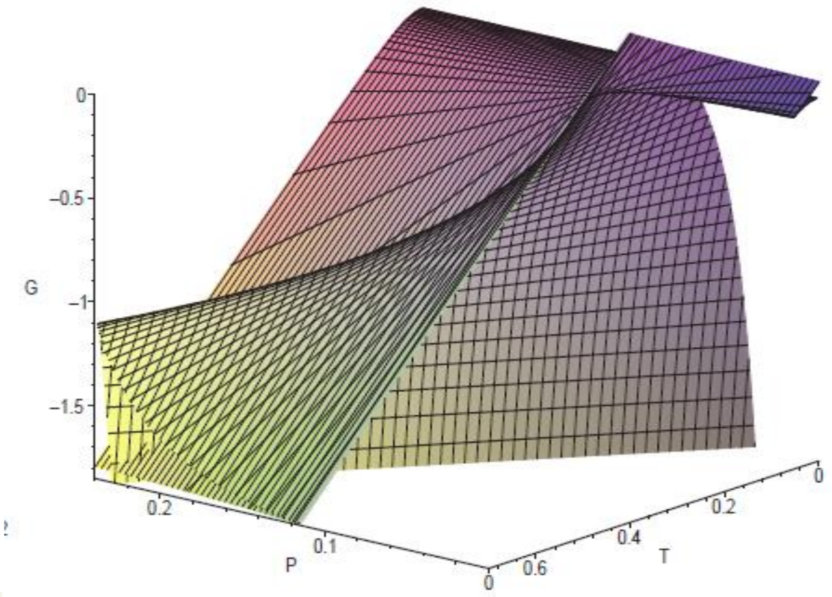
Lecture 2: AdS black holes & black hole chemistry

David Kubizňák
(ITP, Charles University)



Black hole thermodynamics: classical and quantum
Lecture 2
Charles University, 2025

Prelude: Van der Waals fluid & critical phenomena

Johannes Diderik van der Waals (1837-1923)

Van der Waals fluid

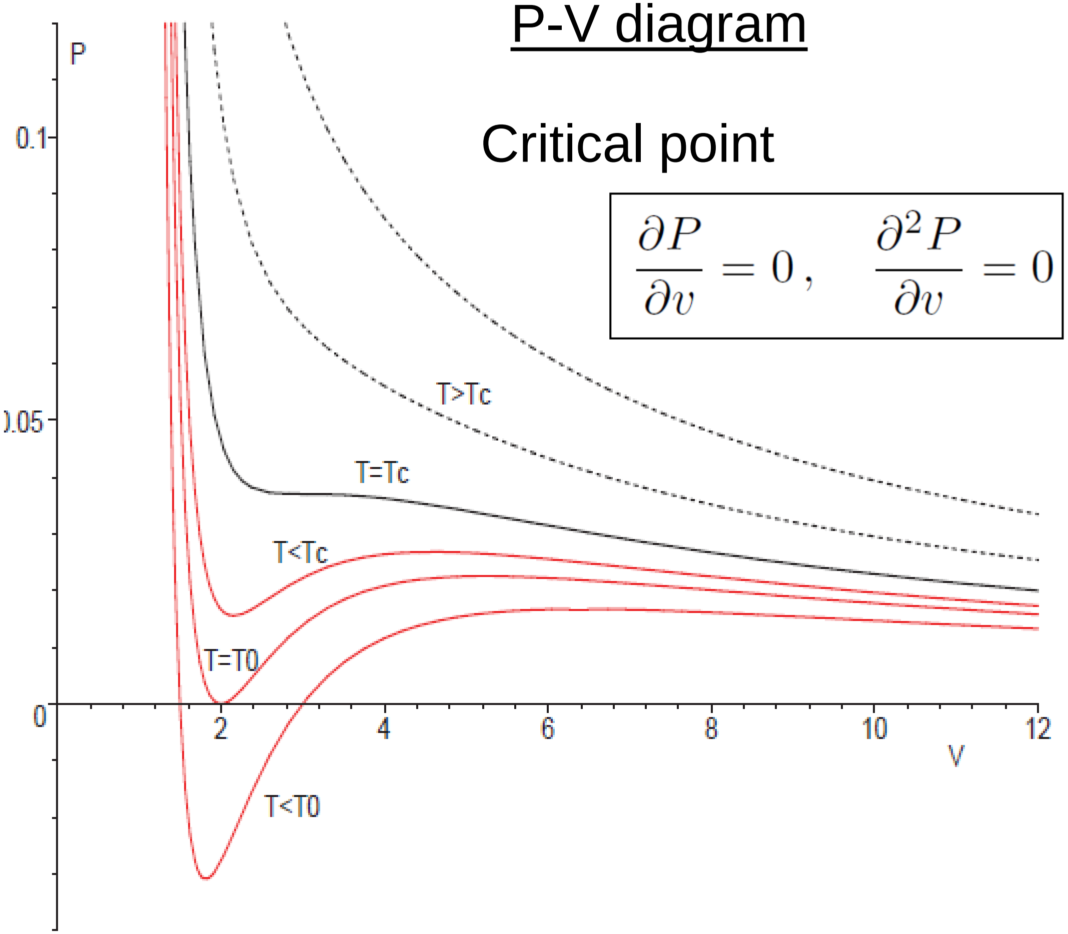
$$\left(P + \frac{a}{v^2}\right)(v - b) = T$$

