Introduction to black hole thermodynamics

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Black hole thermodynamics: classical and quantum
Lecture 1
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Plan for Lecture 1

I. Black holes as thermodynamic objects
  I. From BH mechanics to BH TDs
  II. Black hole evaporation

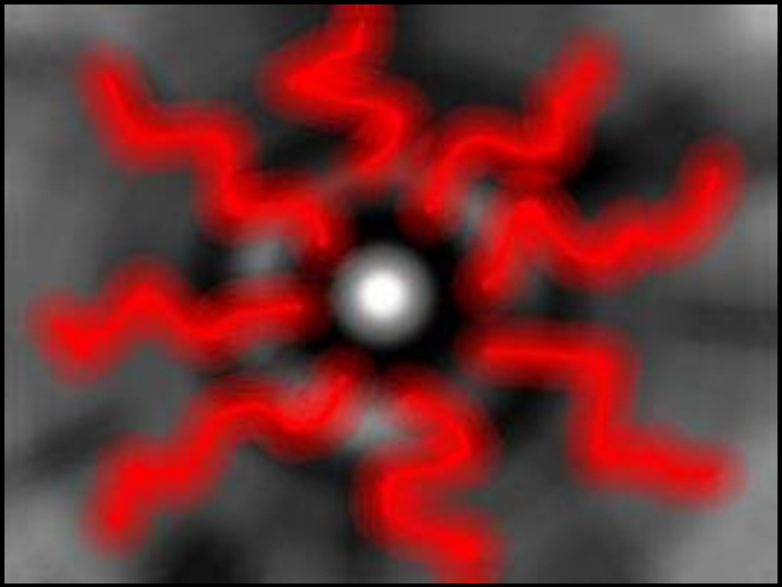
II. Euclidean magic
  I. Temperature from regularity
  II. Entropy from gravitational action

III. Horizon TDs: examples
  I. Schwarzschild BH
  II. Rindler
  III. De Sitter

IV. What is going on? A few words about QFT in CS

V. Summary

I. Black Holes as Thermodynamic Objects



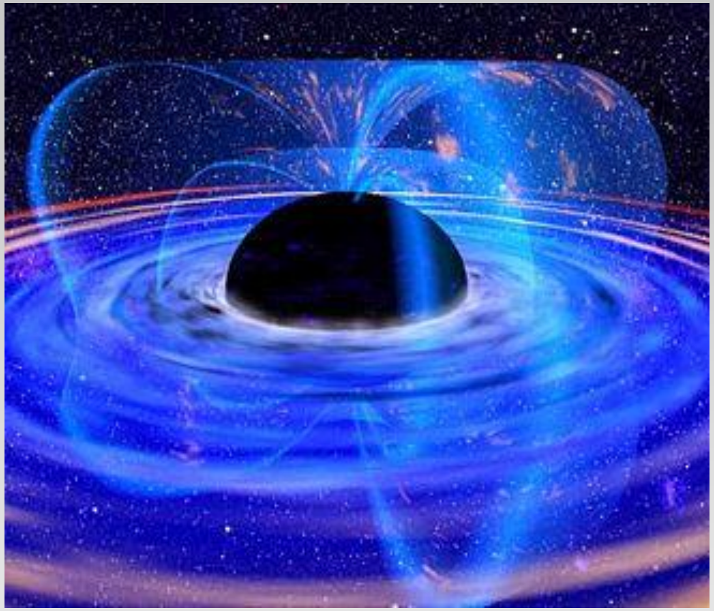
Black holes as thermodynamic objects

If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations-then so much the worse for Maxwell's equations. If it is found to be contradicted by observation-well these experimentalists do bungle things sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation.

Sir Arthur Stanley Eddington

Gifford Lectures (1927), *The Nature of the Physical World* (1928), 74.

Black holes and their characteristics



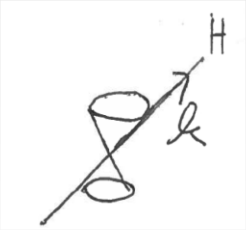
Schwarzschild black hole:

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \qquad f = 1 - \frac{2m}{r}$$

- asymptotic mass (total energy)

$$M = -\frac{1}{8\pi}\int_{S^2_\infty} *dk = m \qquad k = \partial_t$$

- black hole horizon: (radius $r_+ = 2M$)



surface gravity

$$(k^b \nabla_b k^a)_{|H} = \kappa k^a_{|H}$$

$$\Rightarrow \quad \kappa = \frac{f'(r_+)}{2} = \frac{M}{r_+^2} = \frac{M}{(2M)^2} = \frac{1}{4M} = \frac{1}{2r_+}$$

horizon area:

$$A = \int \sqrt{\det \gamma}\, d\theta d\varphi = \int r_+^2 \sin\theta\, d\theta d\varphi = 4\pi r_+^2$$

