Tutorial 1: Solutions

1 Unruh radiation

a) Let us consider a spacetime associated with a uniformly accelerated observer, characterized by the proper acceleration $a = \sqrt{|a_\mu a^\mu|}$, whose coordinates (T, X, y, z) are related to the Minkowski coordinates (t, x, y, z) by

$$t = \Bigl(\frac{1}{a} + X\Bigr)\sinh(aT)\,, \quad x = \Bigl(\frac{1}{a} + X\Bigr)\cosh(aT)\,. \tag{1}$$

Starting from the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2\,, \tag{2}$$

we calculate

$$\begin{aligned} dt &= \sinh(aT)dX + \Bigl(\frac{1}{a} + X\Bigr)a\cosh(aT)dT\,, & (3)\\ dx &= \cosh(aT)dX + \Bigl(\frac{1}{a} + X\Bigr)a\sinh(aT)dT & (4)\end{aligned}$$

Thus,

$$\begin{aligned} -dt^2 + dx^2 &= -(\sinh(aT)dX + \Bigl(\frac{1}{a} + X\Bigr)a\cosh(aT)dT)^2\\ &\quad +(\cosh(aT)dX + \Bigl(\frac{1}{a} + X\Bigr)a\sinh(aT)dT)^2\\ &= -(1+aX)^2dT^2 + dX^2\,, \end{aligned} \tag{5}$$

upon using $\cosh^2(\,) - \sinh^2(\,) = 1$. Thus we recovered the Rindler spacetime:

$$ds^2 = -(1+aX)^2 dT^2 + dX^2 + dy^2 + dz^2 \,. \tag{6}$$

b) Let us now Wick rotate, introducing the Euclidean time, $\tau = iT$. Then we have the following Euclidean metric:

$$ds_E^2 = (1+aX)^2 d\tau^2 + dX^2 + dy^2 + dz^2 \,. \tag{7}$$

To simplify this, let us introduce a new coordinate ρ, by

$$\rho = \frac{1+aX}{a} \quad \Rightarrow \quad d\rho = dX \,, \tag{8}$$

so that

$$ds_E^2 = a^2\rho^2 d\psi^2 + d\rho^2 + \cdots = \rho^2 d\varphi^2 + d\rho^2 \,, \tag{9}$$

upon introducing a new angle coordinate, $\varphi = a\tau$. This looks like a flat space written in polar coordinates, provided the angle φ has a period 2π, otherwise there is a conical singularity at $\rho = 0$, which corresponds to the original Rindler horizon. The reasoning now goes as follows: since the Rindler horizon was originally non-singular, we expect it to be non-singular again. This is achieved by setting (we want to avoid conical singularity)

$$\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \tau \sim \tau + \underbrace{2\pi/a}_{\beta} \quad \Leftrightarrow \quad \boxed{T = \frac{a}{2\pi}\,,} \tag{10}$$

which is the famous <u>Unruh temperature.</u> That is, an accelerated observer sees a thermal bath at a temperature proportional to his acceleration. If you accelerate really fast, you can cook a chicken.

c) Let us next calculate the partition function and derive the entropy of the Rindler horizon. To this purpose we have to calculate the classical action,

$$S_E = \int_\Omega \frac{d^4x\sqrt{g}R}{16\pi G} + \int_{\partial\Omega} \frac{d^3x\epsilon\sqrt{h}K}{8\pi G}\,, \tag{11}$$

(where $\epsilon = -1$ for spacelike and $\epsilon = 1$ for timelike boundary), evaluated for the Rindler Euclidean metric (7).

Of course, since the metric is flat we have $R = 0$ and the first term vanishes. It is the second term that determines the value of the action. This is done as follows. We introduce a boundary at $X = X_0$ =const., calculate the contribution of the second term and then let $X_0 \to \infty$. The boundary has a normal $n^\mu = (0, 1, 0, 0)$ and the corresponding extrinsic curvature is

$$K = \nabla_\mu n^\mu = \frac{1}{\sqrt{g}}(\sqrt{g}n^\mu)_{,\mu} = \frac{a}{1 + aX_0}\,, \tag{12}$$

where we used that $g = (1+aX)^2$. At the same time the boundary metric has the following determinant: $\sqrt{h} = 1 + aX_0$. We thus have ($\epsilon = -1$)

$$S_E = -\underbrace{\int d\tau}_{\beta} \int \frac{dxdy\sqrt{h}K}{8\pi G} = -\frac{a\beta}{8\pi G}\underbrace{\int dydz}_{A} = -\frac{a\beta A}{8\pi G}\,, \tag{13}$$

where A is the (regularized) 'perpendicular Rindler horizon area'.

d) We thus have (setting $G = 1$)

$$F = \frac{S_E}{\beta} = -\frac{a}{2\pi}\frac{A}{4} = -T\frac{A}{4}\,, \tag{14}$$

and so

$$\boxed{S = -\frac{\partial F}{\partial T} = \frac{A}{4}\,,} \tag{15}$$

which is the Bekenstein result. Note also that

$$E = \frac{\partial(\beta F)}{\partial \beta} = 0\,, \tag{16}$$

and that $F = M - TS = -TS$ as the energy of the spacetime is zero.

2 Bekenstein's universal bound

Bekenstein bound provides an upper limit on the thermodynamic entropy of classical and quantum systems:

$$S \leq \frac{2\pi k_B R E}{\hbar c}\,, \tag{17}$$

where E is the total energy of the system, and R is the radius of a sphere enclosing it.

a) Let us qualitatively outline proof of this statement for small E. To this purpose, we consider a system with (small) energy E and (arbitrary) entropy S, contained in a box of radius R. We then consider a black hole with (large) mass M and the same radius R. Its entropy is $S_{\rm BH} \propto M^2$. Let's next lower the system to the black hole, obtaining a black hole of mass $M + E$. Since the total entropy cannot decrease by the generalized second law, we must have (omitting all the pre-factors)

$$S + M^2 \leq (M + E)^2 \approx M^2 + 2ME + O(E^2)\,. \tag{18}$$

Hence we have,

$$S \leq 2ME \approx RE\,, \tag{19}$$

using the fact that the original black hole had a size $R \propto M$.

b) Note that the above bound is saturated for Schwarzschild black hole entropy,

$$S = S_{\rm BH} = \pi r_+^2 = 2\pi r_+ M = \frac{A}{4}\,. \tag{20}$$

We can thus understand this as a a universal bound on the amount of information in a given spatial region with a boundary of area A:

$$\boxed{S \leq \frac{A}{4}\,,} \tag{21}$$

as measured in Planck units. This means that the upper bound is bounded holographically – by the area of the region instead of its volume.