Homework 1: Hawking–Page Transition

Due: 11am Thursday, Oct 31, 2024

As shown first by Hawking and Page in 1983, [1], thermodynamics of black holes in AdS space is not only (contrary to that of their asymptotically flat cousins) *well defined* (one can have black holes with positive specific heat) but it is also rather interesting – for example it features various intriguing *phase transitions*. Such phase transitions then correspond to the associated phase transitions of the dual CFT.

Let us show this explicitly for the Schwarzschild-AdS metric:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2}\,. \tag{1}$$

a) By employing the Euclidean trick in the lecture, show that the black hole temperature is

$$T = \frac{f'(r_+)}{4\pi} = \frac{\ell^2 + 3r_+^2}{4\pi\ell^2 r_+}\,. \tag{2}$$

(Hint: solve the horizon equation $f(r_+) = 0$ for M.)

b) Show that for fixed ℓ, there exists a minimal temperature T_{min} for which these black holes can exist. Calculate $r_+ = r_m$ for which this happens, and show that

$$T_{\text{min}} = \frac{\sqrt{3}}{2\pi}\frac{1}{l}\,. \tag{3}$$

Show that for fixed $T > T_0$ there are two possible solutions for r_+. Correspondingly we talk about two branches of *'small'* and *'large'* AdS black holes – why?

c) To find which branch is thermodynamically preferred, let us calculate the entropy S and the specific heat C. Of course, this can be done via Euclidean path integral, employing the AdS counterterms, as discussed in the lecture (please do it if you do not believe me). Instead, let us use the area law for the entropy (valid for Einstein's gravity). Verify that with this the first law of thermodynamics:

$$\delta M = T\delta S \tag{4}$$

is satisfied. Moreover, by calculating the specific heat: $C = T\left(\frac{\partial S}{\partial T}\right)_\ell$, show that large black holes are locally thermodynamically stable (have $C > 0$) whereas small black holes are 'Schwarzschild-like' (with $C < 0$).

d) Let us further believe in the laws of thermodynamics. Show that the free energy then takes the following form (again this can be justified by calculating Z):

$$F = M - TS = \frac{r_+(\ell^2 - r_+^2)}{4\ell^2}\,. \tag{5}$$

Using Mathematica, plot parametrically (using r_+ as parameter) the free energy diagram $F = F(T, \ell)$, and identify the corresponding branches of black holes.

e) Compare the obtained (black hole) free energy with the free energy of *thermal AdS* (AdS filled with thermal radiation), approximately given by

$$F_{\text{AdS}} \approx 0\,. \tag{6}$$

Since the *global minimum* of the two corresponds to the stable phase, show that exists a critical temperature

$$T = T_{\text{HP}} = \frac{1}{\pi\ell}\,, \tag{7}$$

the so called *Hawking–Page temperature*, at which there is a first-order phase transition from thermal AdS (which is stable for $T < T_{\text{HP}}$) to large black hole phase (which dominates above T_{HP}).

Via the AdS/CFT correspondence, this phase transition has an interpretation of the *confinement/deconfinement* phase transition of the dual quark–gluon plasma [2]. Even more interesting phase transitions occur for charged and rotating AdS black holes, e.g. [3].

References

[1] S.W. Hawking and D.N. Page, *Thermodynamics of Black Holes in Anti-de Sitter Space*, Commun. Math. Phys. *87*, 577(1983).

[2] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys.*2*, 505 (1998).

[3] D. Kubiznak, R. B. Mann and M. Teo, *Black hole chemistry: thermodynamics with Lambda*, Class. Quant. Grav. **34** (2017) no.6, 063001 [arXiv:1608.06147 [hep-th]].