Schwarzschild characteristics: summary

Horizon: $$r_+ = 2M$$

Mass: $$M$$

Surface gravity: $$\kappa = \frac{f'(r_+)}{2} = \frac{1}{2r_+}$$

Horizon area: $$A = \int \sqrt{\det \gamma}\, d\theta d\varphi = 4\pi r_+^2\,.$$

Good idea:

$$dM = \frac{dr_+}{2}\,, \quad dA = 8\pi r_+ dr_+$$

1st law of black hole mechanics:

$$dM = \frac{\kappa}{2\pi}\frac{dA}{4}$$

Laws of black hole mechanics

- Bardeen, Carter, Hawking (1973)

- **Zeroth law:** The surface gravity κ is constant on the black hole horizon.
- **First law:**

$$\boxed{dM = \frac{\kappa}{2\pi}\frac{dA}{4} + \underbrace{\Omega dJ + \Phi dQ}_{\text{work terms}} \, .} \qquad (5.8)$$

Here, Ω is the angular velocity of the black hole horizon, and Φ is its 'electrostatic potential'.

- **Second law:** Classically, the area of the horizon never decreases (provided the null energy condition holds).

$$\boxed{dA \geq 0 \, .} \qquad (5.9)$$

- **Third law:** It is impossible to reduce κ to zero in a finite number of steps.

- Essentially equivalent to **gravitational dynamics**
- Despite the resemblance with laws of TDs, **classical** BHs **are black**

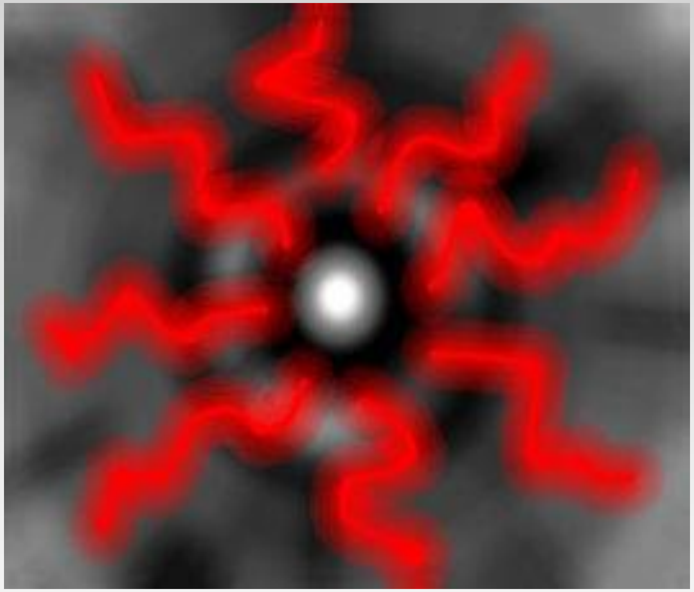
Black hole thermodynamics?

$$dM = \frac{\kappa}{2\pi}\frac{dA}{4}$$ Bekenstein? ⟺ $$dE = TdS$$

Hawking (1974):

$$T = \frac{\kappa}{2\pi}$$ ⇨ $$S = \frac{A}{4}$$

derived using QFT in curved spacetime



Other approaches: Euclidean path integral approach (Gibbons & Hawking-1977), tunnelling, LQG, string theory,…

Classical laws of black hole mechanics become laws of "normal" thermodynamics

Black hole entropy

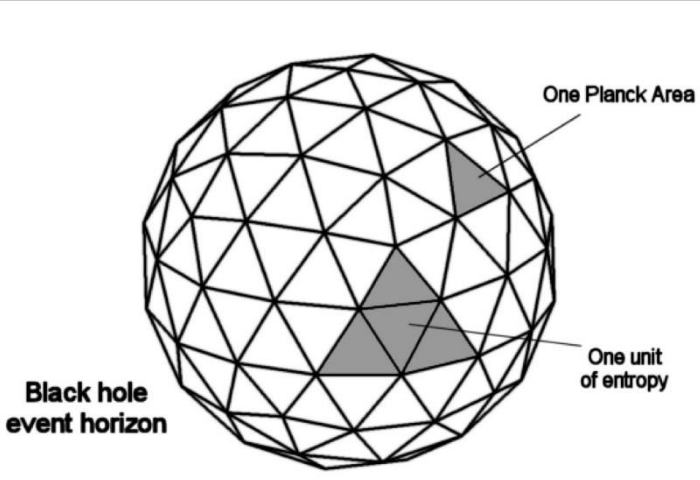
relativity

Stat. mech

$$S = \frac{A}{4}\frac{c^3 k_B}{\hbar G_N}$$

gravity

QM



- Is huge: $$S = \frac{k_B}{4}\frac{A}{l_P^2}\,, \quad l_P = \sqrt{\frac{G\hbar}{c^3}}$$

