Solutions T5

1 Viscous fluids

We have seen in the lecture, that the first derivative correction to the perfect fluid is described by the following tensor:

$$\tau^{\mu\nu} = -2\eta\sigma^{\mu\nu} - 2\zeta\theta P^{\mu\nu}\,, \tag{1}$$

where η is the shear viscosity, ζ is the bulk viscosity, $P^{\mu\nu} = u^\mu u^\nu + g_0^{\mu\nu}$ is the projection operator, and

$$\sigma^{\alpha\beta} = P^{\alpha\gamma}P^{\beta\delta}\Big(\nabla_{(\gamma}u_{\delta)} - \frac{1}{d-1}P_{\gamma\delta}\theta\Big)\,, \quad \theta = \nabla_\gamma u^\gamma\,. \tag{2}$$

a) The splitting above corresponds to splitting to symmetric traceless part and the trace. It follows that only the first part can survive for CFTs, for which the trace has to vanish: $\tau^\mu_\mu = 0$.

To see more concretely, we have

$$\mathrm{Tr}P^{\mu\nu} = \mathrm{Tr}(u^\mu u^\nu) + \mathrm{Tr}g_0^{\mu\nu} = d-1\,. \tag{3}$$

Thus,

$$\mathrm{Tr}\sigma^{\mu\nu} \sim \theta - \frac{1}{d-1}\theta\mathrm{Tr}P_{\mu\nu} = 0\,. \tag{4}$$

On the other hand, the second term yields $-\zeta\theta(d-1)$. If the whole the $\tau^{\mu\nu}$ has to be traceless, we thus have to have

$$\zeta = 0\,. \tag{5}$$

b) Consider the fluid in the rest frame, $u^\mu = (1, 0, 0, 0)$ on the following background:

$$g^0_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & h^0_{xy}(t) & 0 \\ 0 & h^0_{xy}(t) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{6}$$

We then have

$$g_0^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -h^0_{xy}(t) & 0 \\ 0 & -h^0_{xy}(t) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \det g_0 = -1 + O(h^2) . \tag{7}$$

We also have

$$P^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -h^0_{xy}(t) & 0 \\ 0 & -h^0_{xy}(t) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

Similarly we find

$$\theta = \nabla \cdot u = \frac{1}{\sqrt{\det g_0}} (\sqrt{\det g_0} u^a), a = \frac{1}{\sqrt{\det g_0}} (\sqrt{\det g_0} u^t), t = O(h^2) . \tag{9}$$

From (1) we thus have

$$\begin{aligned} \delta\langle\tau^{xy}\rangle &= -2\eta P^{x\alpha} P^{y\beta} \nabla_{(\alpha} u_{\beta)} = -2\eta P^{xx} P^{yy} \nabla_{(x} u_{y)} + O(h^2) \\ &\approx -\eta(\nabla_x u_y + \nabla_y u_x) . \end{aligned} \tag{10}$$

c) To proceed further, we have to evaluate the Christoffel symbols. We have

$$\nabla_x u_y = \underbrace{\partial_x u_y}_{0} - \Gamma^t_{xy} u_t = \Gamma^t_{xy} = \frac{1}{2} g_0^{tt} (g^0_{tx,y} + g^0_{tx,y} - g^0_{xy,t}) = \frac{1}{2} h^0_{xy,t} , \tag{11}$$

and similarly, $\nabla_y u_x = \frac{1}{2} h^0_{xy,t}$. Thus we have

$$\delta\langle\tau^{xy}\rangle = -\eta h^0_{xy,t} . \tag{12}$$

Going to the Fourier space, $h^0_{xy}(t) = \int d\omega e^{i\omega t} h^0_{xy}(\omega)$, we just recover

$$\delta\langle\tau^{xy}(\omega, q = 0)\rangle = -i\omega\eta h^0_{xy}(\omega) . \tag{13}$$

2 CFT thermodynamics from gravitational dual

We have learned in the lecture that thermodynamic quantities of the strongly coupled $\mathcal{N} = 4$ SYM can be derived from those of the planar Schwarzschild-AdS_5 black hole (we suppress the S^5 here):

$$ds^2 = \frac{\ell^2}{z^2}\Big(-f dt^2 + \delta_{ij} dx^i dx^j + \frac{dz^2}{f}\Big), \quad f = 1 - \Big(\frac{z}{z_0}\Big)^4 . \tag{14}$$

