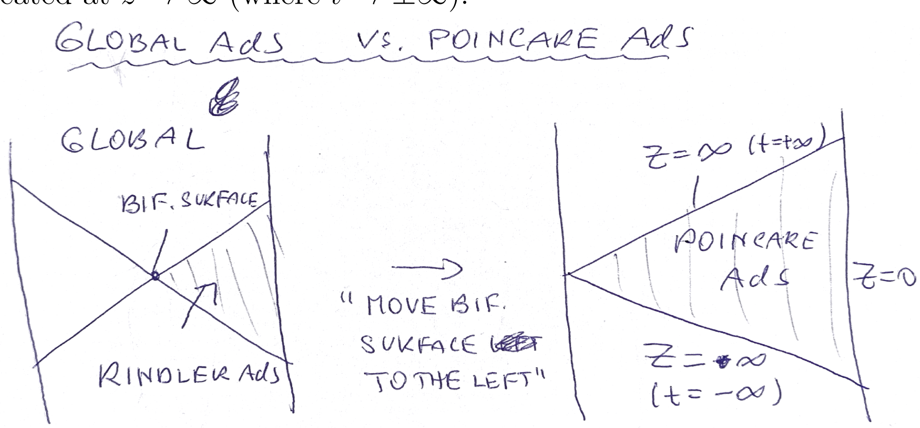
$$(t, \vec{x}, z) \;\rightarrow\; \lambda(t, \vec{x}, z)\,. \tag{2.57}$$

In order to continue the metric to the boundary we have to ensure finiteness by multiplying (2.56) by $\Omega^2(z, x^\mu)$. For example, we may choose

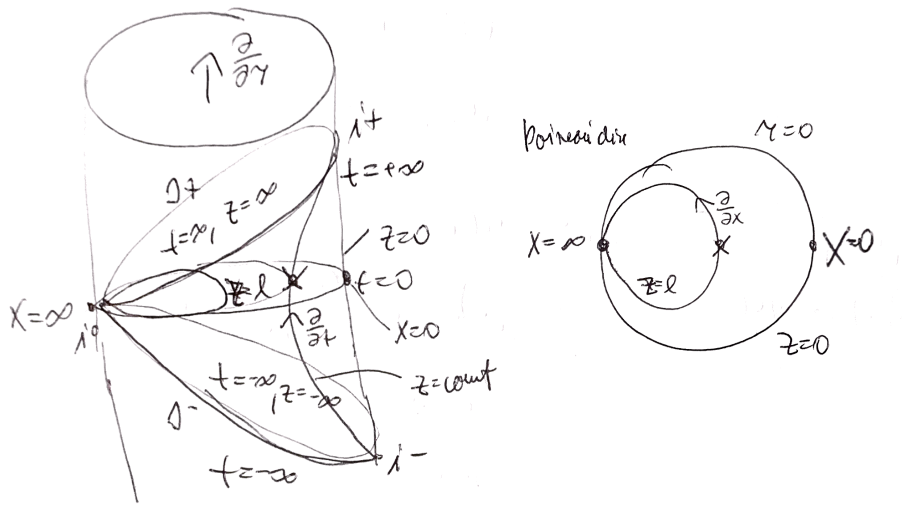
$$\Omega^2 = \frac{z^2}{\ell^2}\omega^2(x^\mu) \quad \Rightarrow \quad ds^2|_{\partial AdS} = \omega^2(x^\mu)\eta_{\mu\nu}dx^\mu dx^\nu\,, \tag{2.58}$$

which is a class of boundary metrics that are related by conformal transformations. This is why we say that the boundary is conformal.

Poincare coordinates are in many respects similar to Rindler coordinates, with ∂_t playing the role of the boost vector. In particular, we have a Poincare horizon located at $z \to \infty$ (where $t \to \pm\infty$):



Sometimes it is useful to consider Poincare disc obtained by setting τ=const. Together with the relation between global and Poincare coordinates this is displayed in the following figure:



Two more coordinate systems

- The following coordinate system is used in brane world scenarios:

$$ds^2 = dr^2 + \ell^2 e^{2r/\ell} \eta_{\mu\nu} dx^\mu dx^\nu \,, \tag{2.59}$$

which is obtained from Poincare by $z = \exp(-r/\ell)$. Here the conformal boundary is at $r \to \infty$ and the horizon at $r \to -\infty$.

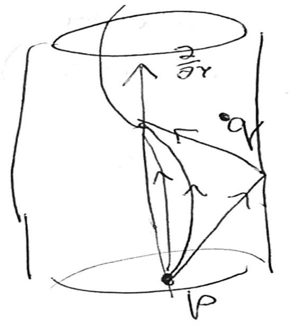
- Fefferman–Graham metric is obtained by setting $z^2 = \rho$, to get

$$\boxed{ds^2 = \ell^2 \Big(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} \eta_{\mu\nu} dx^\mu dx^\nu \Big)} \,. \tag{2.60}$$

This will play a role for the holographic renormalization, and calculation of the holographic stress tensor.

Geodesics

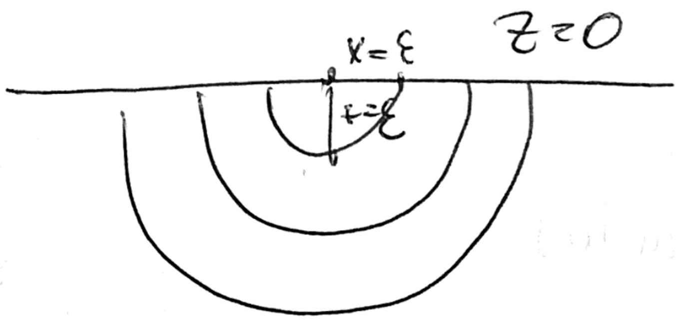
Due to the gravitational pull of AdS towards its origin, the timelike geodesics oscillate with period $\tau = 2\pi$ around the origin, as displayed in the following figure:



Note that despite p and q being timelike separated, no geodesic connects them (one has to accelerate to get to q).

Scale/radius duality: UV/IR correspondence

As obvious from the following picture: UV CFT (with huge energies able to probe small distances on the boundary) corresponds to IR gravity (large distances from the origin in AdS):



In other words UV cutoff in CFT corresponds to IR (large distance) cutoff in AdS: UV CFT behavior is not sensitive to the interior of the bulk.

Chapter 3: Basics of the AdS/CFT correspondence

In the previous chapter we have conjectured the AdS/CFT correspondence:

$$\boxed{Z_{\mathcal{N}=4\text{SYM in } d=3+1}[J] = Z_{\text{Type IIB ST on } AdS_5\times S^5}[J]\,.} \tag{3.1}$$

This is a very strong statement – includes all quantum dof.

For practical calculations we want to weaken the statement and approximate the r.h.s. with the WKB approximation (classical Type IIB SUGRA on $AdS_5 \times S^5$). As discussed previously, this implies $\lambda \to \infty, N_c \to \infty$. So we have:

$$\boxed{Z^{\lambda\to\infty, N_c\to\infty}_{\mathcal{N}=4\,SU(N_c)\,\text{SYM in }d=3+1}[J] \approx \exp\Big(-S_{\text{Type IIB SUGRA on } AdS\times S^5}[J]\Big) + O\Big(\frac{1}{\lambda^{3/2}}, \frac{1}{N_c^2}\Big)\,.} \tag{3.2}$$

However, what plays the role of the source on the gravity side? As we shall try to argue in the following, it is the boundary value of a bulk field.

3.1 State–operator correspondence

- Let us start building dictionary between the two theories: AdS_{d+1} and CFT_d. This is done by 'matching' the symmetries between the field theory operators and string states.

- On the CFT side: operators are characterized by spin and scaling dimension Δ. Remind

$$x \to \lambda x: \qquad O_\Delta \to \lambda^{-\Delta} O_\Delta\,. \tag{3.3}$$

Specifically, primary scalar operator has

$$\langle O_\Delta(x) O_\Delta(y)\rangle \propto \frac{1}{|x-y|^{2\Delta}}\,. \tag{3.4}$$

To find what this corresponds to on AdS side, we "match symmetries".
Example: For a scalar field operator O_Δ, the corresponding bulk field is a massive scalar field ϕ obeying:

$$\Box_{\text{AdS}}\phi = m^2\phi\,, \tag{3.5}$$

with the specific mass:[1]

$$m^2\ell^2 = \Delta(\Delta - d)\,. \tag{3.6}$$

[1] In case of AdS_5/CFT_4 this mass (eigenvalue of $\Box_{Ads}$) originates from a KK reduction of the scalar modes of the 5-form and metric perturbations on S^5, see [?].

Similarly, we map $T_{\mu\nu}$ operator in CFT to AdS metric fluctuations, and so on. This is summarized in the following table:

operator (CFT)	field (AdS)	m vs. Δ
scalar O_Δ	ϕ	$m^2\ell^2 = \Delta(\Delta - d)$
$T_{\mu\nu}$	$h^{\mu\nu}$	$m^2\ell^2 = 0 \;\; \Delta = d$
J_μ	A^μ	$m^2\ell^2 = (\Delta - 1)(\Delta + 1 - d)$

- More concretely, consider AdS in Poincare coordinates (2.56), upon which we have the following action for the scalar ϕ:

$$S = -\frac{C}{2}\int dz d^d x\sqrt{-g}(g^{ab}\partial_a\phi\partial_b\phi + m^2\phi^2)\,, \tag{3.7}$$

where $m^2\ell^2 = \Delta(\Delta - d)$ and $C \propto N_c^2$.[2] This yields the following Klein–Gordon equation (c.f. Tutorial):

$$(\Box - m^2)\phi = 0\,, \quad \Box = \frac{1}{\ell^2}\Big(z^2\partial_z^2 - (d-1)z\partial_z + z^2\eta^{\mu\nu}\partial_\mu\partial_\nu\Big)\,. \tag{3.8}$$

Considering a plane wave ansatz $\phi(z,x) = \phi_k(z)\exp(ik_\mu x^\mu)$, this yields

$$z^2\partial_z^2\phi_k - (d-1)z\partial_z\phi_k - (m^2\ell^2 + k^2 z^2)\phi_k = 0\,. \tag{3.9}$$

Close to the boundary, $z \to 0$, we have two independent solutions

$$\phi_k \sim \begin{cases} z^{\Delta_+} & \text{normalizable}\,, \\ z^{\Delta_-} & \text{non-normalizable}\,, \end{cases} \tag{3.10}$$

where

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2} \tag{3.11}$$

are the roots of (3.6). Thus, near boundary we have

$$\boxed{\phi \sim \phi_0(x)z^{\Delta_-} + \phi_+(x)z^{\Delta_+} + \ldots\,.} \tag{3.12}$$

We have

$$\Delta_- = d - \Delta_+\,. \tag{3.13}$$

Typically, we associate Δ_+ with the dimension of the CFT operator:

$$\boxed{\Delta = \Delta_+\,.} \tag{3.14}$$

Normalizable mode ϕ_+ is then an expectation value for the dual scalar field operator O_{Δ_+}: $\langle O_{\Delta_+}\rangle = \phi_+$, and non-normalizable mode ϕ_0 is a source for this operator. That is for an operator with dimension Δ we have the corresponding source:

$$\boxed{\phi_0 = \lim_{z\to 0}\phi(z,x)z^{\Delta - d}\,.} \tag{3.15}$$

[2] We have $G_{10} \sim g_s^2\ell^8 l_s^8/\ell^8 \sim g_s^2\ell^8/(N_c^2 g_s^2) \sim \ell^8/N_c^2$. From here $G_5 \sim G_{10}/\mathrm{Vol}(S^5) \sim \ell^3/N_c^2$.

- Breitenlohner–Freedman (BF) bound. In flat space, fields with negative m^2 have an upside down potential and are unstable. In AdS, however, small negative m^2 is not a problem. Namely, when

$$\boxed{m^2 \geq m_{\rm BF}^2 = -\frac{d^2}{4\ell^2}} \tag{3.16}$$

the field is still stable. To prove this, let us consider the action (3.7) for $\phi = \phi(z)$, upon which we recover

$$S \sim \int dz d^d x \frac{1}{z^{d+1}} \left(z^2 \partial_z \phi \partial_z \phi + m^2 \ell^2 \phi^2 \right) . \tag{3.17}$$

(For normalizable modes this action is finite near $z = 0$.) Let's now set $\phi = z^{d/2}\varphi$, and $y = \log z$, upon which we have

$$S \propto \int dy d^d x \Big(\partial_y \varphi \partial_y \varphi + \underbrace{[m^2\ell^2 + d^2/4]}_{m_{\rm eff}^2} \varphi^2 \Big) , \tag{3.18}$$

which looks like a scalar field in flat space with the above effective mass – yielding $m_{\rm eff}^2 \geq 0$ for stability.

- Remark: In the following range:

$$-\frac{d^2}{4} < m^2\ell^2 \leq -\frac{d^2}{4} + 1 , \tag{3.19}$$

the identification of the source and expectation value can be interchanged.

3.2 Correlation functions

- We have already established that there is a 1-1 correspondence between the CFT operator O and gravity field ϕ. Within this correspondence, the boundary value ϕ_0 plays the role of the source of O. Namely, on the CFT side we can calculate the (Euclidean) generating function as[3]

$$Z[\phi_0] = \int D\phi e^{-S_E + \int d^d x \phi_0(x) O(x)} = \langle e^{\int \phi_0 O} \rangle \equiv e^{-W[\phi_0]} , \tag{3.20}$$

and the connected correlation functions are then obtained by (think of W as a free energy!)