Parameters a and b characterize the fluid. a measures the **attraction** between particles (a>0) and b corresponds to "**volume of fluid particles**".

P-V diagram

Critical point

$$\frac{\partial P}{\partial v} = 0\,, \qquad \frac{\partial^2 P}{\partial v} = 0$$



Critical point:

$$T_c = \frac{8a}{27b} \qquad v_c = 3b \qquad P_c = \frac{a}{27b^2}$$

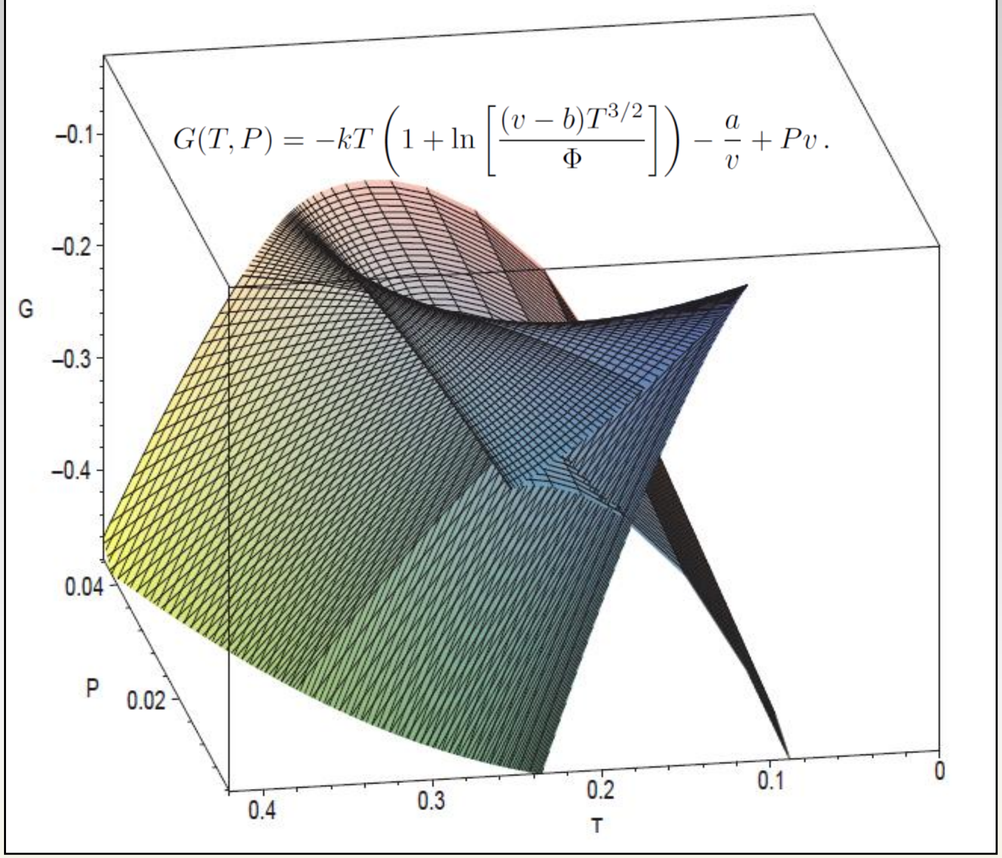
Universal critical ratio

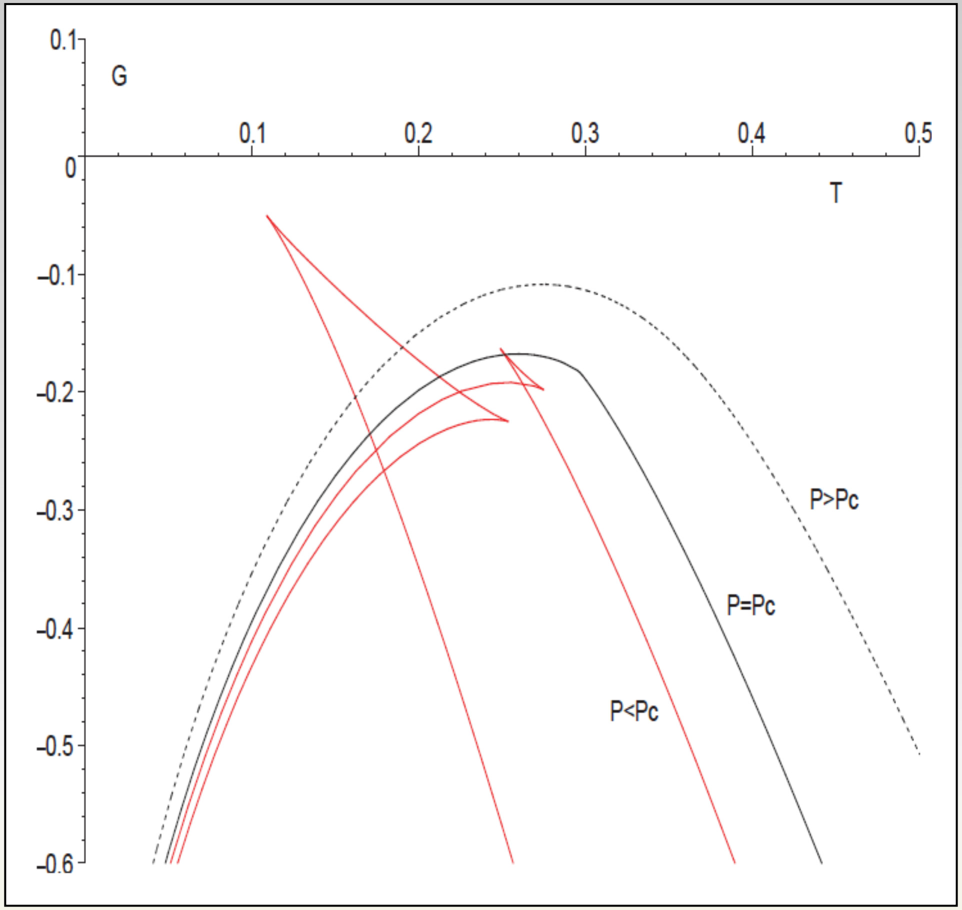
$$\rho_c = \frac{P_c v_c}{T_c} = \frac{3}{8}$$