- Is **holographic**: $$S \propto A$$

- Bekenstein's (universal) bound: $$S \leq \frac{A}{4}$$

Black hole evaporation

- Hawking temperature for Schwarzschild

$$T = \frac{\hbar c^3}{8\pi k_B G M} \propto \frac{1}{M} \qquad \sim 6 \times 10^{-8} \frac{M_\odot}{M} K$$

- Effective Stefan-Boltzmann law:

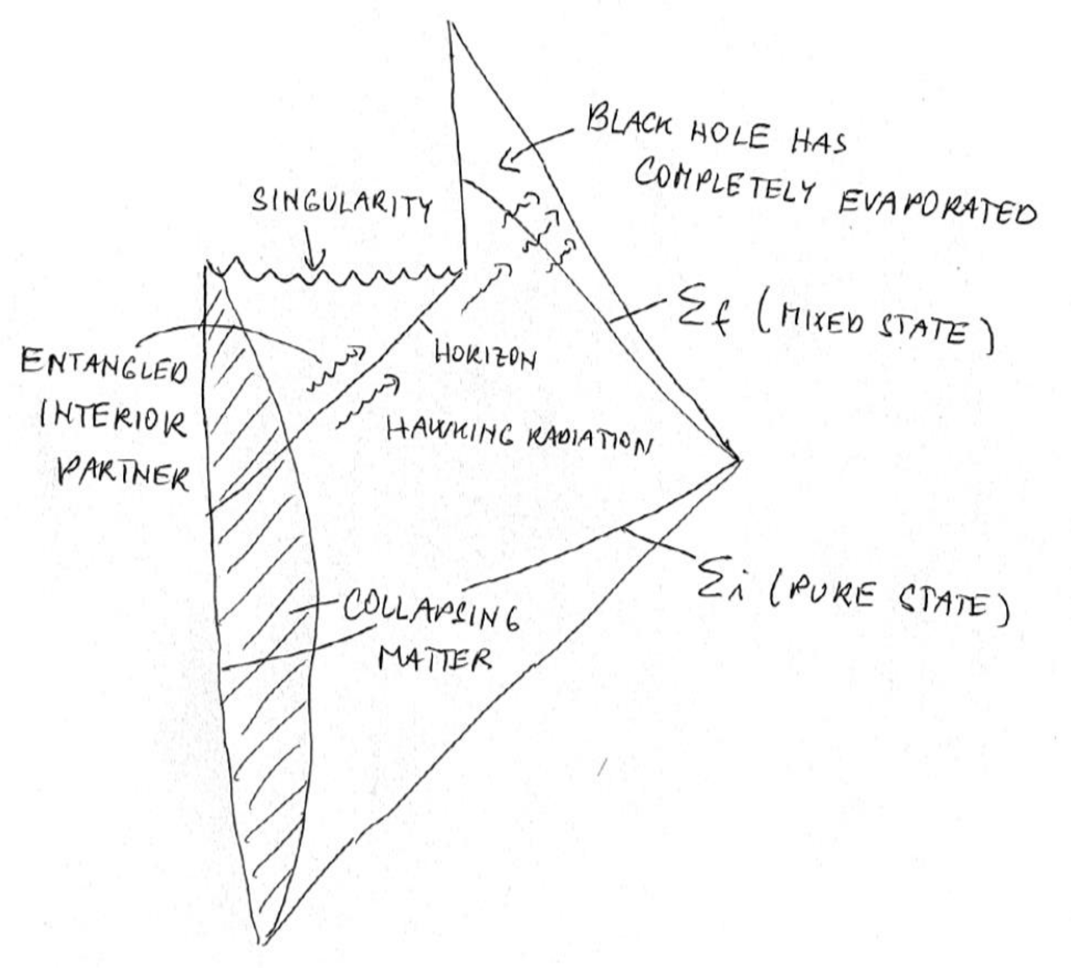
$$\frac{dM}{dt} \propto -\sigma T^4 A \propto -\frac{1}{M^2}$$

- BH completely evaporates $t_{\rm evap} \approx \left(\frac{M}{M_\odot}\right)^3 \times 10^{71}\, s$

- Note also that:
  i) negative **specific heat** $C = T\frac{\partial S}{\partial T} = -\frac{1}{8\pi T^2}$

  ii) **Generalized 2nd law:** $S_{\rm TOT} = S_{\rm BH} + S_{\rm outside} \geq 0$

Black hole info paradox (Hawking 1976)



- Thermal Hawking radiation leads to black hole evaporation.
- If BH completely evaporates, we violated unitary evolution of QM (evolved from the pure state in the beginning to a mixed state at the end) – **info loss** (see later in the course).
- Contradicts the intuition from **AdS/CFT correspondence**

$$Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}$$

II. Euclidean magic

- G. Gibbons, S. Hawking*, Action integrals and partitions functions in quantum gravity*, Phys. Rev D 15, 2752, 1977.
- G. Gibbons and S. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, Phys. Rev D 15, 2738, 1977.