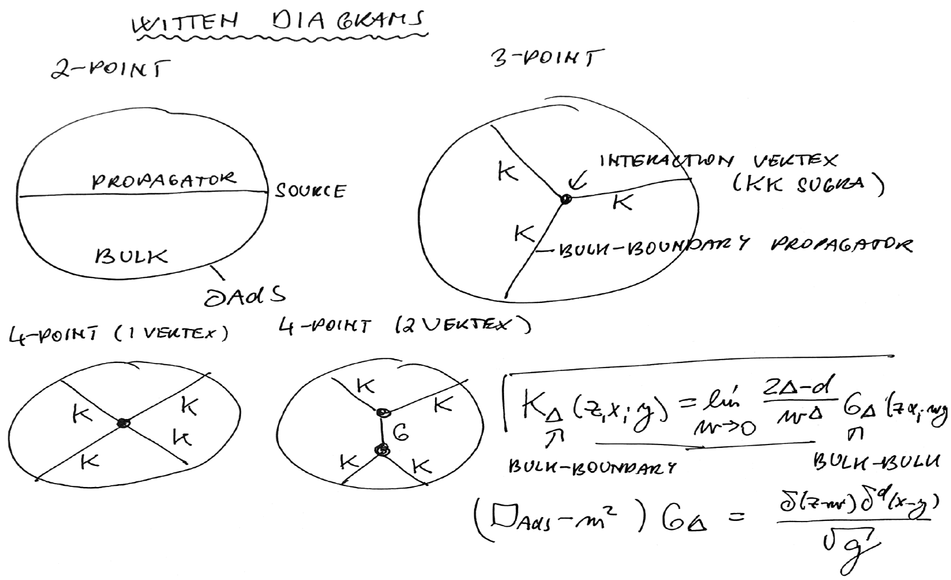
$$\langle O(x_1) \dots O(x_n) \rangle = - \frac{\delta^n W}{\delta\phi_0(x_1)\dots\delta\phi_0(x_n)} \Big|_{\phi_0 = 0} . \tag{3.21}$$

[3]Note that 1-pt expectation values may be non-trivial in the presence of the corresponding source, or in thermal state.

The AdS/CFT dictionary then prescribes:

$$\boxed{W[\phi_0] = S_{\text{SUGRA}}[\phi]\Big|_{\lim_{z\to 0}\phi(z,x)z^{\Delta-d}=\phi_0(x)}}\,. \tag{3.22}$$

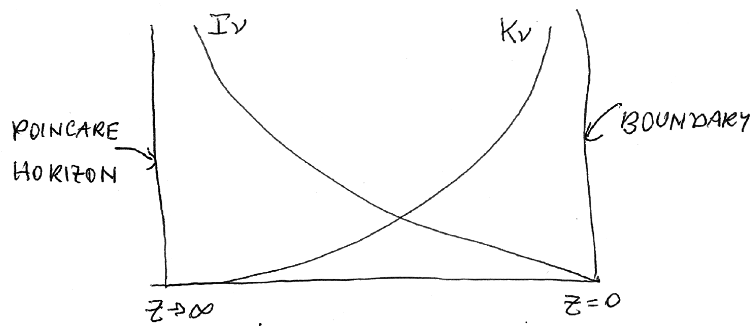
- Thus we have the following recipe: To calculate the correlation functions for operator O with dimension Δ we:
  1. Determine bulk ϕ dual to O
  2. Solve (KK reduced) SUGRA EOM for ϕ subject to boundary conditions $\phi(z,x) \sim z^{d-\Delta}\phi_0(x)$ for $z \to 0$.
  3. Plug this back to the action (obtain the Hamilton's function), whose derivatives w.r.t. ϕ_0 yield the correlation function via (3.21).
- To evaluate (3.22), one needs to calculate tree level diagrams on the gravity side – known as Witten diagrams:



- Example: 2-point function. We want to calculate the 2-point function $\langle O(x)O(y)\rangle$ of a scalar operator O of dimension Δ.
  Step 1: We identify the corresponding bulk field with the test scalar field described by the action (3.7).
  Step 2: We need to solve the corresponding EOM (3.9). This is a Bessel equation and has the following solution:

$$\phi(z,k) = A_k z^{d/2} K_\nu(z|k|) + B_k z^{d/2} I_\nu(z|k|)\,, \quad \nu = \Delta - d/2 = \sqrt{d^2/4 + m^2\ell^2}\,. \tag{3.23}$$

Schematically, they behave as follows:



Regularity on the Poincare horizon ($z \to \infty$) implies $B_k = 0$ and so $\phi(z,k) \to 0$ in the interior boundary. At the same time, $K_\nu(z) \sim z^{-\nu}$ for $z \to 0$ and thence

$$\text{as } z \to 0: \qquad \phi(z,k) \sim z^{d-\Delta} A_k\,. \tag{3.24}$$

Thence A_k is related to ϕ_0. Thus we have the following normalized solution:

$$\phi(z,x) = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \phi(z,k)\,, \quad \phi(z,k) = \frac{z^{d/2} K_\nu(z|k|)}{\epsilon^{d/2} K_\nu(\epsilon|k|)} \phi_0(k) \epsilon^{d-\Delta}\,. \tag{3.25}$$

Step 3. Let us plug the obtained solution back to the action (3.7). On shell, this action reduces to

$$S = -\frac{C}{2} \int d^d x \underbrace{\sqrt{g} g^{zz}}_{(\ell/z)^{d-1}} \phi(z,x) \partial_z \phi(z,x) \Big|_{z=\epsilon}^{z=\infty}\,, \tag{3.26}$$

where we have integrated by parts, and thrown away the boundary terms for large x, assuming the field vanishes sufficiently quickly at 'spatial infinity'. Moreover, K_ν vanishes at $z \to \infty$, and we ma also drop the upper limit. Thus, the action (3.26) now reads

$$S[\phi_0] = -\frac{C\ell^{d-1}}{2\epsilon^{d-1}} \int \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} (2\pi)^d \delta(p+k) \phi(z,k) \partial_z \phi(z,p) \Big|_{z=\epsilon}\,. \tag{3.27}$$

Using finally that $\int d^d x O(x) \phi_0(x) = \int \frac{d^2 k}{(2\pi)^d} O(k) \phi_0(-k)$, we have

$$\begin{aligned} \langle O(k) O(p) \rangle_\epsilon &= -(2\pi)^{2d} \frac{\delta^2 S[\phi_0]}{\delta\phi_0(-k)\delta\phi_0(-p)} = \ldots \\ &= -\frac{(2\pi)^d \delta^d(k+p) C \ell^{d-1}}{\epsilon^{2\Delta-d}} \Big(\frac{d}{2} + \frac{\epsilon|k| K'_\nu(\epsilon|k|)}{K_\nu(\epsilon|k|)} \Big)\,. \end{aligned} \tag{3.28}$$

In order to do the expansion of K_ν for small ϵ (which depends on ν) we consider the case when ν is a positive integer, in which case

$$\begin{aligned} \langle O(k) O(p) \rangle_\epsilon \;\propto\; & \delta(k+p) \Big[\frac{\#}{\epsilon^{2\Delta-d}} + \frac{\#}{\epsilon^{2\Delta-d-2}} + \cdots + \# \log\epsilon \\ & + \# |k|^{2\nu} \log(|k|) + O(\epsilon^2) \Big]\,. \end{aligned} \tag{3.29}$$

The divergent terms in the first line are scheme dependent and will be removed by holographic renormalization; only the first term in the second line contributes. Going back to the position space, one can recover that (if you do not believe it try!)

$$\langle O(x)O(y)\rangle \propto \frac{1}{|x-y|^{2\Delta}}\,, \tag{3.30}$$

which agrees with (3.4).

3.3 Holographic renormalization

This is a systematic method for dealing with near boundary divergences – you will study this in detail in your homework.

Scalar field

- As a toy example, we start with a scalar field, employing the (Euclidean) Fefferman–Graham coordinates (with boundary located at $\rho = 0$):

$$ds^2 = g_{ab}dx^a dx^b = \ell^2\Big(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\delta_{\mu\nu}dx^\mu dx^\nu\Big)\,. \tag{3.31}$$

We are interested in the scalar field action and EOM:

$$S_0 = \frac{C}{2}\int d\rho d^d x\sqrt{g}\Big(g^{ab}\partial_a\phi\partial_b\phi + m^2\phi^2\Big)\,, \tag{3.32}$$

$$(\Box - m^2)\phi = 0\,,\quad \Box\phi = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b\phi)\,. \tag{3.33}$$

- To solve the EOM, we expand ϕ around the boundary:

$$\boxed{\phi(\rho,x) = \rho^{(d-\Delta)/2}\Big(\underbrace{\phi_0(x) + \rho\phi_2(x) + \rho^2\phi_4(x) + \dots}_{\varphi(\rho,x)}\Big)\,.} \tag{3.34}$$

EOM then yields (see tutorial)

$$[\Delta(\Delta-d) - m^2\ell^2]\varphi + \rho\Box_0\varphi + 2(d-2\Delta+2)\rho\partial_\rho\varphi + 4\rho^2\partial_\rho^2\varphi = 0\,, \tag{3.35}$$

where $\Box_0 = \delta^{\mu\nu}\partial_\mu\partial_\nu$. We now solve order by order in ρ to recover:

$$m^2\ell^2 = \Delta(\Delta-d)\,,\quad \phi_{(2n)} = \frac{1}{2n(2\Delta-d-2n)}\Box_0\phi_{(2n-2)}\,. \tag{3.36}$$

Note: This breaks when the denominator becomes zero, in which case we need to add at that order a logarithmic term: $\rho^k\log(\rho)\chi_{2k}$, in which case we find

$$\chi_{2k} = -\frac{1}{2^{2k}\Gamma(k)\Gamma(k+1)}(\Box_0)^k\phi_0\,, \tag{3.37}$$

while ϕ_{2k} is no longer determined by EOM.

- The on-shell action needs to be regularized by introducing the cut-off at $\rho = \epsilon$:

$$S_r = -\frac{C}{2}\int d^dx\sqrt{g}g^{\rho\rho}\phi\partial_\rho\phi\Big|_{\rho=\epsilon}. \tag{3.38}$$

(We have integrated by parts, using the fact that ϕ vanishes both at $x \to \infty$ and $\rho \to \infty$, together with EOM.) This then gives

$$S_r = CL^{d-1}\int d^dx\Big(\epsilon^{-\Delta+\frac{d}{2}}a_0 + \epsilon^{-\Delta+\frac{d}{2}+1}a_2 + \cdots - \log\epsilon a_{2\Delta-d}\Big), \tag{3.39}$$

where

$$\begin{aligned} a_0 &= -\frac{1}{2}(d-\Delta)\phi_0^2\,, \quad a_2 = -\frac{d-\Delta+1}{2(2\Delta-d-2)}\phi_0\Box_0\phi_0\,, \\ a_{2\Delta-d} &= -\frac{d}{2^{2k+1}\Gamma(k)\Gamma(k+1)}\phi_0(\Box_0)^k\phi_0\,. \end{aligned} \tag{3.40}$$

- Since $\Delta > d/2$, S_r diverges. To subtract these divergences we introduce counterterms. These counterterms better be covariantly expressed on the boundary – constructed from the induced metric on the boundary:

$$\gamma_{\mu\nu} = \frac{\ell^2}{\epsilon}\delta_{\mu\nu}\,, \quad \Box_\gamma = \gamma^{\mu\nu}\partial_\mu\partial_\nu\,, \tag{3.41}$$

from the boundary field $\phi(\epsilon, x)$, and its derivatives: $\Box_\gamma\phi(\epsilon, x)\,, \ldots$. For this we need to invert (3.34), that is to write ϕ_{2n} in terms of $\phi(\epsilon, x)$. To second order in ϵ this inversion reads

$$\begin{aligned} \phi_0 &= \epsilon^{-(d-\Delta)/2}\Big(\phi(\epsilon,x) - \frac{1}{2(2\Delta-d-2)}\Box_\gamma\phi(\epsilon,x)\Big)\,, \\ \phi_2 &= \epsilon^{-(d-\Delta)/2-1}\frac{1}{2(2\Delta-d-2)}\Box_\gamma\phi(\epsilon,x)\,. \end{aligned} \tag{3.42}$$

Plugging this back to (3.39), we can express the divergences in terms of the boundary field $\phi_b(x) = \phi(\epsilon, x)$. In order to cancel these divergences, we introduce the following counterterms:

$$\boxed{S_{ct} = \frac{C}{\ell}\int d^dx\sqrt{\gamma}\Big(\frac{d-\Delta}{2}\phi_b^2(x) + \frac{1}{2(2\Delta-d-2)}\phi_b(x)\Box_\gamma\phi_b(x) + \ldots\Big)\,.} \tag{3.43}$$

As these are covariantly written on the boundary, they can be evaluated in any coordinates! The total action with these counterterms,

$$S = S_0 + S_{ct} \tag{3.44}$$

is then finite.