Gibbs free energy

$$dG = -SdT + vdP$$

Swallow tail behavior





System wants to minimize Gibbs free energy

→

Phase transitions

Maxwell’s equal area law & phase diagrams

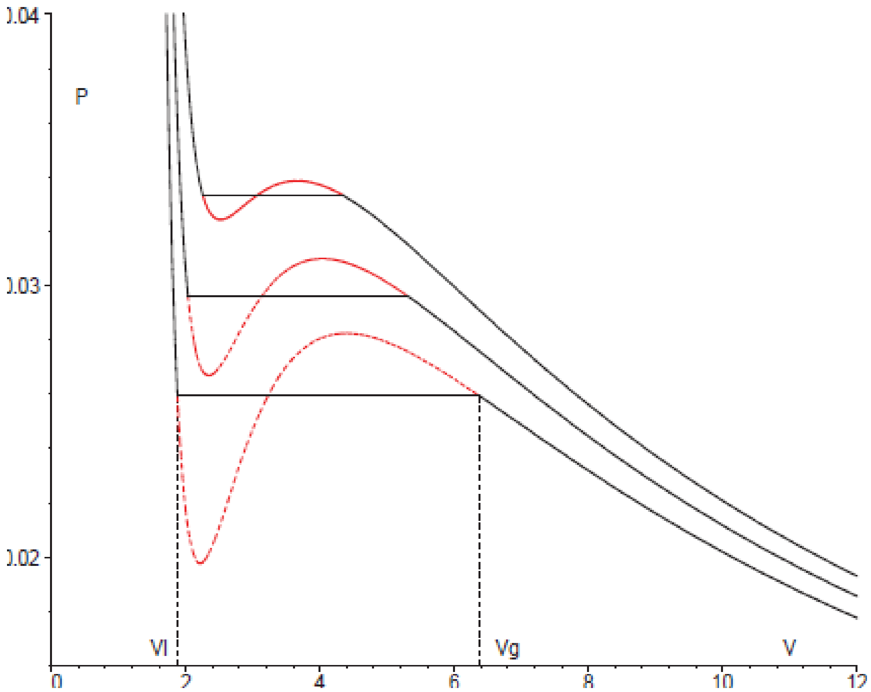


FIG. 2. **Maxwell’s equal area law.** The ‘oscillating’ (dashed) part of the isotherm $T < T_c$ is replaced by an isobar, such that the areas above and below the isobar are equal one another.

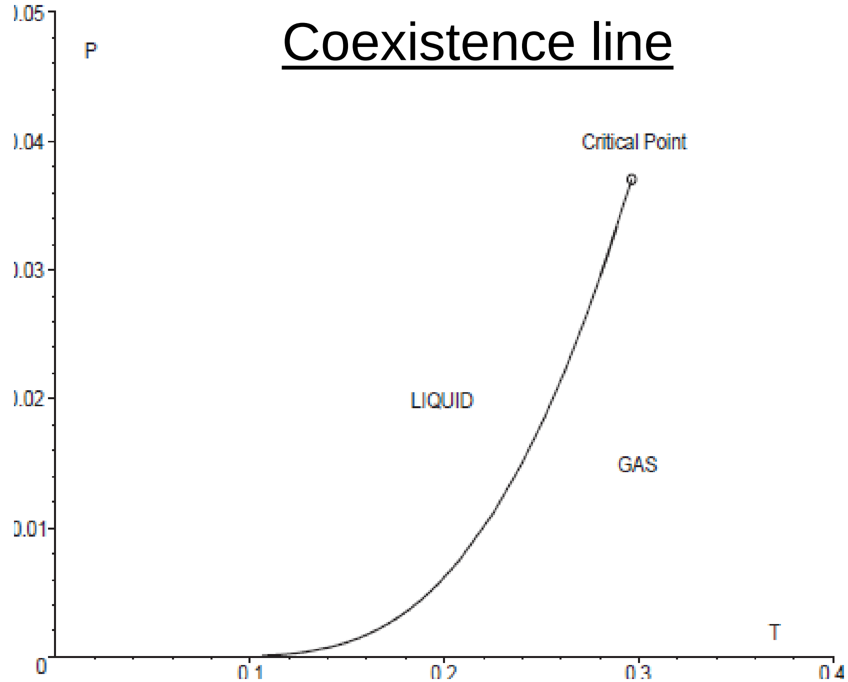


FIG. 5. **Coexistence line of liquid–gas phase.** Fig. displays the coexistence line of liquid and gas phases of the Van der Waals fluid in (P, T)-plane. The critical point is highlighted by a small circle at the end of the coexistence curve.

$$dG = -SdT + vdP$$

Both phases have the same Gibbs free energy

$$\oint vdP = 0$$

Critical exponents

• Describe the behaviour of physical quantities near the critical point
• Universal (depend on dimensionality and/or range of interactions)
• In d ≥ 4 spatial dimensions they can be calculated using MFT (each dof couples to the average of the other dof).

For example, for a fluid with critical Tc, vc, and Pc.

