Homework 2: Holographic renormalization

Due: 10am Thursday December 12, 2024

When discussing thermodynamics of AdS black holes, we have encountered *gravitational counter terms* whose purpose was to yield the on-shell action finite and to tune its particular value. These counter terms can be systematically derived by a method known as the *holographic renormalization*. Following [1], the goal of the present homework is to show you how.

For simplicity we concentrate on deriving counter terms for the massive scalar field, given by the standard action

$$S_0 = \frac{C}{2}\int d^{d+1}x\sqrt{g}\Big(g^{ab}\partial_a\phi\partial_b\phi + m^2\phi^2\Big)\,, \tag{1}$$

in a $(d+1)$-dimensional (Euclidean) AdS space, written in *Fefferman–Graham coordinates*:

$$ds^2 = g_{ab}dx^a dx^b = \ell^2\Big(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\delta_{\mu\nu}dx^\mu dx^\nu\Big)\,, \tag{2}$$

yet another convenient coordinate system for AdS spacetime, obtained from the Poincare coordinates by setting $z^2 = \rho$. In particular, the boundary is located at $\rho = 0$, whereas $\rho \to \infty$ corresponds to the Poincare horizon.

1 Equations of motion

a) Show that the scalar equation can be written as

$$(\Box - m^2)\phi = 0\,, \quad \Box\phi = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b\phi)\,, \tag{3}$$

and write it explicitly for the metric (2).

b) Perform the following expansion of the scalar near the boundary:

$$\phi(\rho, x) = \rho^{(d-\Delta)/2}\Big(\underbrace{\phi_0(x) + \rho\phi_2(x) + \rho^2\phi_4(x) + \dots}_{\varphi(\rho,x)}\Big)\,. \tag{4}$$

Thence show that the EOM can be written as

$$[m^2\ell^2 - \Delta(\Delta - d)]\varphi - \rho\Box_0\varphi - 2(d - 2\Delta + 2)\rho\partial_\rho\varphi - 4\rho^2\partial_\rho^2\varphi = 0\,, \tag{5}$$

where $\Box_0 = \delta^{\mu\nu}\partial_\mu\partial_\nu$.

c) Argue that at the two leading orders in ρ (in principle one can go to higher orders as well) give the following relations:

$$m^2\ell^2 = \Delta(\Delta - d)\,, \quad \phi_2 = \frac{1}{2(2\Delta - d - 2)}\Box_0\phi_0\,, \tag{6}$$

assuming the denominator of the latter does not vanish (if it does vanish a logarithmic term needs to be added, e.g. [1]). Do you recognize the first relation?

2 Action regularization

a) Argue that, under a certain condition, the (regularized) on-shell action (1) can be written as

$$S_r = -\frac{C}{2}\int d^dx\sqrt{g}g^{\rho\rho}\phi\partial_\rho\phi\Big|_{\rho=\epsilon}^{\infty}\,, \tag{7}$$

where we have introduced the cutoff close to the boundary, at $\rho = \epsilon$. In what follows we further assume that we deal with solutions that vanish (sufficiently fast) at the Poincare horizon so that the upper boundary does not contribute.

b) Show that explicitly we get

$$S_r = C\ell^{d-1}\int d^dx\Big(\epsilon^{-\Delta+\frac{d}{2}}a_0 + \epsilon^{-\Delta+\frac{d}{2}+1}a_2 + \dots\Big)\,, \tag{8}$$

where

$$a_0 = -\frac{1}{2}(d-\Delta)\phi_0^2\,, \quad a_2 = -\frac{d-\Delta+1}{2(2\Delta - d - 2)}\phi_0\Box_0\phi_0\,. \tag{9}$$

c) Argue that $\Delta > d/2$. That is, you have shown that the action diverges and needs to be renormalized.

3 Action renormalization

a) To subtract the above divergences we introduce *counter terms*. These counter terms better be covariantly expressed on the boundary – constructed from the *induced metric* on the boundary:

$$\gamma_{\mu\nu} = \frac{\ell^2}{\epsilon}\delta_{\mu\nu}\,, \quad \Box_\gamma = \gamma^{\mu\nu}\partial_\mu\partial_\nu\,, \tag{10}$$

and from the boundary field $\phi(\epsilon, x)$, and its derivatives such as $\Box_\gamma\phi(\epsilon, x)$. For this we need to invert the Fefferman–Graham expansion (4).

Show that, to second order in ϵ this inversion reads

$$\begin{aligned} \phi_0 &= \epsilon^{-(d-\Delta)/2}\Big(\phi(\epsilon, x) - \frac{1}{2(2\Delta - d - 2)}\Box_\gamma\phi(\epsilon, x)\Big)\,, \\ \phi_2 &= \epsilon^{-(d-\Delta)/2-1}\frac{1}{2(2\Delta - d - 2)}\Box_\gamma\phi(\epsilon, x)\,. \end{aligned} \tag{11}$$

b) Thence show that, plugging this back to (8), we can express the divergences in terms of the boundary field $\phi_b(x) = \phi(\epsilon, x)$ and introduce the following counter terms that cancel them:

$$S_{ct} = \frac{C}{\ell}\int d^dx\sqrt{\gamma}\Big(\frac{d-\Delta}{2}\phi_b^2(x) + \frac{1}{2(2\Delta - d - 2)}\phi_b(x)\Box_\gamma\phi_b(x) + \dots\Big)\,. \tag{12}$$

While derived in FG coordinates, the resulting expression for the counter terms is *covariantly written* on the boundary, and can be evaluated in any coordinates! The total action with these counter terms,

$$S = S_0 + S_{ct}\,, \tag{13}$$

is then finite and can be used for calculating the correlation functions.

c) Argue that in particular for a 1-point function we have

$$\langle O(x)\rangle = -\frac{\delta S}{\delta\phi_0(x)} = -\lim_{\epsilon\to 0}\Big(\frac{\ell^d}{\epsilon^{\Delta/2}}\frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta\phi(\epsilon, x)}\Big)\,. \tag{14}$$

Using S instead of S_0 in the calculation of the 2-point function in the lecture would lead directly to the expected finite result.

4 Holographic stress tensor

While we have not done the renormalization of the gravitational action, can you argue the following relation between the boundary energy momentum tensor $\tau_{\mu\nu}(h)$ discussed in the third lecture and the expectation value of the CFT stress tensor?

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{\det g_0}}\frac{\delta S}{\delta g_0^{\mu\nu}(x)} = \lim_{\epsilon\to 0}\left(\frac{\ell^{d-2}}{\epsilon^{d/2-1}}\tau_{\mu\nu}(h)\right). \tag{15}$$

Hint: Replace $\delta_{\mu\nu}dx^\mu dx^\nu$ in (2) with arbitrary curved metric $g_0 + \rho g_1 + \dots$.

Summary. If you got all the way here, you now understand the way the *holographic renormalization* works for scalar fields in AdS, and how to calculate the expectation value of the CFT energy momentum tensor: CONGRATULATIONS :)

References

[1] Ammon, Martin, and Johanna Erdmenger. *Gauge/gravity duality: Foundations and applications.* Cambridge University Press, 2015.