- Since it is the ϕ_0 which is the source of the operator O, the expectation value of O is then given by

$$\boxed{\langle O(x)\rangle = -\frac{\delta S}{\delta\phi_0(x)} = \lim_{\epsilon\to 0}\Big(\frac{\ell^d}{\epsilon^{\Delta/2}}\frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta\phi(\epsilon,x)}\Big)\,.} \tag{3.45}$$

Other correlation functions can also be straightforwardly calculated. For example

$$\langle O(x)O(y) = -\frac{\delta^2 S}{\delta\phi_0(x)\delta\phi_0(y)} \tag{3.46}$$

immediately yields the result (3.30).

Gravity

- Similar to scalar field we start from the Einstein–Hilbert action (supplemented by the York–Gibbons–Hawking term):

$$S_0 = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{g}\Big(R + \frac{d(d-1)}{\ell^2}\Big) - \frac{1}{8\pi G}\int d^d x\sqrt{\gamma}\mathcal{K}\,, \tag{3.47}$$

and perform the Fefferman–Graham expansion of the metric near the boundary:

$$ds^2 = \ell^2\Big(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\big[\underbrace{g_{0\mu\nu}(x) + \rho g_{2\mu\nu}(x) + \rho^2 g_{4\mu\nu}(x)\dots}_{g_{\mu\nu}(\rho,x)}\big]dx^\mu dx^\nu\Big)\,, \tag{3.48}$$

considering the asymptotically AdS manifolds. (If the boundary is even-dimensional, additional logarithmic term appears: $\rho^{d/2}\log\rho h_{d\mu\nu}$.)

- Inserting this ansatz into Einstein equations then determines g_d in terms of g_0. For example, we have

$$g_{2\mu\nu} = \frac{\ell^2}{d-2}\Big(R_{\mu\nu} - \frac{1}{2(d-1)}Rg_{0\mu\nu}\Big)\,. \tag{3.49}$$

- We next plug these back to the Einstein–Hilbert action and identify the divergent terms,

$$S_{\rm reg} = -\frac{1}{16\pi G}\int d^d x\sqrt{\det g^0}(\epsilon^{-d/2}a_0 + \epsilon^{d/2+1}a_2 + \cdots - \log\epsilon a_d) + \text{finite}\,, \tag{3.50}$$

where a's are expressed in terms of g^0.

- Writing these in terms of the boundary metric $\gamma_{\mu\nu} = \frac{\ell^2}{\rho} g_{0\mu\nu}$ (and boundary curvature invariants $\mathcal{R}, \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}, \dots$), and choosing the counterterms to cancel these divergencies, we find

$$S_{ct} = \frac{1}{8\pi G_{d+1}} \int d^d x \sqrt{\gamma} \Big(\underbrace{\frac{d-1}{\ell}}_{1st} + \underbrace{\frac{\ell}{2(d-2)}\mathcal{R}}_{2nd} + \underbrace{\frac{\ell^3}{2(d-2)^2}\Big(\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} - \frac{1}{d-1}\mathcal{R}^2\Big)}_{3rd} + \dots \Big)\,, \tag{3.51}$$

where the 3rd counter cancels the logarithmic divergence present in $d=4$ dimensions; we have seen the 1st and 2nd conterterms before for $d=3$. The total action $S = S_0 + S_{ct}$ is then finite.

- Since $g_{0\mu\nu}$ is the source for the quantum operator $T_{\mu\nu}(x)$, we have

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{\det g_0}} \frac{\delta S}{\delta g_0^{\mu\nu}(x)}\,. \tag{3.52}$$

However, since

$$\gamma_{\mu\nu}(x) = \lim_{\epsilon\to 0} \frac{l^2}{\epsilon} g_{0\mu\nu}\,, \tag{3.53}$$

we have

$$\boxed{\langle T_{\mu\nu}(x)\rangle = \lim_{\epsilon\to 0}\Big(\frac{\ell^{d-2}}{\epsilon^{d/2-1}}\tau_{\mu\nu}\Big)\,,} \tag{3.54}$$

where $\tau_{\mu\nu}$ is the boundary stress tensor discussed previously:

$$\tau_{\mu\nu} = -\frac{2}{\sqrt{\gamma}}\frac{\delta S}{\delta \gamma^{\mu\nu}} = \frac{1}{8\pi}\Big(\mathcal{K}h_{\mu\nu} - \mathcal{K}_{\mu\nu} + \ell\mathcal{G}_{\mu\nu} - \frac{2}{\ell}\gamma_{\mu\nu}\Big)\,. \tag{3.55}$$

It is the $\langle T_{\mu\nu}(x)\rangle$ that is finite and can be used to calculate properties of the CFT/bulk spacetime.

3.4 Conformal anomaly*

- In field theory we have

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g_0}}\frac{\delta W}{\delta g_0^{\mu\nu}(x)}\,, \tag{3.56}$$

where $g_0^{\mu\nu}$ is a classical (fixed) background field – a source for $T_{\mu\nu}$.

- Conformal anomaly arises when energy momentum tensor does not remain traceless under quantum corrections:

$$\langle T^\mu_\mu\rangle \neq 0\,. \tag{3.57}$$

For example in 2d, we have

$$\boxed{\langle T^{\mu}{}_{\mu}(x)\rangle = \frac{c}{24\pi}R_0\,,} \tag{3.58}$$

where c is the central charge. Note that $\int d^2x\sqrt{g}R_0 = 2\pi\chi$ is an Euler characteristic (R_0 is a topological density in $d=2$).

- Similarly, in 4d, we have

$$\boxed{\langle T^{\mu}{}_{\mu}(x)\rangle = \frac{c}{16\pi^2}C_0^{abcd}C^0_{abcd} - \frac{a}{16\pi^2}\mathcal{G}_0\,,} \tag{3.59}$$

where $\mathcal{G}_0$ is the Gauss–Bonnet topological density.

Specifically, for $\mathcal{N}=4$ $SU(N_c)$ SYM one has

$$\boxed{c = a = \frac{1}{4}(N_c^2-1)\,,} \tag{3.60}$$

and the conformal anomaly reads

$$\langle T^{\mu}_{\mu}\rangle = \frac{c}{8\pi^2}\Big(R^0_{\mu\nu}R_0^{\mu\nu} - \frac{1}{3}R_0^2\Big) \;\to\; \frac{N_c^2}{32\pi^2}\Big(R^0_{\mu\nu}R_0^{\mu\nu} - \frac{1}{3}R_0^2\Big)\,, \tag{3.61}$$

in the large N_c limit. One can show that this is independent of λ to all orders in perturbation theory.

- Weyl transformation on the boundary gives the trace of energy momentum tensor. We want to find a diffeomorphism that acts on the following metric

$$ds^2 = \ell^2\Big(\frac{d\rho^2}{\rho^2} + \frac{1}{\rho}g_{\mu\nu}(\rho,x)dx^\mu dx^\nu\Big)\,, \quad \lim_{\rho\to 0} g_{\mu\nu}(\rho,x) = g_{0\mu\nu}(x)\,, \tag{3.62}$$

and reduces to Weyl on the boundary. This is known as PBH transformation:

$$\rho = \rho'(1-2\sigma(x'))\,, \quad x^\mu = x'^\mu + a^\mu(x',\rho')\,. \tag{3.63}$$

Requiring $g'_{\rho\rho} = g_{\rho\rho}\,,\ g'_{\rho\mu} = g_{\rho\mu} = 0$ imposes $\partial_\rho a^\mu = \frac{\ell^2}{2}g^{\mu\nu}\partial_\nu\sigma$. Under PBH, we have

$$g_{\mu\nu} \to g_{\mu\nu} + 2\sigma(1-\rho\partial_\rho)g_{\mu\nu} + \nabla_\mu a_\nu + \nabla_\nu a_\mu\,, \tag{3.64}$$

which reduces to Weyl on the boundary provided $a_\mu \to 0$ there.

- We now apply this diffeo to the total action S:

$$\delta(S_0+S_{ct}) = 2\int d^dx\sigma(x)\Big(\epsilon\frac{\delta}{\delta\epsilon} - g^{0\mu\nu}\frac{\delta}{\delta g^{0\mu\nu}}\Big)(S_0+S_{ct})\,. \tag{3.65}$$

The counterterms ensure the finiteness of the AdS action. However, the presence of the $-\log\epsilon$ term in even dimensions spoils the invariance under PBH transformation. Namely, focusing on $d=4$, we have:

$$\begin{aligned}\delta S_{ct} &= 2\int d^dx\sigma\Big(\epsilon\frac{\delta}{\delta\epsilon}-g^{0\mu\nu}\frac{\delta}{\delta g^{0\mu\nu}}\Big)S_{ct}=2\int d^dx\sigma\epsilon\frac{\delta}{\delta\epsilon}S_{ct}\\ &= -\frac{\ell^3}{64\pi G_5}\int d^4x\sqrt{g_0}\Big(R^0_{\mu\nu}R_0^{\mu\nu}-\frac{1}{3}R_0^2\Big)\,. \end{aligned}\tag{3.66}$$

Note that, while in the previous section, we wanted to express the counterterms in terms of the (divergent) boundary metric γ. (That is why the counterterms can cancel the divergencies of the action.) Here, instead, everything is calculated with the help of the (finite) CFT metric g_0. Thus we schematically have

$$\begin{aligned}\delta(S_0+S_{ct}) &= \Big(\frac{\delta}{\delta\epsilon}-\frac{\delta}{\delta g^0}\Big)(S_0+S_{ct})=\underbrace{\Big(\frac{\delta}{\delta\epsilon}-\frac{\delta}{\delta g^0}\Big)S_0}_{\text{0 by diffeo invariance}}+\underbrace{\Big(\frac{\delta}{\delta\epsilon}-\frac{\delta}{\delta g^0}\Big)S_{ct}}_{\text{-anomaly}}\\ &= \underbrace{\frac{\delta}{\delta\epsilon}(S_0+S_{ct})}_{\text{0 by construction}}-\underbrace{\frac{\delta}{\delta g^0}(S+S_{ct})}_{-\langle T^\mu_\mu\rangle}\,. \end{aligned}\tag{3.67}$$

Thus we found:

$$\langle T^\mu_\mu\rangle=\frac{\ell^3}{64\pi G_5}\Big(R^0_{\mu\nu}R_0^{\mu\nu}-\frac{1}{3}R_0^2\Big)=\frac{N_c^2}{32\pi^2}\Big(R^0_{\mu\nu}R_0^{\mu\nu}-\frac{1}{3}R_0^2\Big)\,,\tag{3.68}$$

using that

$$G_5=\frac{G_{10}}{\mathrm{Vol}(S^5)}=\frac{\pi\ell^3}{2N_c^2}\,.\tag{3.69}$$

3.5 Wilson loops

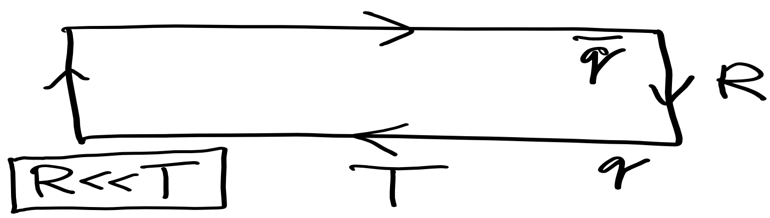
Let us now add probes (quarks) to the system.

- Wilson loops are non-local gauge invariant operators that describe the parallel transport of a quark along the closed path $\mathcal{C}$. The corresponding (infinitely heavy test) quark field ψ picks up a phase: $\psi(x+\mathcal{C})=W(\mathcal{C})\psi(x)$. Wilson loop operator is given by

$$\boxed{\mathcal{W}(\mathcal{C})=\frac{1}{N_c}\mathrm{Tr}W(\mathcal{C})=\frac{1}{N_c}\mathrm{Tr}\Big(P\exp(i\oint_{\mathcal{C}}dx^\mu A_\mu)\Big)\,.}\tag{3.70}$$

Here, everything is in the fundamental representation of the $SU(N_c)$ group.