- Exponent α governs the behaviour of the specific heat at constant volume,

$$C_v = T\frac{\partial S}{\partial T}\Big|_v \propto |t|^{-\alpha} \,. \quad \text{(A2)}$$

- Exponent β describes the behaviour of the *order parameter* $\eta = v_g - v_l$ (the difference of the volume of the gas v_g phase and the volume of the liquid phase v_l) on the given isotherm

$$\eta = v_g - v_l \propto |t|^{\beta} \,. \quad \text{(A3)}$$

- Exponent γ determines the behaviour of the *isothermal compressibility* κ_T,

$$\kappa_T = -\frac{1}{v}\frac{\partial v}{\partial P}\Big|_T \propto |t|^{-\gamma} \,. \quad \text{(A4)}$$

- Exponent δ governs the following behaviour on the critical isotherm $T = T_c$:

$$|P - P_c| \propto |v - v_c|^{\delta} \,. \quad \text{(A5)}$$

$$\alpha = 0\,, \quad \beta = \frac{1}{2}\,, \quad \gamma = 1\,, \quad \delta = 3$$

The same critical exponents derived for ferromagnets, superfluidity,..
Problem: MFT neglects fluctuations, to go beyond one needs to use RG techniques

Plan for Lecture 2

I. Black holes in AdS

II. Black hole chemistry

III. TD phase transitions of AdS black holes
   I. VdW behavior of charged AdS black holes
   II. Hawking-Page transition
   III. Reentrant phase transitions
   IV. Isolated critical point

IV. Summary

If you want know more:

- DK, R.B. Mann, *P-V criticality of charged AdS black holes*, JHEP 07 (2012) 033; ArXiv:1205:0559.
- DK, R.B. Mann, M. Teo, *Black hole chemistry: thermodynamics with Lambda*, CQG 34 (2017) 063001, Arxiv:1608.0614.

I) Black Holes in Anti de Sitter (AdS)

Lecture 1: asymptotically flat BHs

- First law of black hole thermodynamics:

$$\delta M = T\delta S + \sum_i \Omega_i \delta J_i + \Phi \delta Q$$

- Smarr-Gibbs-Duhem relation:

$$\frac{d-3}{d-2} M = TS + \sum_i \Omega_i J_i + \frac{d-3}{d-2}\Phi Q$$

Basic properties:

- Thermodynamic ensemble not well defined (C<0)!
- Where is the standard PdV term?
- TD behaviour interesting, yet not exactly analogous to everyday thermodynamics!

Instead: Consider BHs in AdS!

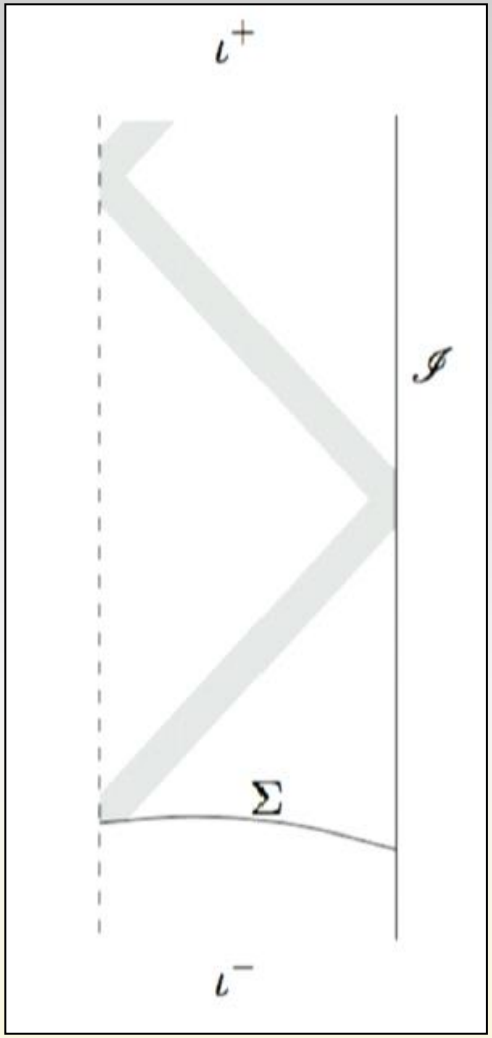
Global AdS4: a few basic facts

- **Anti de Sitter (AdS) space** = maximally symmetric solution of EE with negative Lambda:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad \Lambda = -\frac{3}{\ell^2}$$

- metric:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \,, \qquad f = 1 + \frac{r^2}{\ell^2}$$



- Pull of the negative cosmological constant implies that AdS acts like a **confining box**

- There is a timelike conformal boundary (due to reflective BCs, **nonlinearities do not decay**)

Bizon and Rostworowski, PRL107, 031102 (2011).