Euclidean trick (Gibbons & Hawking 1977)

• **Thermal Green functions** have periodicity in **Euclidean** time

$$\tau = it$$

$$G(\tau) = G(\tau + \beta)\,, \quad \beta = 1/T\,.$$

(Conversely, periodicity of G defines a thermal state. A thermometer interacting with the given field for a long time will register this temperature.)

• **Quantum fields** in the vicinity of black holes have this property (as seen by distant static observers).

• What about the **gravitational field** itself? Consider **Euclideanized Schwarzschild**:

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2$$

Euclidean trick (Gibbons & Hawking 1977)

- **Near horizon** expand:

$$f = \underbrace{f(r_+)}_{0} + \underbrace{(r - r_+)}_{\Delta r} \underbrace{f'(r_+)}_{2\kappa} + \cdots = 2\kappa \Delta r$$

$$ds^2 = 2\kappa \Delta r d\tau^2 + \frac{dr^2}{2\kappa \Delta r} + r_+^2 d\Omega^2$$

- Change variables:

$$d\rho^2 = \frac{dr^2}{2\kappa \Delta r} \quad \Leftrightarrow \quad d\rho = \frac{dr}{\sqrt{2\kappa \Delta r}} \quad \Leftrightarrow \quad \Delta r = \frac{\kappa}{2}\rho^2$$

$$ds^2 = \kappa^2 \rho^2 d\tau^2 + d\rho^2 + r_+^2 d\Omega^2 = \rho^2 d\varphi^2 + d\rho^2 + \ldots .$$

$$\varphi = \kappa\tau$$

… looks like **flat space** in polar provided:

φ has a period 2π.

(otherwise conical singularity exists at ρ=0)

Euclidean trick (Gibbons & Hawking 1977)

- **Original manifold** non-singular:

$$\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \tau \sim \tau + \underbrace{2\pi/\kappa}_{\beta} \quad \Leftrightarrow \quad \boxed{T = \frac{\kappa}{2\pi}\,,}$$

… which is the **Hawking's temperature**.

Gravitational partition function

$$\boxed{Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}}$$

(using WKB approximation)

- **Free energy**:

$$F = -\frac{1}{\beta}\log Z \approx \frac{S_E}{\beta} \quad \Rightarrow \quad S = -\frac{\partial F}{\partial T}$$

Euclidean trick (Gibbons & Hawking 1977)

- **Gravitational action**:

$$S_E = \int_\Omega \frac{d^4x\sqrt{g}R}{16\pi G} + \int_{\partial\Omega} \frac{d^3x\epsilon\sqrt{h}K}{8\pi G} + \text{counter terms}$$

Einstein-Hilbert action (gives Einstein equations)

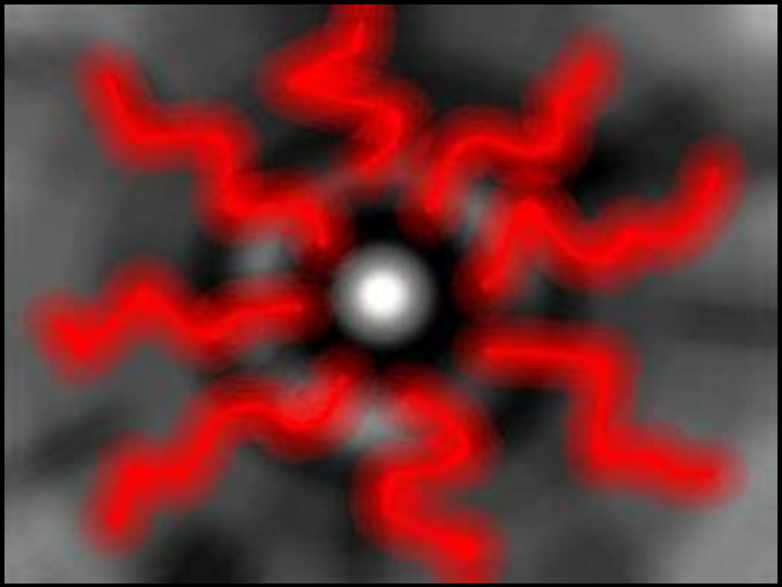
York-Gibbons-Hawking term (yields well posed variational principle with Dirichlet BCs)

Counter terms: “renormalize” the value of the action (In AdS given covariantly by *holographic renormalization*. In flat space no covariant prescription exists!)