The expectation value $\langle\mathcal{W}(\mathcal{C})\rangle$ along certain paths, such as the one displayed in the following picture, provides an order parameter for the confinement/deconfinement phase transition:

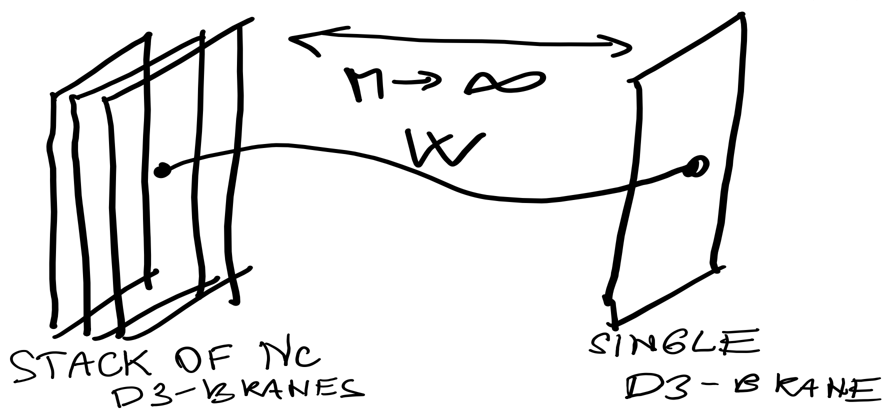


It corresponds to 'creating a quark-antiquark pair, pulling them R distance apart, letting them interact, and annihilating them much later. We thus have

$$\langle \mathcal{W}(\mathcal{C}) \rangle \propto \exp\Big(-TV(R)\Big) , \tag{3.71}$$

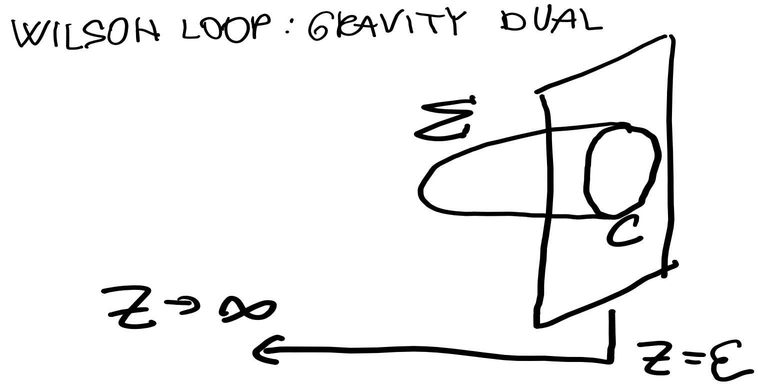
and thus $\langle \mathcal{W}(\mathcal{C}) \rangle \approx Z[J]/Z[0] \sim \langle f|e^{-HT}|i\rangle \sim e^{-E_0 T}$ yields the (static) quark-antiquark potential $V(R)$. In the confined phase we expect $V \sim R$; in this case the Wilson loop follows the '*area law*': $\langle \mathcal{W}(\mathcal{C}) \rangle \sim \exp(-\text{Area}(\mathcal{C}))$. On the other hand, in the unconfined phase $V \sim 1/R$ (Coulomb law).

- Fundamental representation. In the above we arrived, through the $D3$-brane construction at SYM (where all the fields are massless and in the adjoint representation of the $SU(N_c)$). Intuitively, we have N_c^2 possibilities how to stretch strings between a stack of N_c $D3$-branes, giving rise to the adjoint representation of the $SU(N_c)$. To construct the fundamental representation with infinitely heavy quarks, we instead consider a situation described in the following figure:



Namely, we have $(N_c + 1)$ $D3$-branes, where one of them is separated far away from the stack of the remaining N_c branes. The quark is now described by a classical long string that stretches between the separated brane and the remaining stack; the large distance gives rise to the large mass of the quark. Moreover, we have now N_c possibilities where to end the string – corresponding to the N_c 'components' of the fundamental representation of $SU(N_c)$.

- Gravitational dual. In order to model the Wilson loop, we want the above heavy particles (described by the long strings stretching from the separated brane to the stack) to move around the closed path $\mathcal{C}$. The stack of N_c $D3$-branes is replaced by $AdS_5 \times S^5$, and the single $D3$ brane is located on the boundary of AdS_5 (at $z = \epsilon$). Expectation value of the Wilson loop is then given by the semi-classical partition function of the macroscopic string in $AdS_5 \times S^5$ (all possible embeddings) whose worldsheet Σ ends on the path of the Wilson loop at the boundary, see figure:



At large N_c and large λ, this is given by the saddle point approximation:

$$\boxed{\langle \mathcal{W}(\mathcal{C}) \rangle = e^{-S_{\text{NG,min}}}\,, \quad S_{\text{NG}} = \frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{\det \gamma_{AB}}\,.} \tag{3.72}$$

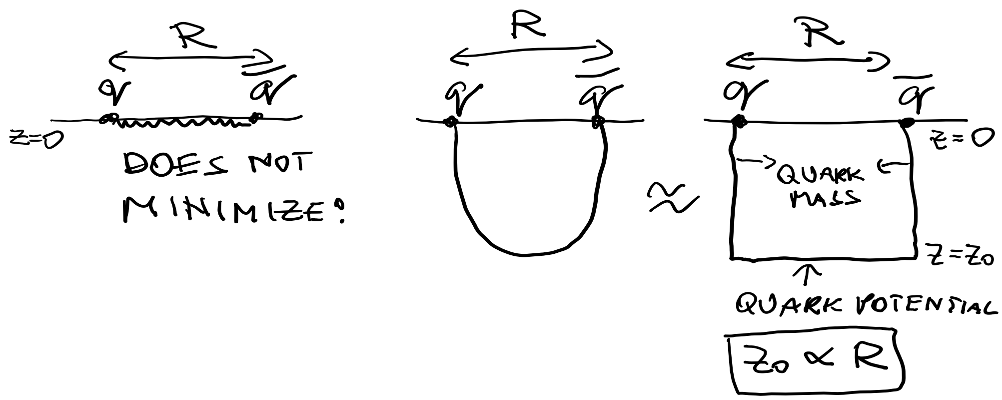
By comparing to (3.71) we thus have the following formula for the quark-antiquark potential:

$$V(R) = \frac{1}{T} S_{NG}(R)\,. \tag{3.73}$$

The divergence of the latter corresponds to the self energy of the pointlike charge, and can be removed by the due counterterm (e.g. to parallel strings stretching from the boundary).

- Qualitative results. Let us now discuss the following interesting cases (you will calculate the exact unconfined potential in a tutorial).

  – Unconfined phase. To minimize the NG action in the pure AdS space, let us make the following rectangular approximation, described in the picture:



Here, the vertical parts of the string contribute to the quark's mass, while it is the horizontal part which determines the quark potential. The induces metric for this part of the string is:

$$\gamma = \frac{\ell^2}{z^2}(d\tau^2 + dx^2) \quad \Rightarrow \quad \det\sqrt{\gamma_{AB}} = \frac{\ell^2}{z^2}\,. \tag{3.74}$$

Intuitively, for separation R of the quarks, the horizontal string is at

$$z_0 \propto R \tag{3.75}$$

distance. We thus have

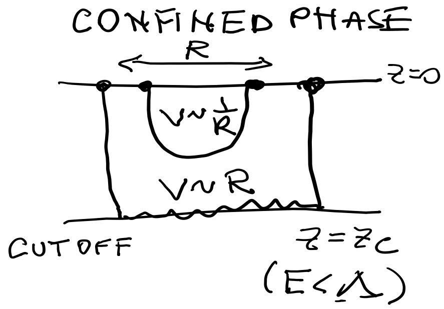
$$S_{\mathrm{NG}} \sim \frac{1}{\alpha'}\frac{\ell^2}{z_0^2}\int d\tau dx = \frac{\ell^2}{l_s^2}\frac{TR}{z_0^2} = T\frac{\sqrt{2\lambda}}{R} = TV(R)\,. \tag{3.76}$$

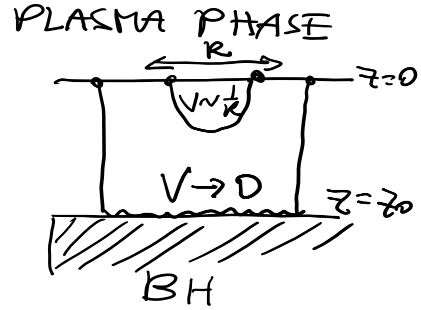
So we derived the unconfined (Coulomb) potential

$$\boxed{V = \frac{\sqrt{2\lambda}}{R}\,.} \tag{3.77}$$

Note that perturbatively, one would expect $V \propto \lambda/R$; the $\sqrt{\lambda}$ is the strong coupling effect.

– Confined phase does not exist for SYM (which is scale invariant). So we need to perturb AdS, and introduce a scale, that would correspond to transition (low) energy scale Λ at which the confinement happens. So we consider the AdS with cutoff z_c, as displayed in the following figure:





In this case, for sufficiently large separation R, we reach the cutoff and cannot go any further (CFT low energy modes 'go' deeply in the bulk, and see this cutoff). In that case, the action reads

$$S_{\rm NG} \sim \frac{1}{\alpha'}\frac{\ell^2}{z_c^2}\int d\tau dx = \frac{\ell^2}{l_s^2}\frac{TR}{z_c^2} = T\frac{\sqrt{2\lambda}R}{z_c^2} \quad \Rightarrow \quad \boxed{V \propto R}\,, \tag{3.78}$$

which is the confining potential.

– The plasma phase (or SYM at finite temperature) corresponds to the AdS black hole case, with the metric

$$ds^2 = \frac{\ell^2}{z^2}\left(-f dt^2 + \frac{dz^2}{f} + \delta_{ij}dx^i dx^j\right)\,, \quad f = 1 - \frac{z^4}{z_+^4}\,. \tag{3.79}$$

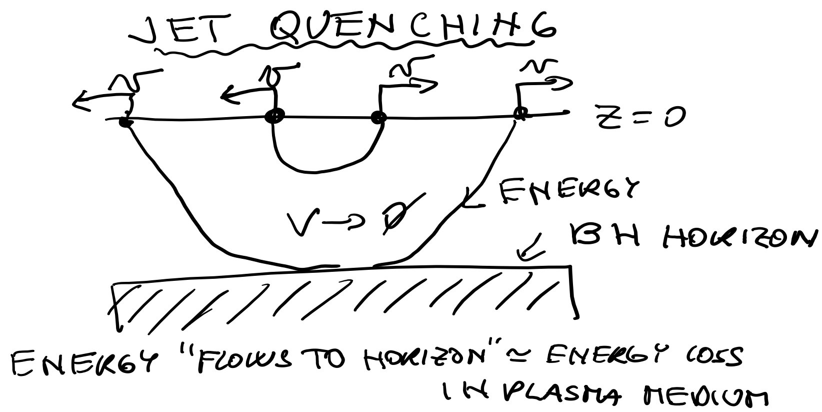
So in this case, when the separation is large enough, the string reaches all the way to the horizon, where $f=0$, and we have

$$\det\gamma_{AB} = \sqrt{f}\frac{\ell^2}{z_+^2} \to 0\,. \tag{3.80}$$

So in this case the potential vanishes, $V\to 0$, which corresponds to Debye screening.

– To study the plasma/confinement phase transition (at finite temperature), we can either study the soliton/planar BH phase transition, where one of the directions parallel to the horizon are compactified (introducing the scale z_c by hand). An alternative is the standard Hawking–Page phase transition (you studied in your homework) present for spherical black holes (where the extra scale corresponds to the 'radius' of the spherical horizon).

– Jet quenching. As the produced hadron jets propagate through the plasma medium, they loose energy and are eventually absorbed by the plasma. This corresponds to the following AdS dual:


JET QUENCHING
Z = 0
ENERGY
V → 0
BH HORIZON
ENERGY "FLOWS TO HORIZON" ≃ ENERGY LOSS
IN PLASMA MEDIUM

Chapter 4: Advanced Topics

4.1 AdS/CFT at finite temperature

- To discuss the finite temperature SYM, we return back to the non-extremal black $D3$-brane (2.20) and take its near-horizon limit $r \ll \ell$. In order to remain outside of the horizon we have to have $r > r_0$ – so we take $r_0 < r \ll \ell$ and f remains fixed. We thus have (setting $z = \ell^2/r$):

$$\boxed{ds^2 = \underbrace{\frac{\ell^2}{z^2}\Big(-f dt^2 + \frac{dz^2}{f} + \delta_{ij} dx^i dx^j\Big)}_{\text{Schwarzschild}-AdS_5} + \underbrace{\ell^2 d\Omega_5^2}_{S^5}\,, \quad f = 1 - \left(\frac{z}{z_0}\right)^4,} \tag{4.1}$$

  where the black hole has a planar horizon. This provides the gravity dual of the strongly coupled SYM at finite temperature

$$T = \frac{|f'(z_0)|}{4\pi} = \frac{1}{\pi z_0}\,. \tag{4.2}$$

- Note that the metric is invariant under

$$x^\mu \to \lambda x^\mu\,, \quad z \to \lambda z\,, \quad z_0 \to \lambda z_0\,. \tag{4.3}$$

  So one can always scale the horizon radius – this means that *all temperatures are equivalent* and the physics is the same (except at zero temperature). Thence there is no characteristic temperature, nor phase transitions (the original SYM is a CFT and the only scale is that of T). This is no longer true in the spherical case, where we observe the Hawking–Page transition as discussed in the introduction.

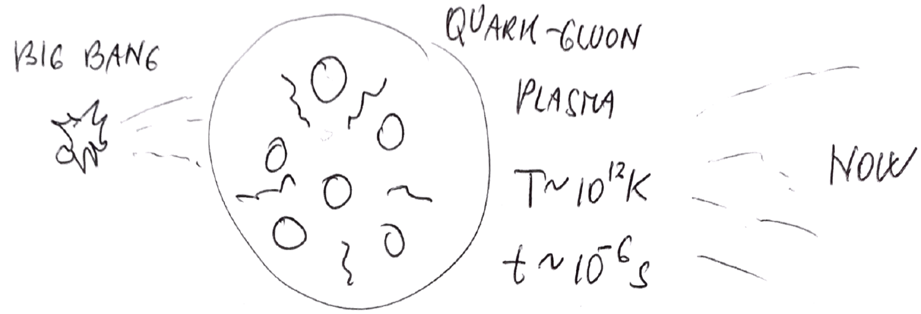
- In your tutorial, you shall translate the properties of the Schwarzschild–AdS_5 to those of the SYM at strong coupling, and compare them to the weak coupling results.

- Note also, that the finite temperature gauge theory corresponds to stationary black hole solutions in the bulk. This corresponds to equilibrium. In order to departure from equilibrium, we will have to perturb this background.