Schwarzschild-AdS black hole

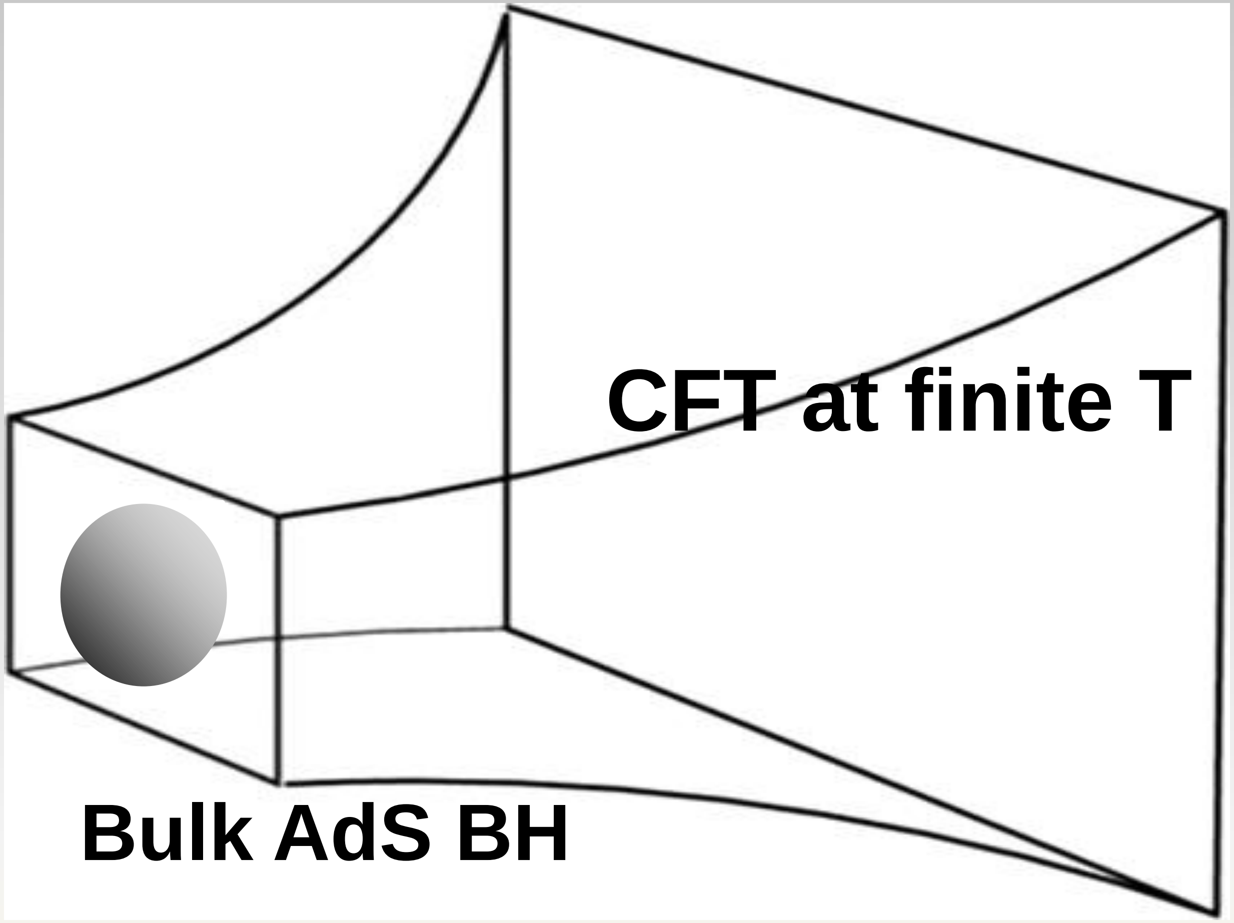
$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_k^2 \qquad f = k - \frac{2m}{r} + \frac{r^2}{\ell^2}$$

- It is an **Einstein space** (vacuum with Lambda solution)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad \Lambda = -\frac{3}{\ell^2}$$

- Choices of $k = 0, \pm 1$ correspond to various **horizon topologies** (with $d\Omega_k^2$ the corresponding metric).
- One can add, charge, or rotation – having charge-AdS, or Kerr-AdS black holes in 4 and higher dimensions.
- As we shall see, these have rather interesting properties.

