Finite temperature

1 Viscous fluids

We have seen in the lecture, that the first derivative correction to the perfect fluid is described by the following tensor:

$$\tau^{\mu\nu} = -2\eta\sigma^{\mu\nu} - 2\zeta\theta P^{\mu\nu}\,, \tag{1}$$

where η is the shear viscosity, ζ is the bulk viscosity, $P^{\mu\nu} = u^\mu u^\nu + g_0^{\mu\nu}$ is the projection operator, and

$$\sigma^{\alpha\beta} = P^{\alpha\gamma}P^{\beta\delta}\Big(\nabla_{(\gamma}u_{\delta)} - \frac{1}{d-1}P_{\gamma\delta}\theta\Big)\,, \quad \theta = \nabla_\gamma u^\gamma\,. \tag{2}$$

a) Argue that for any CFT, we have to have $\zeta = 0$.

b) Consider the fluid in the rest frame, $u^\mu = (1,0,0,0)$ on the following background:

$$g^0_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & h^0_{xy}(t) & 0 \\ 0 & h^0_{xy}(t) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Show that to the linear order in perturbation, we have

$$\delta\langle\tau^{xy}\rangle \sim -\eta(\nabla_x u_y + \nabla_y u_x)\,. \tag{4}$$

c) Evaluate the Christoffel symbol and go to the Fourier space to show that

$$\delta\langle\tau^{xy}(\omega, q=0)\rangle = -i\omega\eta h^0_{xy}\,. \tag{5}$$

2 CFT thermodynamics from gravitational dual

We have learned in the lecture that thermodynamic quantities of the strongly coupled $\mathcal{N}=4$ SYM can be derived from those of the planar Schwarzschild-AdS_5 black hole (we suppress the S^5 here):

$$ds^2 = \frac{\ell^2}{z^2}\Big(-f dt^2 + \delta_{ij}dx^i dx^j + \frac{dz^2}{f}\Big)\,, \quad f = 1 - \Big(\frac{z}{z_0}\Big)^4 . \tag{6}$$

This is done via the help of the standard AdS/CFT relations:

$$\frac{\ell^4}{l_s^4} = 2\lambda\,, \quad G_{10} = 8\pi^6 l_s^8 g_s^2 = \frac{\pi^4}{2}\frac{\ell^8}{N_c^2}\,. \tag{7}$$

a) The temperature of the CFT is determined by the black hole temperature T – calculate it!

b) The entropy of the CFT is given by the black hole entropy. We can calculate it both – from the $5d$-point of view (suppressing S_5) or the full $10d$ point of view:

$$S = \frac{\text{Area}_5}{4G_5} = \frac{\text{Area}_{10}}{4G_{10}}\,. \tag{8}$$

Use this to derive the expression for the G_5.

Assuming further that $x^i \in (0, L_i)$ and $V_3 = L_1L_2L_3$, calculate the entropy density $s = S/V_3$. Show that it is of the expected CFT form – confirming that the black hole entropy yields the entropy of the dual CFT. Moreover, show that it is proportional to N_c^2, thence we are in the unconfined (plasma) phase.

c) By calculating the Euclidean action (employing all the counterterms) one obtains the following free energy:

$$F = -\frac{\pi^2}{8}N_c^2 V_3 T^4\,. \tag{9}$$

Use this to calculate the pressure P, the energy density ϵ, and the entropy density s:

$$P = -\frac{\partial F}{\partial V_3}\,, \quad \epsilon = \frac{1}{V_3}\frac{\partial(\beta F)}{\partial \beta}\,, \quad s = -\frac{1}{V_3}\frac{\partial F}{\partial T}\,, \tag{10}$$

and compare these to your previous tutorial dimensional analysis. Thence you have shown how to calculate the thermodynamic quantities of the strongly coupled CFT at finite temperature.

3 CFT at weak coupling: free thermal gas

Let us finally calculate the entropy density of the SYM at small coupling. Recall from (for example your cosmology/statistical physics course) that we have

$$s_{\text{boson}} = \frac{2\pi^2}{45}T^3\,, \quad s_{\text{fermion}} = \frac{7}{8}s_{\text{boson}}\,. \tag{11}$$

How many fermionic and bosonic dof do we have for the large N_c, $SU(N_c)$ SYM? Thence show that for the total entropy we find

$$s_{\text{free}} = \frac{4}{3}s_{\text{BH}}\,. \tag{12}$$

Do we have a match?