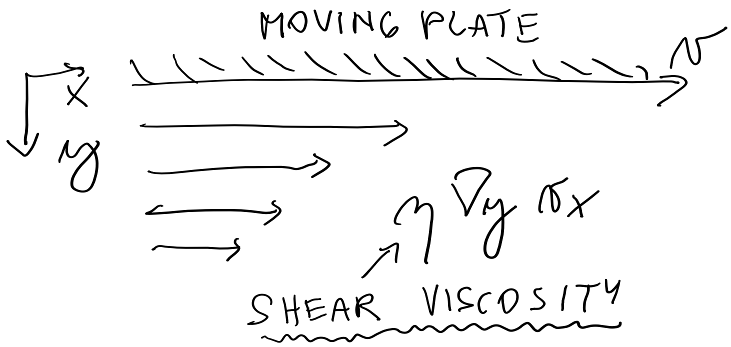
4.2 Shear viscosity of QGP

Physics of quark gluon plasma

- Quark-Gluon Plasma (QGP) describe the state of matter in the early Universe ($T \approx 10^{13} K, t \approx 10^{-6} s$):



- Strongly coupled (not a CFT!)
- Was recreated in RHIC and LHC accelerators – well described by the hydro regime. In particular, it features a measurable shear viscosity:



- The experimental results show that QGP has the following shear viscosity to entropy density ratio:

$$\boxed{\frac{\eta}{s} = \frac{1}{4\pi} \times (1, 2.5)\,.} \tag{4.4}$$

  Only some cold atomic gases have a similarly small value, while other liquids (such as water or liquid helium) have this ratio much (10^3 times) higher. Note also that, since the temperature of QGP is large, η itself is huge (it is an energy momentum transfer).

- Remark. Considering ϕ^4 theory at weak coupling, one has:

$$l_{\text{mfp}} \sim \frac{1}{g^2 T}\,, \quad \eta \sim \rho l_{\text{mfp}} \sim \frac{T^3}{g^2}\,, \quad s \sim T^3 \quad \Rightarrow \quad \boxed{\frac{\eta}{s} \sim \frac{1}{g^2}}\,. \tag{4.5}$$

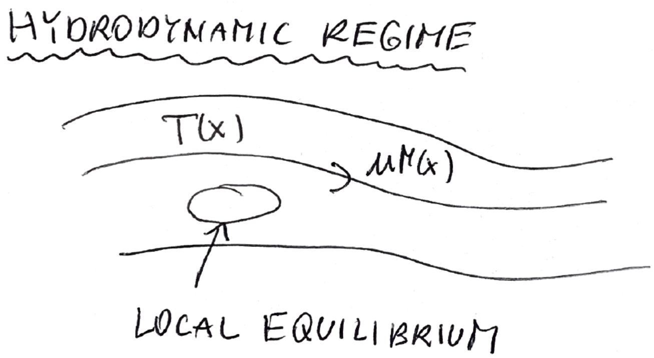
  That is, for small g, the ratio is large. (Of course, this result does not make sense for $g \to 0$ as in that case $l_{\text{mfp}} \to \infty$ and the hydro description breaks down. Using naively the same formula in strong coupling regime, $g \to \infty$, yield $\eta/s \approx 0$. As we shall see, this does not happen – there is a saturation of η/s for strong coupling. (Moreover, already for $g \sim 1$, for which $\eta/s \sim 1$, the perturbative expansion breaks down!)

Hydrodynamics

- Hydrodynamics= *effective field theory* describing the long-range ($\vec{k} \to 0$), low-energy ($\omega \to 0$) fluctuations around equilibrium. Roughly, we require

$$\lambda_{\text{fluctuations}} \gg l_{\text{mfp}} \, , \tag{4.6}$$

if the 'mean free path' l_{mfp} can be defined. That is, the system is locally in equilibrium:



- The macroscopic dynamics is governed by conservation laws, such as

$$\boxed{\nabla_\mu T^{\mu\nu} = 0 \, .} \tag{4.7}$$

Here, $T^{\mu\nu}$ is a function of d variables $T(x)$ and $u^\mu(x)$, supplemented by constitutive relations expressing $T^{\mu\nu}$ in terms of these variables. These relations introduce the transport coefficients and are constructed in such a way to satisfy the second law of thermodynamics:

$$T^{\mu\nu} = T^{\mu\nu}(T, u^\mu) = T^{\mu\nu}\Big(\rho(T(x)), P(T(x)), \underbrace{\eta, \zeta, \dots}_{\text{transp. coeffs.}}\Big) \, . \tag{4.8}$$

Such transport coefficients (coefficients of the EFT expansion), depend on details of the miscroscopic theory and cannot be determined in the framework of hydrodynamics.

- Example. To give an example as to how constitutive relations are constrained by second law, let us consider the heat diffusion (with heat flow q^i):

$$\partial_t \rho + \partial_i q^i = 0 \, , \quad d\rho = T ds \, . \tag{4.9}$$

Combining these, we get

$$0 = \partial_t s + \frac{1}{T}\partial_i q^i = \partial_t s + \partial_i\Big(\frac{q^i}{T}\Big) + \frac{\partial_i T q^i}{T^2}\,. \tag{4.10}$$

Integrating these, we thus get

$$\partial_t S = \int d^3x \partial_t s = -\int d^3x \frac{\partial_i T q^i}{T^2} = \kappa \int d^3x \Big(\frac{\partial_i T}{T}\Big)^2 \geq 0\,, \tag{4.11}$$

upon choosing the constitutive relation $q^i = -\kappa \partial^i T$, $\kappa \geq 0$.

- To describe the fluid in local thermal equilibrium, we expand $T^{\mu\nu}$ using the derivative expansion:

$$T^{\mu\nu} = \underbrace{(P+\rho)u^\mu u^\nu + P\gamma^{\mu\nu}}_{\text{0th-order: perfect fluid}} + \underbrace{\tau^{\mu\nu}}_{\text{1st-order}} + \dots\,, \tag{4.12}$$

where $\gamma_{\mu\nu}$ is the 'background metric' where the fluid lives, and we expanded to the nth-order in derivatives of the fluid dynamical fields, corresponding to the convergent series in $(l_{\text{mfp}}/\lambda_{\text{fluctuations}})^n$.

In particular, at the first derivative order in the Landau's frame (where $T^\mu{}_\nu u^\nu = -\rho u^\mu$, that is, there is no energy flow in the rest frame), we have[1]

$$\boxed{\tau^{\mu\nu} = -2\eta\sigma^{\mu\nu} - 2\zeta\theta P^{\mu\nu}\,.} \tag{4.13}$$

Here, we have introduced two transport coefficients: η the shear viscosity, and ζ the bulk viscosity; $P^{\mu\nu} = u^\mu u^\nu + \gamma^{\mu\nu}$, and

$$\sigma^{\alpha\beta} = P^{\alpha\gamma}P^{\beta\delta}\Big(\nabla_{(\gamma}u_{\delta)} - \frac{1}{d-1}P_{\gamma\delta}\theta\Big)\,, \quad \theta = \nabla_\gamma u^\gamma\,. \tag{4.14}$$

The positivity of the entropy current $s^\mu = su^\mu$ ($\nabla_\mu s^\mu \geq 0$) requires that

$$\eta \geq 0\,, \quad \zeta \geq 0\,. \tag{4.15}$$

As you will prove in your tutorial, for CFTs we also have

$$\zeta = 0\,. \tag{4.16}$$

[1] Keeping only the first order is physically problematic. For example, we have unphysical instabilities or the frame dependence, see however, recent [?].

Linear response theory

- The (macroscopic) transport coefficients are an input in hydrodynamics. They can be calculated from microscopic physics with the help of linear response theory. Namely, they are related to Green's functions via the Green–Kubo relations.

- Linear response theory. Consider a system in equilibrium, described by the density matrix ρ_0, which is at some point perturbed by an external source ϕ_0:

$$\delta S = \int d^d x \phi_0(t,x) O(x) \,. \tag{4.17}$$

  Then, it can be shown that to linear order in ϕ_0, we have

$$\boxed{\delta\langle O(t,x)\rangle = -\int dt' d^{d-1}x' G_R^{OO}(t-t', x-x')\phi_0(t',x')}\,, \tag{4.18}$$

  where the retarded Green's function G_R^{OO} is defined as

$$G_R^{OO}(t-t', x-x') = -i\theta(t-t')\langle[O(t,x), O(t',x')]\rangle \,, \tag{4.19}$$

  where the average is calculated using the unperturbed equilibrium density matrix ρ_0, and all operators evolve in time according to the interaction picture, that is according to H_0. In the Fourier space we then have

$$\boxed{\delta\langle O(k)\rangle = -G_R^{OO}(k)\phi_0(k) \,.} \tag{4.20}$$

- Kubo formula. To give an example of a Kubo formula, let us consider Ohm's law:

$$\delta\langle J^x\rangle = \sigma E_x^0 \,, \tag{4.21}$$

  where σ is the transport coefficient called conductivity. Considering a gauge where $A_t^0 = 0$, we have $E_x^0 = -\partial_t A_x^0$, which upon Fourier transforming yields $i\omega A_x^0$. So we have

$$\delta\langle J^x\rangle = i\omega\sigma A_x^0 \quad \Leftrightarrow \quad \delta\langle J^x\rangle = -G_R^{xx} A_x^0 \,, \tag{4.22}$$

  where the latter comes from the linear response theory. We thus find

$$\sigma(\omega) = -\frac{G_R^{xx}(\omega, q=0)}{i\omega} \,, \tag{4.23}$$

  which is the promised Kubo relation. (In particular, the DC conductivity is obtained by taking the limit $\omega \to 0$ of the previous.)

- Thermal internal force. Sometimes we are not interested in response to external source ϕ_0, but rather to a thermal internal force. To use the linear response theory, we 'model' the internal force as arising from some external source. For example, in the case of fluids we imagine that the perturbation of $T^{\mu\nu}$ arises from 'ficticious' gravitational force – metric perturbation.

Calculation of the shear viscosity

- Step 1: macroscopic hydro description. You will show in your tutorial that a fictitious gravitational field h^0_{xy} induces the following perturbation of the 1st-order energy momentum tensor (4.13) of our fluid (setting $\vec{k} = 0$):

$$\delta\tau^{xy}(\omega) = -i\omega\eta h^0_{xy}(\omega)\,. \tag{4.24}$$

- Step 2: microscopic description. Using the linear response theory formula (4.20), we have

$$\delta\langle\tau^{xy}(\omega)\rangle = -G_R^{xy,xy}(\omega)h_{xy}(\omega)h^0_{xy}(\omega) \tag{4.25}$$

  Comparing to (4.24), we thus derived the following Kubo formula for the viscosity:

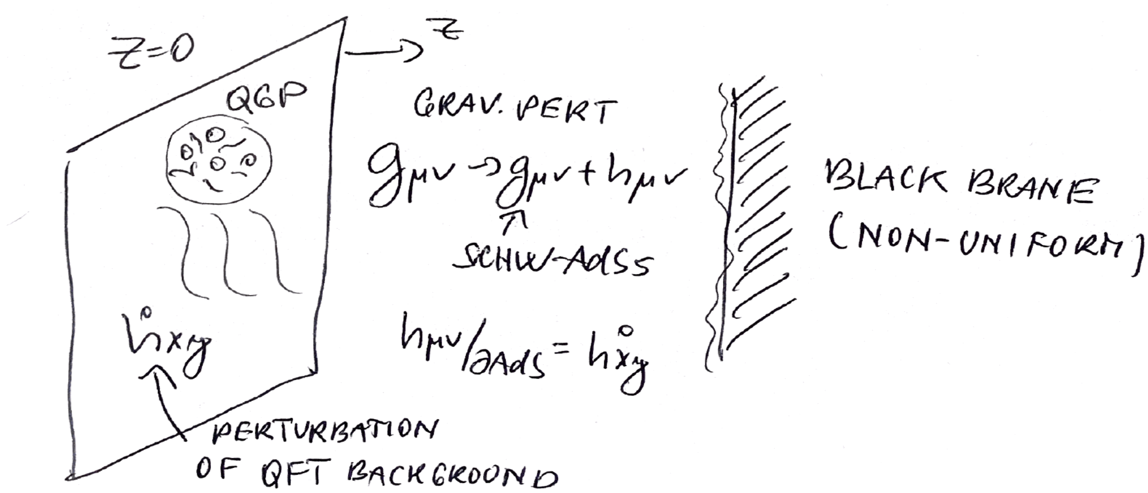
$$\eta = \lim_{\omega\to 0}\frac{1}{i\omega}G_R^{xy,xy}(\omega)\,, \tag{4.26}$$

  where

$$G_R^{xy,xy}(\omega,\vec{0}) = \int dt d^3x e^{-i\omega t}\theta(t)\langle[\tau^{xy}(t,\vec{x}),\tau^{xy}(0,\vec{0})]\rangle\,. \tag{4.27}$$

  Thus, to calculate the viscosity we need to calculate the 2-point function $\langle\tau^{xy}\tau^{xy}\rangle$. We shall do this using the AdS/CFT correspondence.