- The prescription confirms **Bekenstein’s area law!**

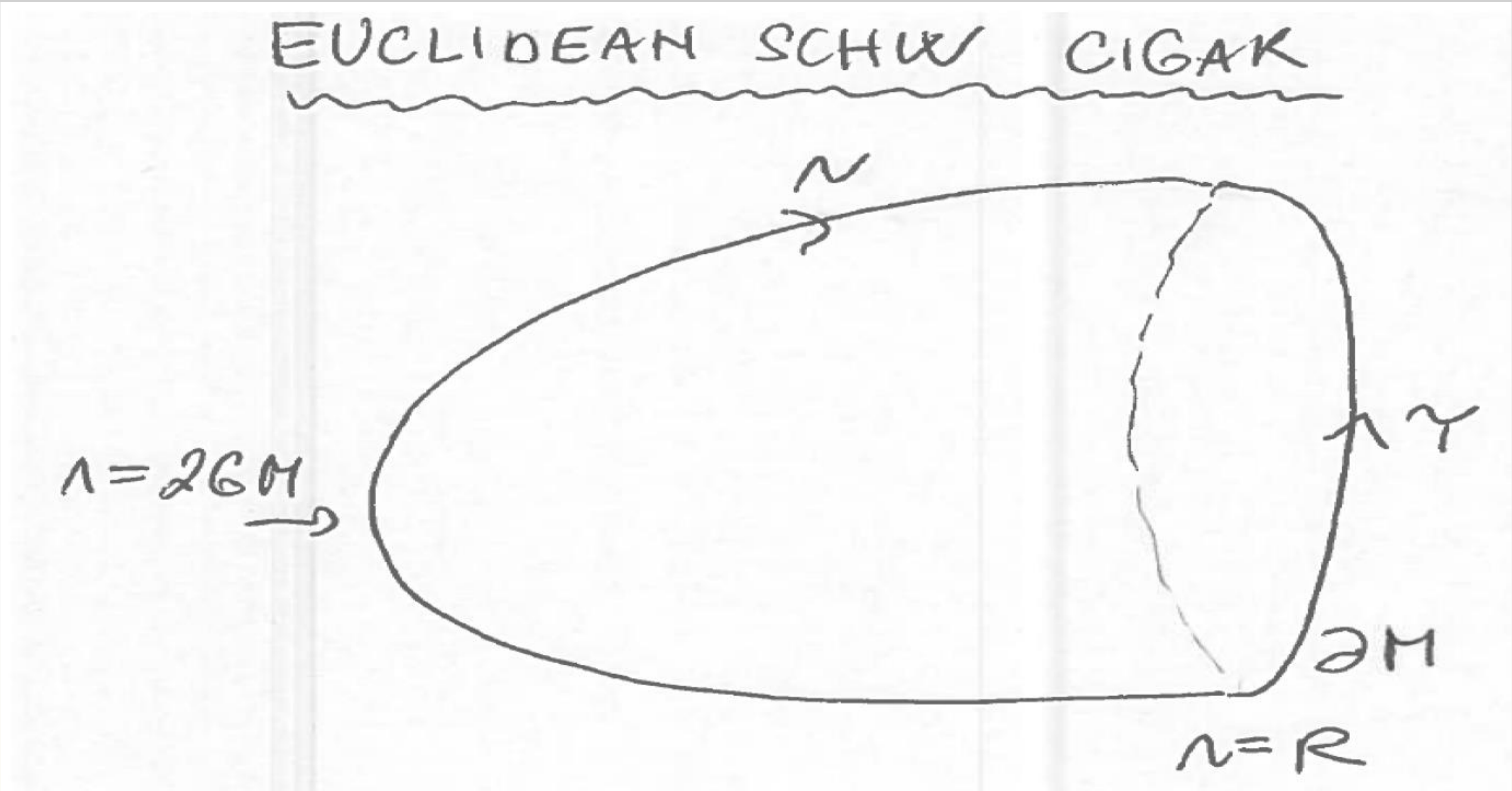
$$S = -\frac{\partial F}{\partial T} = \frac{A}{4}$$

III. Horizon TDs: examples



Example 1: Euclideanized Schwarzschild

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2M}{r}$$



- **Absence** of conical singularity implies:

$$T = \frac{1}{\beta} = \frac{f'(r_+)}{4\pi} = \frac{1}{8\pi M}$$

Example 1: Euclideanized Schwarzschild

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \,, \quad f = 1 - \frac{2M}{r}$$

- **Background subtraction:**

$$S_E = -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}R - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}(K - K_0)$$

- Ricci flat: $R_{\mu\nu} = 0$ ⇨ $S_{\mathrm{EH}} = 0$
- Boundary at: $r = R$ $\quad d\gamma^2 = f(R)d\tau^2 + R^2 d\Omega^2$