AdS/CFT duality (at finite temperature)



Bulk:
Hawking temp T
BH entropy S
BH mass M

⟺

Boundary:
CFT at finite T
CFT entropy S
CFT energy E

Gravitational action in AdS

$$S_E = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left[R+\frac{6}{\ell^2}\right]+\frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-h}\mathcal{K}$$
$$-\frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-h}\left[\frac{2}{\ell}+\frac{\ell}{2}\mathcal{R}(h)\right],$$

- The 2nd line are the covariant **AdS counterterms** (c.f. `vague' background subtraction in AF case)

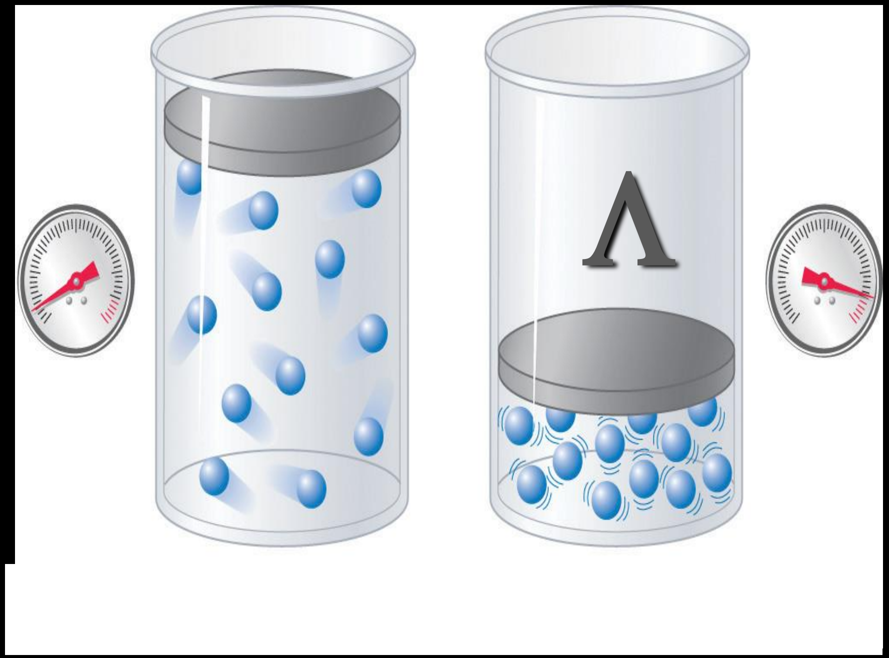
- Variation yields:

$$\delta S_E = -\frac{1}{2}\int_{\partial\Omega} d^3x\sqrt{-h}\tau_{ab}\delta h^{ab}$$ + bulk EOM

here

$$8\pi\tau_{ab} = \mathcal{K}h_{ab} - \mathcal{K}_{ab} + \ell G_{ab}(h) - \frac{2}{\ell}h_{ab}$$

is up to trivial (infinite) scaling the **holographic stress tensor**



II. Black hole chemistry

Black hole chemistry

Simple idea:

- Consider an asymptotically **AdS black hole spacetime**

- Identify the cosmological constant with a **thermodynamic pressure**

$$P = -\frac{\Lambda}{8\pi G}\,, \quad \Lambda = -\frac{(D-1)(D-2)}{2l^2}$$

- Allow this to be a "**dynamica**l" quantity

(Teitelboim and Brown – 1980's)

Immediate consequences

- **Extended black hole thermodynamics:**

D.Kastor, S.Ray, and J.Traschen, *Enthalpy and the Mechanics of AdS Black Holes*, Class. Quant. Grav. 26 (2009) 195011.

$$\delta M = T\delta S + \Theta\, \delta P + \dots$$

- Introduces the standard **-PdV term** into black hole thermodynamics

- Black hole mass M no longer identified with energy but rather interpreted as **enthalpy**

$$U = M + \epsilon V = M - PV$$

Immediate consequences

- **Black hole volume:**

Schwarzschild(-AdS):

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,\ldots} \Rightarrow V = \frac{4}{3}\pi r_+^3$$

• More involved for more complicated black holes

• The fact this this provides a good definition of volume is supported by the **Reverse Isoperimetric Inequality** conjecture:

- M. Cvetic, G.W Gibbons, DK, C.N. Pope, *Black hole enthalpy and an entropy inequality for the thermodynamic volume,* Phys. Rev. D84 (2011) 024037, [arXiv:1012.2888].
- M. Amo, A.M. Frassino, R.A. Hennigar, *New Inequalities in Extended black hole thermodynamics*, Arxiv:2307.03011.