- Step 3: Employ AdS/CFT to calculate $G_R(\omega)$, namely to evaluate the correlator $\langle\tau^{xy}\tau^{xy}\rangle$. To do this, we consider the bulk black brane spacetime, whose gravitational perturbation reduces to the h^0_{xy} on the boundary:



  By expanding the York–Einstein–Hilbert–Λ action for $g_{\mu\nu} + h_{\mu\nu}$ (where $h_{\mu\nu} = \delta^x_{(\mu}\delta^y_{\nu)}h_{xy}$), we get the following quadratic action for the perturbation h_{xy}:

$$S_q[h_{xy}] = \frac{N^2}{8\pi^2\ell^3}\int d^4x dz\sqrt{-g}\Big(-\frac{1}{2}g^{\mu\nu}\partial_\mu h_{xy}\partial_\nu h_{xy}\Big)\,. \tag{4.28}$$

This is an action for a massless scalar h_{xy}, which we have to solve on the Schwarzschild-AdS background. Expressing the solution in terms of the boundary data h^0_{xy} and plugging back to this action, we find the generating functional $S_q[h^0_{xy}]$, which then yields

$$\langle \tau^{xy}\tau^{xy}\rangle = -\frac{\delta S_q[h^0]}{\delta h^0_{xy}\delta h^0_{xy}} \,. \tag{4.29}$$

This is then used to calculate the Green function. The calculation yields

$$G_R(\omega) = \frac{\pi N_c^2 T^3}{8} i\omega \,, \tag{4.30}$$

giving

$$\eta = \frac{\pi}{8} N_c^2 T^3 \,. \tag{4.31}$$

Since you will derive in your tutorial $s = \frac{\pi^2}{2} N_c^2 T^3$, we have the following famous result:

$$\boxed{\frac{\eta}{s} = \frac{1}{4\pi}\,,} \tag{4.32}$$

which is a great victory for the gauge/gravity correspondence.

- Remark. This result is conjectured to provide a universal lower bound on the real fluid viscosity (irrespective of the presence of other fields, or whether the conformal symmetry or supersymmetry are broken). It makes strongly coupled holographic fluids the most perfect fluids (next to ideal fluids).

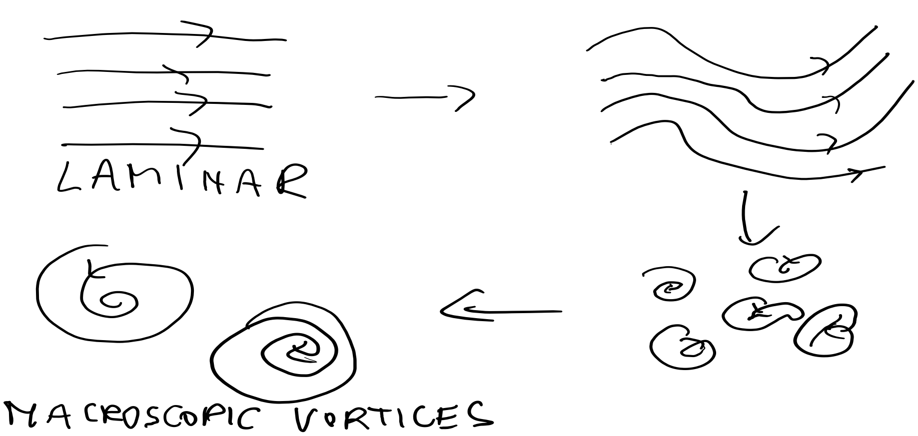
4.3 Turbulent gravity*

- Fluid/gravity correspondence is a map between two classical systems: viscous conformal fluids on the AdS boundary and the long-wavelength perturbations of the AdS black branes in GR. This map can be derived in the derivative expansion of Einstein equations and is thus independent of the AdS/CFT correspondence.

- On the fluid side, one can define the Reynolds number

$$R \sim \frac{\rho v L}{\eta} \,, \tag{4.33}$$

where L is the distance scale, and v the velocity fluctuation. For large enough R, the flow is no longer laminar, and we observe a turbulent behavior, characterized by the energy cascade to long wavelengths. We observe a formation of macroscopic, long-lived, vortices, as displayed in the following picture [?]:



- On the gravity side, this corresponds to long-lived hydrodynamic quasi-normal modes of black branes, as opposed to exponentially decaying quasinormal modes in the 'laminar regime'.

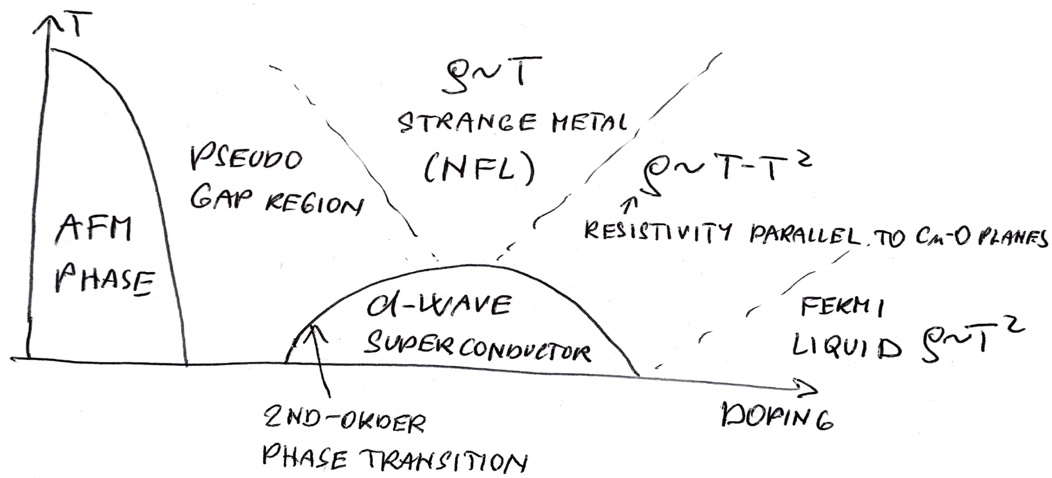
  In other words, turbulent behavior is present in Einstein's theory in the dynamical settings.

- (Un)related remark. It is now well known that AdS is unstable towards forming black holes – this corresponds to thermalization of the dual CFT. However, it was shown that certain initial data do not result in black hole formation [?]. These define dynamical CFT configurations that never equilibrate.

4.4 Holographic superconductors

- Motivation: The standard superconductors are described by the BCS theory. They correspond to weakly coupled Fermi liquid, cannot have critical temperature above $T_c \sim 30 - 40K$, and are the s-wave superconductors.

  On the other hand, the high-T_c superconductors are strongly coupled, correspond to d-wave ($l = 2$), and have very complicated phase diagrams which cannot be described by the BCS theory:



For these reasons, people model these systems with the help of holographic superconductors [?, ?].

- In order to describe a 2nd-order phase transition to superconducting phase, the entropy has to remain continuous. Thus, in the bulk side we seek a phase transition from one black hole phase to another – need to evade BH no hair theorems. Due to the (2+1) geometry of high-T_c superconductors we focus on 4d planar black holes.

- On the field theory side, the superconductor is characterized by zero resistivity, a response to the $U(1)$ current, suggesting we need a gauge field in the bulk. We also need a scalar order parameter, $\langle O \rangle$, corresponding to the "mascroscopic wave function" in Ginsburg–Landau's theory – thus we need a charged scalar field in the bulk. The simplest phenomenological theory in the bulk is thus

$$S = \int d^4x \sqrt{-g} \Big(R - 2\Lambda - \frac{1}{4} F_{MN}^2 - |D_M \psi|^2 - V(\psi) \Big) , \tag{4.34}$$

where

$$D_M = \nabla_M - ieA_M , \quad V(\psi) = m^2 |\psi|^2 . \tag{4.35}$$

Our aim is now to show that this systems exhibits the following behavior: For $T > T_c$, we have "normal" Reissner–Nordstrom (RN) phase, characterized by $\psi = 0$. At T_c, the RN phase becomes unstable w.r.t. a scalar field condensation, and the system undergoes a 2nd-order phase transition to a hairy black hole, with $\psi \neq 0$, which is a preferred phase below T_c.

- Probe limit: formally can be achieved by $\psi \to \psi/e, A_M \to A_M/e$ and $e \to \infty$. That is, our gravity background is Schw-AdS$_4$:

$$ds^2 = \Big(\frac{r_0}{\ell}\Big)^2 \frac{1}{u^2}(-f dt^2 + dx^2 + dy^2) + \ell^2 \frac{du^2}{f u^2} , \quad f = 1 - u^3 , \tag{4.36}$$

which has the following temperature:

$$T = \frac{3r_0}{4\pi\ell^2} \tag{4.37}$$

The normal $T > T_c$ phase is characterized by

$$\psi = 0\,, \quad A = \mu(1-u)dt\,, \tag{4.38}$$

At T_c this becomes unstable, as ψ becomes tachyonic. Namely, its effective mass is

$$m^2_{\text{eff}} = m^2 + g^{MN}A_M A_N = m^2 - \left(\frac{\mu}{T}\right)^2 \frac{9(1-u)u^2}{16\pi^2\ell^2(1+u+u^2)}\,; \tag{4.39}$$

for large μ/T, that is small enough T, this becomes negative. In what follows we choose

$$m^2 \equiv -\frac{2}{\ell^2}\,, \tag{4.40}$$

which is with the BF bound $m^2 \geq -(d-1)^2/(4\ell^2) = -9/(4\ell^2)$.

- In the superconducting phase we assume

$$\psi = \psi(u)\,, \quad A = A_t(u)dt\,. \tag{4.41}$$

This then yields the following equations:

$$A_t'' - \frac{2|\psi|^2}{u^2 f}A_t = 0\,, \tag{4.42}$$

$$\frac{u^2}{f}\Big(\frac{f}{u^2}\psi'\Big)' + \Big(\frac{A_t^2}{\tau^2 f^2} - \frac{\ell^2 m^2}{u^2 f}\Big)\psi = 0\,, \tag{4.43}$$

where $\tau = \frac{4\pi}{3}T$.

- Close to T_c ψ is small and does not backreact on A_t. We thus have $A = \mu(1-u)dt$ there, satisfying the first equation, and only have to solve the ψ equation. Close to the boundary this behaves as

$$u \to 0: \qquad \psi \sim \psi_0 u^{\Delta_-} + \psi_+ u^{\Delta_+}\,, \tag{4.44}$$

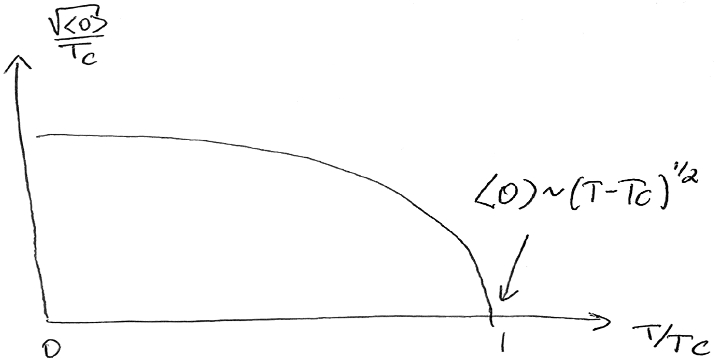
where, as always,

$$\Delta_\pm = \frac{d-1}{2} \pm \sqrt{\frac{(d-1)^2}{4} + m^2\ell^2} = (1,2)\,. \tag{4.45}$$

Of course, ψ_0 has the meaning of teh external source. Since this is a spontaneous symmetry breaking, we set the source $\psi_0 = 0$. At the same time ψ_+ plays the role of the order parameter

$$\langle O \rangle = \psi_+\,. \tag{4.46}$$

Numerical integration then yields the following figure:



It can also be shown that the order-parameter critical exponent $\beta = 1/2$:

$$\langle O \rangle = (T - T_c)^{1/2} \,. \tag{4.47}$$

- Conductivity is a response to external current. We thus add

$$A_x = A_x(u) e^{-i\omega t} \,, \tag{4.48}$$

where on the boundary

$$u \to 0 : \qquad A_x \sim A_x^0 (1 + A_x^1 u) \,, \tag{4.49}$$

Thus we have

$$\langle J^x \rangle = A_x^1 A_x^0 = \sigma E_x^0 = i\omega \sigma A_x^0 \quad \Rightarrow \quad \sigma(\omega) = \frac{A_x^1}{i\omega} \,, \tag{4.50}$$

where the latter equality comes from the Ohm's law. The A_x equation writes as

$$\frac{1}{f}(f A_x')' + \Big(\frac{\omega^2}{\tau^2 f^2} - \frac{2\psi^2}{u^2 f} \Big) A_x = 0 \,. \tag{4.51}$$

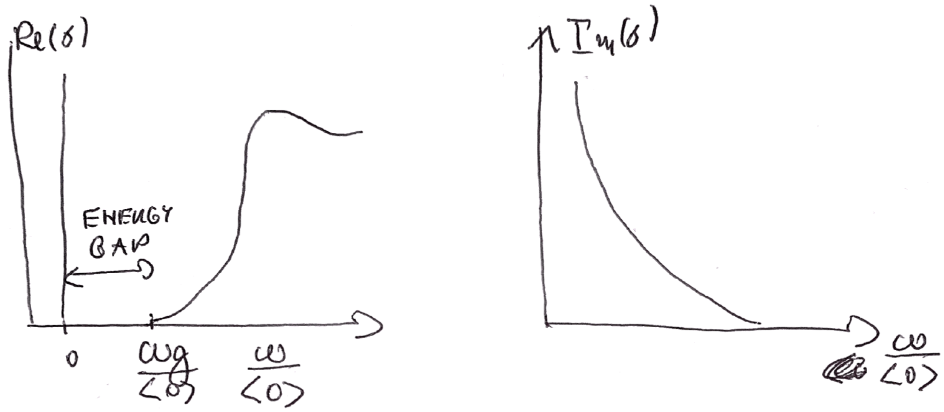
At high T, $\psi = 0$ and

$$A_x^1 = 0 + \frac{i\omega}{2\pi T} + O(\omega^2) \,. \tag{4.52}$$

Thus $\mathcal{R}(\sigma)$ is finite and $Im(\sigma)$ has no $1/\omega$ pole. However, below T_c where psi^2 is non-vanishing, we find

$$A_x^1 = \# + O(\omega) + O(\omega^2) \tag{4.53}$$

Thus the DC conductivity $\mathcal{R}(\sigma(\omega \to 0)))$ diverges and $Im(\sigma)$ has $1/\omega$ pole, as displayed on the following numerical figures:



- Response to magnetic field. One cannot see the Meissner effect, as the magnetic field is fixed on the boundary. However, it is easy to show that the magnetic field destroys the superconductivity. Namely, the effective mass of the scalar now becomes

$$m_{\text{eff}}^2 = m^2 + g^{tt}A_t^2 + g^{ii}A_i(x)^2\,. \tag{4.54}$$

The last terms is positive and prevents the field to become tachyonic.