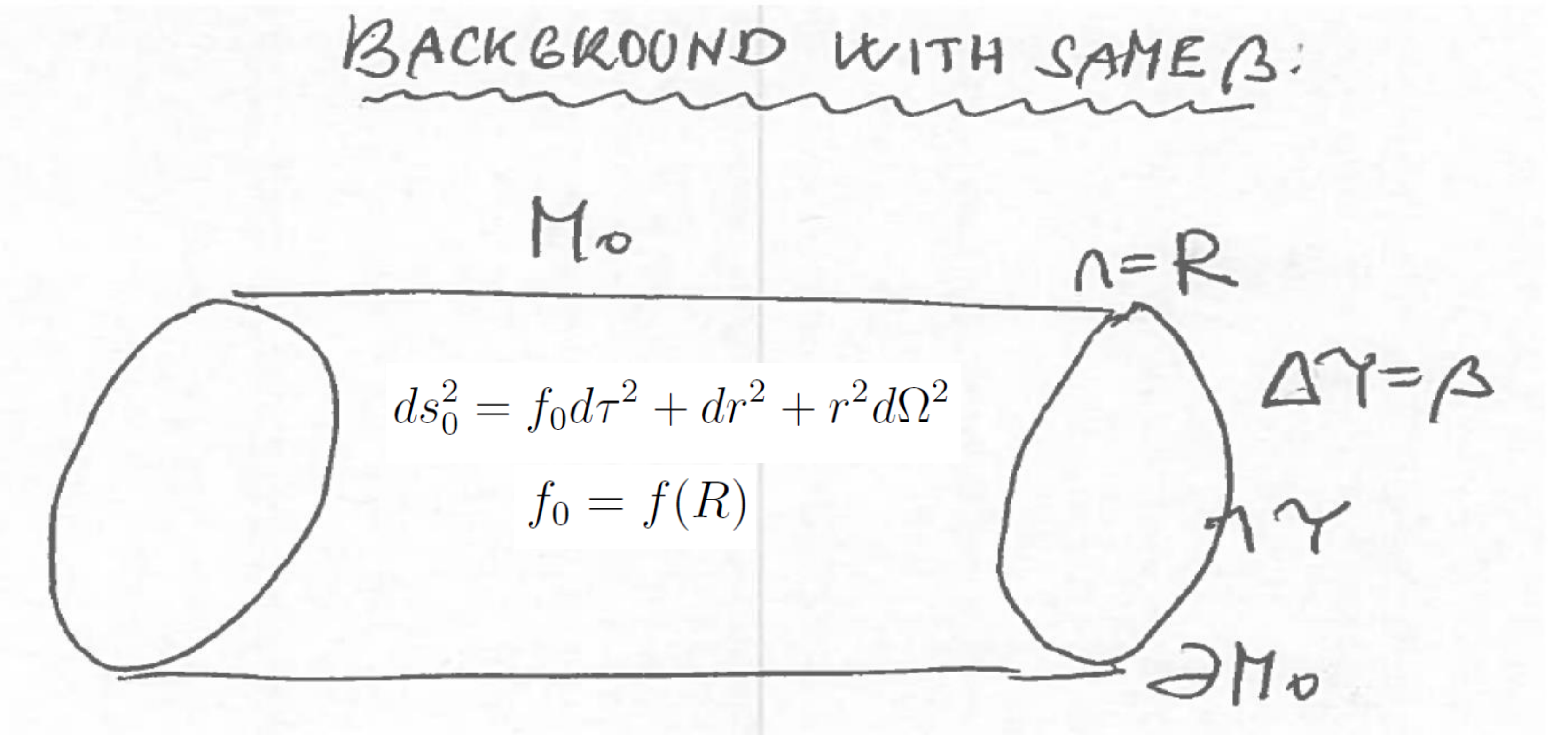
$$K = \nabla_\mu n^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}n^\mu) \qquad n = \sqrt{f}\partial_r\Big|_{r=R}$$

$$\int K\sqrt{h}d^3x = \underbrace{4\pi\beta}_{\int d\tau d\theta d\phi} R^2\left[\frac{2}{R}f(R) + \frac{1}{2}f'(R)\right] = 4\pi\beta(2R - 3M)$$