This is done via the help of the standard AdS/CFT relations:

$$\frac{\ell^4}{l_s^4} = 2\lambda , \quad G_{10} = 8\pi^6 l_s^8 g_s^2 = \frac{\pi^4}{2}\frac{\ell^8}{N_c^2} . \tag{15}$$

a) The temperature of the CFT is determined by the black hole temperature T. This reads

$$T = \frac{|f'(z_0)|}{4\pi} = \frac{1}{\pi z_0} . \tag{16}$$

b) The entropy of the CFT is given by the black hole entropy. We can calculate it both – from the $5d$-point of view (suppressing S_5) or the full $10d$ point of view:

$$S = \frac{\text{Area}_5}{4G_5} = \frac{\text{Area}_{10}}{4G_{10}} . \tag{17}$$

Using this fact, we can derive the expression for the G_5. Namely, we have

$$\text{Area}_{10} = \text{Vol}(S_5)\text{Area}_5 , \quad \text{Vol}(S_5) = \ell^5\omega_5 = \pi^3\ell^5 . \tag{18}$$

Thus we derived the Kaluza–Klein formula:

$$G_5 = \frac{G_{10}}{\text{Vol}(S_5)} = \frac{\pi}{2}\frac{\ell^3}{N_c^2} . \tag{19}$$

Assuming further that $x^i \in (0, L_i)$ and $V_3 = L_1 L_2 L_3$, we can calculate the entropy density $s = S/V3$:

$$s = \frac{S_5}{V_3} = \frac{\text{Area}}{4G_5 V_3} = \frac{\ell^3}{4G_5 z_0^3} = \frac{\pi^2 N_c^2}{2} T^3 , \tag{20}$$

employing the previously derived formula for the temperature. Obviously, this is of the expected form for the CFT derived in the previous tutorial –

confirming that the black hole entropy yields the entropy of the dual CFT. Moreover, it is proportional to N_c^2, thence we are in the unconfined (plasma) phase.

c) By calculating the Euclidean action (employing all the counterterms) one can obtain the following free energy:

$$F = -\frac{\pi^2}{8} N_c^2 V_3 T^4 \,. \tag{21}$$

This then immediately yields the following thermodynamic quantities of the dual CFT:

$$\begin{aligned} P &= -\frac{\partial F}{\partial V_3} = \frac{\pi^2}{8} N_c^2 T^4 \,, \\ \epsilon &= \frac{1}{V_3} \frac{\partial(\beta F)}{\partial \beta} = \frac{3\pi^2 N_c^2}{8} T^4 \,, \\ s &= -\frac{1}{V_3} \frac{\partial F}{\partial T} = \frac{\pi^2 N_c^2}{2} T^3 \,, \end{aligned} \tag{22}$$

which agrees with the previous results. Thence you have shown how to calculate the thermodynmic quantities of the strongly coupled CFT at finite temperature.

3 CFT at weak coupling: free thermal gas

Let us finally calculate the entropy density of the SYM at small coupling. Recall from (for example your cosmology/statistical physics course) that we have

$$s_{\text{boson}} = \frac{2\pi^2}{45} T^3 \,, \quad s_{\text{fermion}} = \frac{7}{8} s_{\text{boson}} \,. \tag{23}$$

The $\mathcal{N} = 4$ $SU(N_c)$ SYM (whose fields are in adjoint representation) has 2 (photons)+ 6 (scalars) $= 8 \times N_c^2$ degrees of bosonic dof. Since the theory is supersymmetric, it has the sam number of fermionic degrees of freedom. Thus we have the following contribution to the entropy:

$$s_{\text{free}} = 8N_c^2 s_{\text{boson}} + 8N_c^2 s_{\text{fermion}} = (8+7) N_c^2 s_{\text{boson}} = \frac{2\pi^2}{3} N_c^2 T^3 \,. \tag{24}$$

When compared to the result calculated in AdS/CFT, we find

$$s_{\text{free}} = \frac{4}{3} s_{\text{BH}} \,. \tag{25}$$

The mismatch is not surprising – it is the difference between strong and weak coupling.