4.5 Holographic entanglement entropy

Entanglement entropy

- Consider a mechanical system characterized by a density matrix ρ. The von Neumann entropy is defined as follows:

$$\boxed{S_{\text{vN}} = -\text{Tr}(\rho \log \rho)\,.} \tag{4.55}$$

This vanishes for a pure state, for which $\rho = \rho^2$, and is maximised for diagonal ρ with equal probabilities.

- Entanglement entropy provides a measure for the entanglement of quantum states. Let the Hilbert space has the product structure for two subsystems A and B:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B\,. \tag{4.56}$$

We define a reduced density matrix for A by

$$\rho_A = \text{Tr}_B \rho\,. \tag{4.57}$$

The entanglement entropy for A is then defined as

$$\boxed{S_A = -\text{Tr}_A(\rho_A \log \rho_A)\,.} \tag{4.58}$$

- Entanglement entropy has a number of properties. If the global system is a pure state and A is a complement of B, we have

$$S_A = S_B \,. \tag{4.59}$$

We also have a strong subaditivity condition:

$$\boxed{S_A + S_B \geq S_{A\cup B} + S_{A\cap B}\,,} \tag{4.60}$$

for any two subsystems A and B, which mean that 'mutual information is monotonic in region size'.

- In QFT (at fixed time $t = t_0$), one finds for a region A a divergent result

$$S = c_0 (R\Lambda)^{d-2} + c_1 (R\Lambda)^{d-3} + \ldots \left[+c_d \log(R\Lambda) \right], \tag{4.61}$$

where c_i are model dependent, R is the 'linear size' of region A, and Λ the UV cutoff; logarithmic term is present in even dimensions. For the leading term, we note the 'area law':

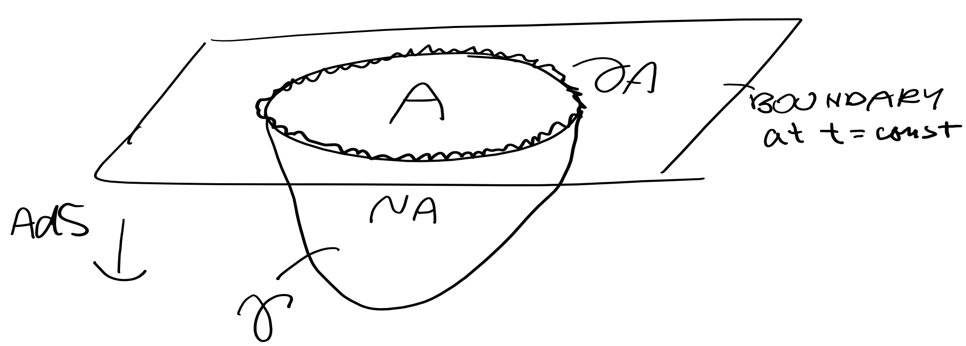
$$\text{Area}(\partial A) \propto R^{d-2} \,. \tag{4.62}$$

In particular, for 2d CFT, one finds

$$S = \frac{c}{3} \log(\Lambda R) \,. \tag{4.63}$$

Ryu–Takayanagi proposal

- A holographic realization of the entanglement entropy is realized as follows: we seek a minimal surface in AdS_{d+1} whose boundary is given by ∂A:



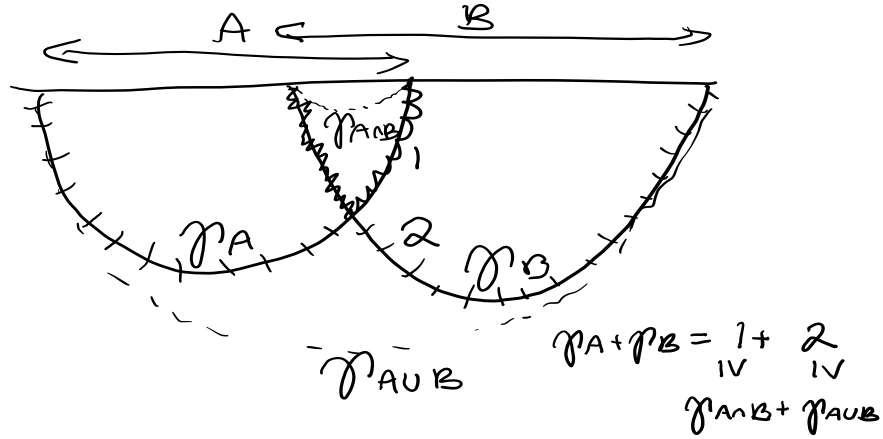
and we have:

$$\boxed{S_A = \min_{\gamma \sim A} \frac{\text{Area}(\gamma)}{4G_{d+1}}\,.} \tag{4.64}$$

Here, γ and A are homologous, $\gamma \sim A$, when there exists a region r_A such that $\partial r_A = A \cup \gamma$ (in other words, surface γ can be smoothly contracted to surface A). Moreover, $\partial A = \partial \gamma$ is the entangling surface.

- One finds
$$S_A = \frac{\ell^{d-1}}{G_{d+1}} \frac{\text{Area}(\partial A)}{\epsilon^{d-2}} + \dots\,, \tag{4.65}$$
where ϵ is the UV cutoff in the radial direction, $\epsilon \propto 1/\Lambda$, or more generally
$$S_A = b_0 \left(\frac{\ell}{\epsilon}\right)^{d-2} + b_1 \left(\frac{\ell}{\epsilon}\right)^{d-4} + \dots\,, \tag{4.66}$$
which is to be compared to (4.61)

- Strong subaditivity is difficult to prove in QI context. However, in the holographic context we have the following simple proof, displayed in the figure:



Covariant proposal

- Let us now consider time dependent case. That is, we consider the case of time varying bulk M and boundary theory that is taken to be in a time-varying state on a fixed background ∂M. Since the boundary metric is non-dynamical we can choose a foliation by equal time slices. These, however, cannot be canonically extended to the bulk. For this reason we can no longer consider minimal surfaces at an instant of time in the bulk. Rather, we need to consider extremal surfaces.

- Extremal surface is a codimension-2 spacelike surface γ. Such a surface has two null normals, n and l; we can normalize $n \cdot l = -1$. Defining the projector to this surfaces by
$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu l_\nu + n_\nu l_\mu\,, \tag{4.67}$$
we can get the extrinsic curvature tensors
$$K^1_{\mu\nu} = \gamma^\gamma_\mu \gamma^\delta_\nu \nabla_\gamma n_\delta\,, \quad K^2_{\mu\nu} = \gamma^\gamma_\mu \gamma^\delta_\nu \nabla_\gamma l_\delta\,. \tag{4.68}$$
The extremality means that
$$K^1 = \gamma^{\mu\nu} K^1_{\mu\nu} = 0 \quad \text{and} \quad K^2 = \gamma^{\mu\nu} K^2_{\mu\nu} = 0\,. \tag{4.69}$$

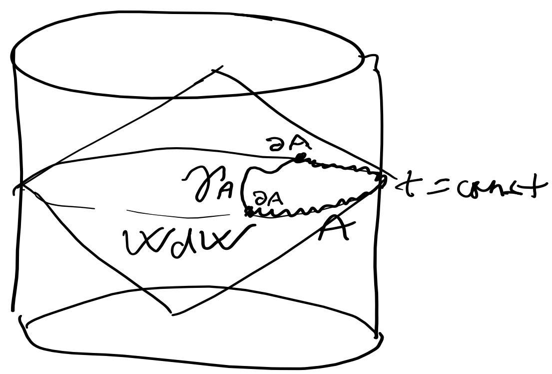
For such surfaces, we have

$$\boxed{\delta \text{Area}(\gamma) = 0\,.} \tag{4.70}$$

- HRT prescription for the entanglement entropy is as follows: Consider extremal surfaces anchored on ∂A: $\partial\gamma_A = \gamma_A|_{\partial M} = \partial A$ that are homologous to A. Then we have

$$\boxed{S_A = \min_{\gamma_A} \frac{\text{Area}(\gamma_A)}{4G_{d+1}}\,.} \tag{4.71}$$

  Note that γ_A can 'move in time', it must, however, lie in the Wheeler de Witt (WdW) patch of $A \cup \bar{A}$:



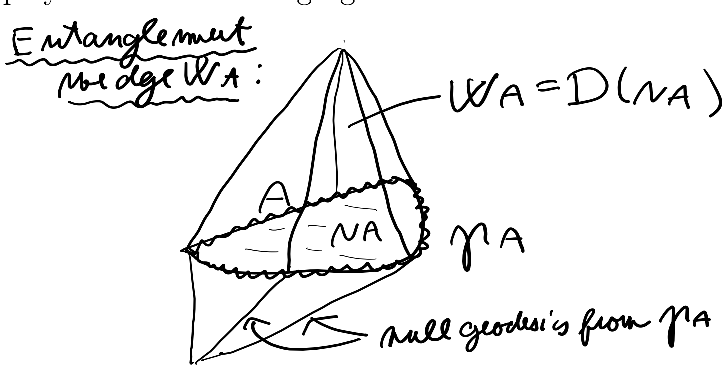
  For the static case, this reduces to the previous definition.

- Bulk reconstruction. The content of the AdS/CFT is that if one knows the full CFT on the boundary, one can recover the full bulk. Now, what if we only know part of CFT, say in region A?

  The answer is, we are able to recover bulk in the entanglement wedge W_A [?]. W_A is formed by the spacelike separated points from the corresponding extremal surface γ_A towards A:

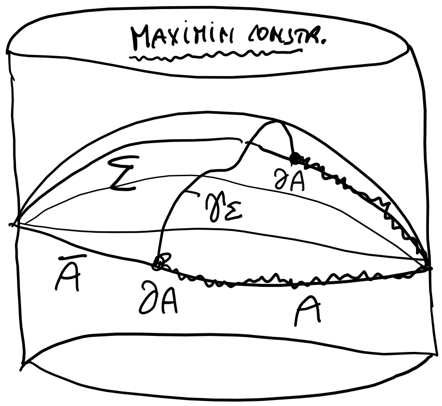
$$\boxed{W_A = D(r_A)}\,, \tag{4.72}$$

  as displayed in the following figure:



This is sometimes dubbed as "dual of ρ_A".

- Maximin construction due A. Wall [?] provides an equivalent definition for the covariant entanglement entropy. The construction goes as follows. Consider a test bulk Cauchy surface Σ, anchored on $A \cup \bar{A}$. In this surface we find a minimal surface γ_Σ:



We then maximize over all such Cauchy surfaces (filling WdW patch):

$$\boxed{S_A = \max_{\Sigma} \min_{\gamma_\Sigma \sim A} \frac{\text{Area}(\gamma_\Sigma)}{4G_{d+1}}\,.} \tag{4.73}$$

(That is, we "minimize in the spatial direction and maximize in the time direction".) It can be shown that the maximin surface is the extremal surface; there isn't a unique Cauchy surface that contains the extremal surface.

4.6 Back to black: information paradox

At the beginning of the course, we have learned about the black hole information paradox: If the black hole is formed from a pure state and completely evaporates, we end up in a mixed thermal state of Hawking radiation – the information seems lost in black hole and we violated the unitarity of time evolution [?].

This conclusion seems to contradict the AdS/CFT picture where the process of the black hole formation and evaporation is encoded in the boundary CFT evolution, which is manifestly unitary.

Let us now discuss some recent progress on this issue, following [?].

Two types of entropies

- For understanding the paradox more closely, we need to distinguish two types of entropies.

- Fine-grained (quantum or von Neumann) entropy. Given the density matric ρ of a quantum system, this is calculated as

$$\boxed{S_{\rm N} = -{\rm Tr}(\rho \log \rho)\,.} \tag{4.74}$$

It quantifies our ignorance about precise quantum state of a system (vanishes for a pure state). It is invariant under time evolution:

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho]\,. \tag{4.75}$$

For a 2-part quantum system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ in a pure state, we have

$$S(A) = S(B)\,, \quad S(A \cup B) = 0\,. \tag{4.76}$$

- Coarse-grained (thermodynamic) entropy. Measuring only a subset of observables $\{A_i\}$ (typically energy), we maximize over all possible density matrices that are consistent with our observations:

$$\boxed{S_{\rm TD} = \max_{\rho}\Big(-{\rm Tr}(\rho \log \rho)\Big)\,, \quad a_i = {\rm Tr}(\rho A_i)\,.} \tag{4.77}$$

Such an entropy arises from "sloppiness" – provides a measure of the total number of dof available to the system. This type of entropy obeys the 2nd law of thermodynamics (increases under unitary time evolution).