Example 1: Euclideanized Schwarzschild

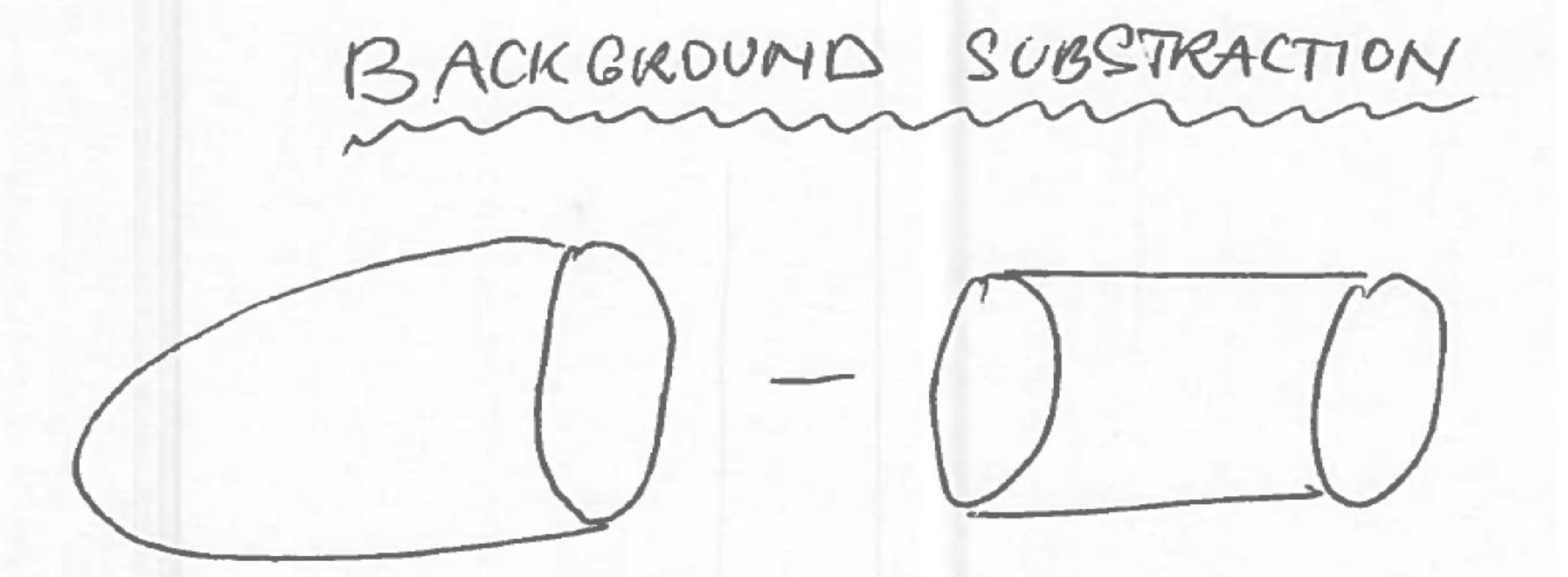
$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \quad f = 1 - \frac{2M}{r}$$

- **Subtract:**



$$\int d^3x \sqrt{h_0} K_0 = 8\pi\beta R\sqrt{f(R)} = 8\pi\beta R\Big(1 - \frac{M}{R} + O(1/R^2)\Big)$$

Example 1: Euclideanized Schwarzschild



$$F = \frac{S_E}{\beta} = \frac{M}{2} = \frac{\beta}{16\pi} = M - TS$$

$$S = -\frac{\partial F}{\partial T} = \frac{\beta^2}{16\pi} = 4\pi M^2 = \pi r_+^2 = \frac{A}{4}$$

… confirmed **Bekenstein's area law!**

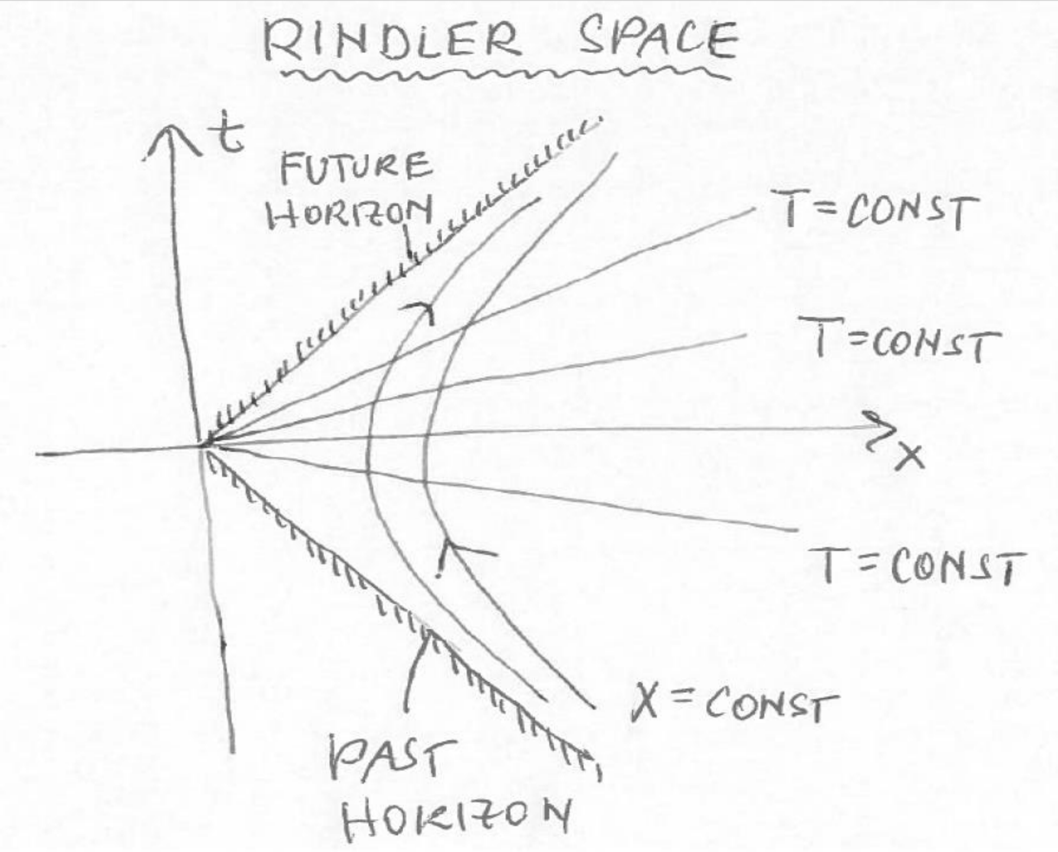
Example 2: Rindler spacetime

- Consider uniformly accelerated observer:

$$a = \sqrt{a_\mu a^\mu} = \text{const}$$

$$t = \frac{1}{a}\sinh(a\tau)\,, \quad x = \frac{1}{a}\cosh(a\tau)$$

- **Rindler frame**:



$$t = \left(\frac{1}{a} + X\right)\sinh(aT)\,,$$

$$x = \left(\frac{1}{a} + X\right)\cosh(aT)$$

Rindler horizon at:

$$X_R = -\frac{1}{a}$$

$$ds^2 = -dt^2 + dx^2 = -(1 + aX)^2 dT^2 + dX^2$$

Example 2: Rindler spacetime

- **Euclidean Rindler**: $\tau = iT$

$$ds_E^2 = (1 + aX)^2 d\tau^2 + dX^2 + dy^2 + dz^2$$

- Zooming on Rindler horizon + change coordinates:

$$\rho = \frac{1 + aX}{a} \quad \Rightarrow \quad d\rho = dX$$

$$ds_E^2 = a^2 \rho^2 d\tau^2 + d\rho^2 + \cdots = \rho^2 d\varphi^2 + d\rho^2$$

$$\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \tau \sim \tau + \underbrace{2\pi/a}_{\beta} \quad \Leftrightarrow \quad \boxed{T = \frac{a}{2\pi},}$$

… which is **Unruh's temperature**.

Example 2: Rindler spacetime

$$ds_E^2 = (1+aX)^2 d\tau^2 + dX^2 + dy^2 + dz^2$$

- Action calculation: $$S_E = \int_\Omega \frac{d^4x\sqrt{g}R}{16\pi G} + \int_{\partial\Omega} \frac{d^3x\epsilon\sqrt{h}K}{8\pi G}$$

nothing

- Boundary: $X = X_0 = \text{const.}$ $\quad n = \partial_X$

$$K = \nabla_\mu n^\mu = \frac{1}{\sqrt{g}}(\sqrt{g}n^\mu)_{,\mu} = \frac{a}{1+aX_0}$$

$$\sqrt{h} = 1 + aX_0$$

$$S_E = -\underbrace{\int d\tau}_{\beta} \int \frac{dxdy\sqrt{h}K}{8\pi G} = -\frac{a\beta}{8\pi G}\underbrace{\int dydz}_{A} = -\frac{a\beta A}{8\pi G}$$

Example 2: Rindler spacetime

$$S_E = -\underbrace{\int d\tau}_{\beta} \int \frac{dxdy\sqrt{h}K}{8\pi G} = -\frac{a\beta}{8\pi G}\underbrace{\int dydz}_{A} = -\frac{a\beta A}{8\pi G}$$

- Free energy: $$F = \frac{S_E}{\beta} = -\frac{a}{2\pi}\frac{A}{4} = -T\frac{A}{4}$$

$$S = -\frac{\partial F}{\partial T} = \frac{A}{4}$$

… which is Bekenstein's result

$$E = \frac{\partial(\beta F)}{\partial \beta} = 0 \quad \Rightarrow \quad F = M - TS = -TS$$

Example 3: de Sitter horizon

- **de Sitter (dS) space** = maximally symmetric solution of EE with positive Lambda:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad \Lambda = \frac{3}{\ell^2}$$

- metric:

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \qquad f = 1 - \frac{r^2}{\ell^2}$$

- Cosmological horizon:

$$f(r_c) = 0 \quad \Rightarrow \quad r_c = \ell$$

- repeating the Euclidean trick we get:

$$T = \frac{|f'(r_c)|}{4\pi} = \frac{1}{2\pi\ell}\,.$$

… which is **Gibbons-Hawking temperature**
(note the absolute value!)

Summary of Lecture 1

1) Black holes are **thermodynamic objects.** They can be assigned **Hawking's temperature** and **Bekenstein's entropy**. Obey the standard laws of TDs.

2) **Euclidean magic** predicts thermodynamics of BH, Rindler, or dS horizons. Namely:

   i. **Regularity** of the **Euclideanized manifold** fixes periodicity of the Euclidean time and yields **Hawking temperature** of the black hole.

   ii. G**ravitational partition function** yields free energy, which in WKB approximation recovers the **Bekenstein's area law** (& other conjugate quantities).

3) **Thermal properties** are associated with particular observers (notion of particles is **ambiguous**) .