The key obvious result is: Von Neumann entropy cannot be bigger than the thermodynamic entropy:

$$\boxed{S_{\rm N} \leq S_{\rm TD}\,.} \tag{4.78}$$

(Possible entanglement is bounded by TD entropy.)

For quantum field in a region Σ,

$$S_{\rm N}(\Sigma) = S_{\rm N}(\rho_\Sigma) \tag{4.79}$$

is divergent at its boundaries and is in general time dependent (as we move the slice forward in time).

- The generalized and Bekenstein–Hawking entropies are thermodynamic entropies:

$$\underbrace{S_{\rm gen}}_{\text{TD entropy}} = \underbrace{\frac{\text{Area of horizon}}{4G}}_{\text{TD entropy}} + \underbrace{S_{\rm outside}}_{\text{quantum entropy}}\,. \tag{4.80}$$

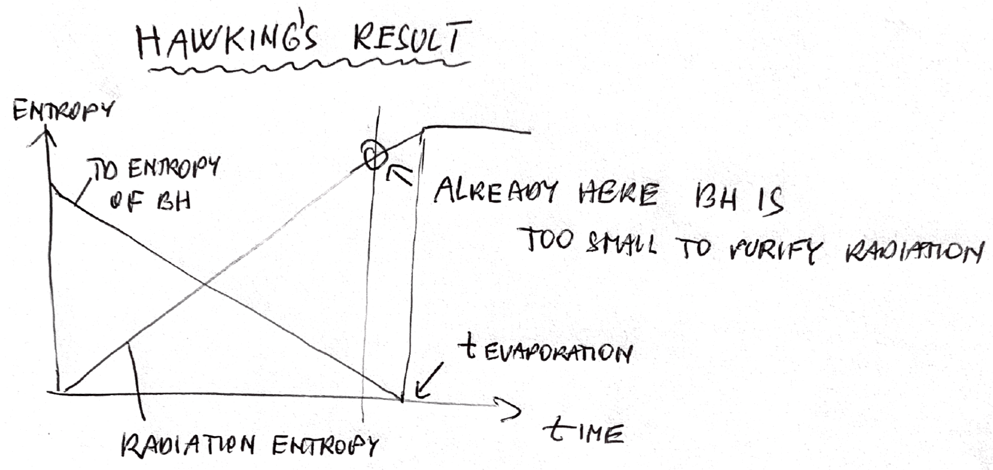
We thus have

$$\Delta S_{\rm gen} \geq 0\,. \tag{4.81}$$

(For example, increases rapidly as the black hole horizon forms in a collapse.)

Page curve

- Hawking 1976: information loss: After BH completely evaporates, we have a mixed state – huge entropy stored in Hawking radiation:



- Page time. If the total system is in pure state, then for fine-grained entropies we have

$$S_{\text{black hole}} = S_{\text{rad}} \,. \tag{4.82}$$

However, we have to have

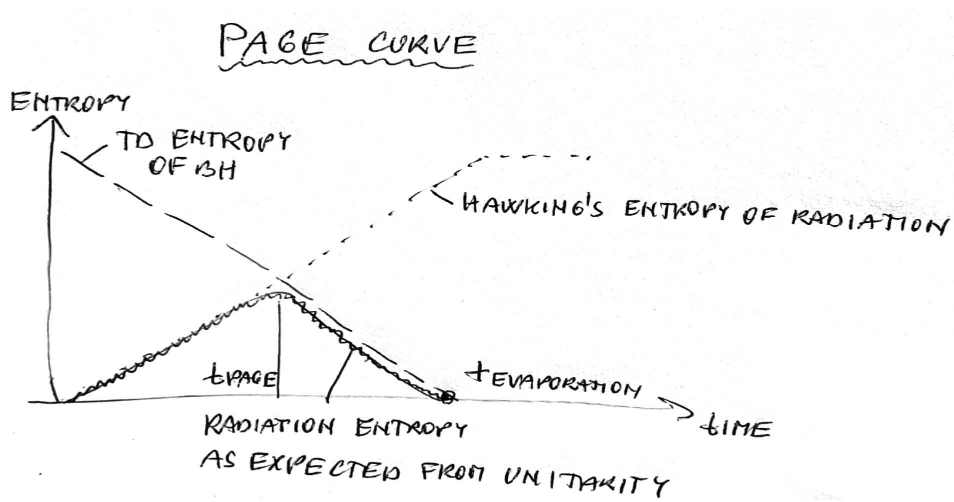
$$S_{\text{black hole}} \leq S_{\text{Bekenstein–Hawking}} = S_{\text{coarse-grained}} \,. \tag{4.83}$$

And thence we have

$$\boxed{S_{\text{rad}} \leq S_{\text{BH-TD}} \,,} \tag{4.84}$$

with the two being equal at the Page time.

- If afterwards the bound saturates we recover the Page curve:



Quantum entropy of gravitating systems

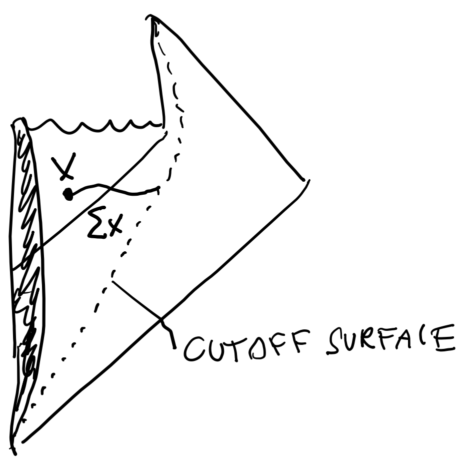
- For a long time people thought that to explain the Page curve it is not enough to use semiclassical theory and one need to use the theory of quantum gravity.

  In past couple of years great progress was achieved, using the semiclassical theory: we can calculate the quantum entropy of radiation/black hole. The key is to use formula for fine-grained entropy of gravitational systems.

- The fine-grained entropy of gravitational systems is given by the following formula:

$$\boxed{S = \min_X \Big[\mathrm{ext}_X \Big(\underbrace{\frac{\mathrm{Area}(X)}{4G} + S_{\mathrm{semicl}}(\Sigma_X)}_{S_{\mathrm{gen}(X)}}\Big)\Big]} \,. \tag{4.85}$$

  Here X is the quantum extremal surface, a (classical) surface which extremizes the generalized entropy (including quantum entropy of fields in the QFT in curved space semiclassical approximation) [?]:



  Compare the prescription to covariant calculation of the holographic entanglement entropy.

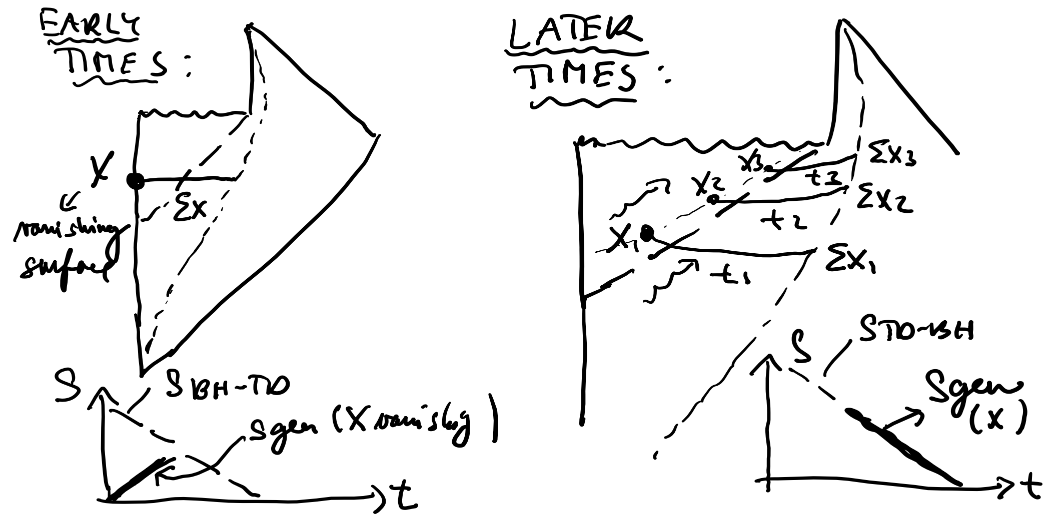
Recovery of Page curve: black hole entropy

- Fairy tale. We have the following stages of the black hole evaporation:

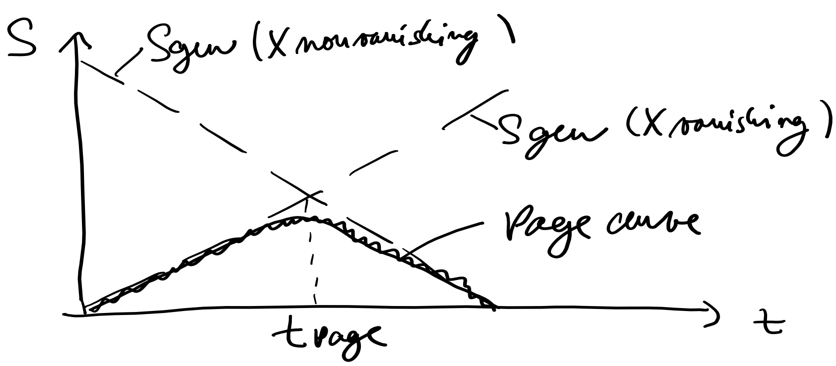
  1. Immediately after the collapse (no Hawking radiation yet), the only extremal surface X is of zero size. This yields zero entropy of BH.
  2. Once the Hawking radiation goes through Σ_X, the entropy starts increasing (in accord with Hawking's calculation).
  3. However, shortly after Hawking radiation starts escaping BH, another non-trivial extremal surface, close to the horizon, appears. The corresponding

entropy is decreasing as the black hole horizon shrinks due to evaporation (according to Page's result).

This is displayed on the following picture:



- Composing the two pictures, we just recover the Page curve for the black hole entropy:

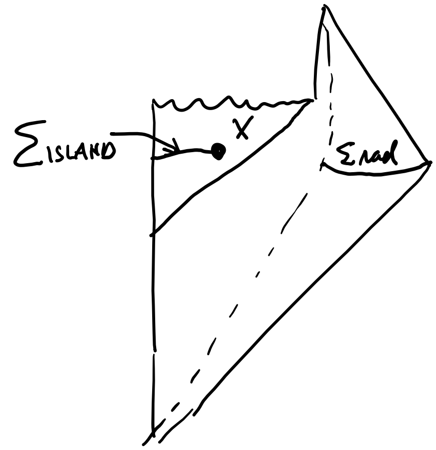


Entropy of radiation

- Entropy of Hawking radiation is given by the Island formula:

$$\boxed{S_{\rm rad} = \min_X\Big[\mathrm{ext}_X\Big(\frac{\mathrm{Area}(X)}{4G} + S_{\rm semicl}(\Sigma_{\rm rad}\cup\Sigma_{\rm island})\Big)\Big]}\,, \tag{4.86}$$

as displayed in the following picture:

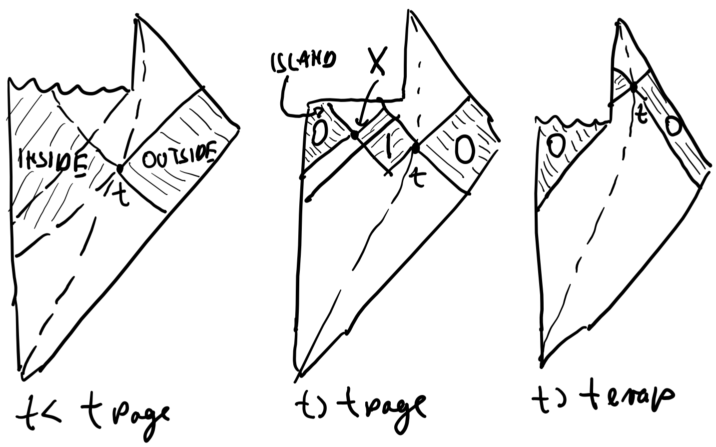


This recovers

$$\boxed{S_{\text{black hole}} = S_{\text{rad}}\,,} \tag{4.87}$$

and the Page curve for the Hawking radiation.

- We have the corresponding <u>entanglement wedges</u>:



"Complicated operations on radiation should be able to extract info from the BH interior." (Is this due to some kind of wormholes?)

- <u>To summarize.</u> The fine-grained (island) entropy formula reproduces Page curve for the entropy of Hawking radiation. Thus, there is no information loss, and the unitarity is restored. In late times, part of the interior of the BH is accessible from outside (due to some kind of wormholes?). If you want to know more, please read [**?**] and references therein.

4.7 What we did not have time to talk about

There are many topics which we left behind in this course. To name a few:

- *Integrability*
- *Holographic RG flows*
- *Complexity* [**?**]
- *Beyond AdS holography*: celestial holography [**?**], Lifshitz,...

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