Immediate consequences

- **Black hole volume:**

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,\ldots}$$

Schwarzschild(-AdS):

$$V = \frac{4}{3}\pi r_+^3$$

Reverse isoperimetric inequality

$$\mathcal{R} = \left(\frac{(d-1)\mathcal{V}}{\omega_{d-2}}\right)^{\frac{1}{d-1}} \left(\frac{\omega_{d-2}}{\mathcal{A}}\right)^{\frac{1}{d-2}}$$

$$\omega_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}$$

Conjecture: for any AdS black hole $\mathcal{R} \geq 1$

"For a black hole of given **thermodynamic volume** V, the entropy is maximised for Schwarzschild-AdS"

Consistency between 1st law and Smarr relation (scaling argument)

Euler's theorem:

$$f(\alpha^p x, \alpha^q y) = \alpha^r f(x,y)$$

⇒

$$r f(x,y) = p\left(\frac{\partial f}{\partial x}\right) x + q\left(\frac{\partial f}{\partial y}\right) y.$$

Mass of black hole: $M = M(A, P)$

since $[P] = L^{-2}\,, \quad [A] = L^2\,, \quad [M] = L$ ⇒

$$M = 2A\left(\frac{\partial M}{\partial A}\right) - 2P\left(\frac{\partial M}{\partial \Lambda}\right)$$

+

$$dM = \kappa dA + V dP$$

Smarr relation: $M = 2(TS - VP)$

Mass plays the role of **enthalpy** rather than internal energy

Immediate consequences

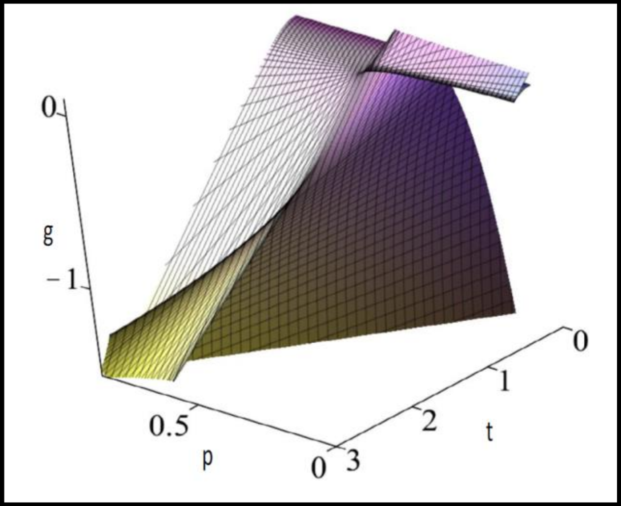
- **Consistent Smarr relation:**

$$\delta M = T\delta S + \sum_i \Omega_i \delta J_i + \Phi \delta Q \quad + V\delta P$$

$$\frac{d-3}{d-2} M = TS + \sum_i \Omega_i J_i + \frac{d-3}{d-2}\Phi Q \quad - \frac{2}{d-2} VP$$

- **Black holes and phase transitions:**

  - AdS black holes can be in **thermal equilibrium**
  - Exhibit interesting **phase transitions**
  - Provide dual description of CFT at finite temperature via **AdS/CFT correspondence**



III. TD phase transitions of AdS black holes

Canonical Example: VdW behavior of charged AdS black holes

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2\,, \qquad A = -\frac{Q}{r} dt$$

$$f = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} + \frac{r^2}{l^2}\,,$$

- Basic thermodynamic quantities:

$$M = \frac{r_+(l^2 + r_+^2)}{2l^2 G} + \frac{Q^2}{2r_+}\,, \qquad T = \frac{3r_+^4 + l^2 r_+^2 - GQ^2 l^2}{4\pi l^2 r_+^3}$$

$$S = \frac{\pi r_+^2}{G}\,, \quad V = \frac{4\pi r_+^3}{3}\,, \quad \phi = \frac{Q}{r_+}\,, \quad P = -\frac{1}{8\pi}\Lambda = \frac{3}{8\pi}\frac{1}{l^2}$$

$$\boxed{F = M - TS = \frac{3GQ^2 l^2 + l^2 r_+^2 - r_+^4}{4G r_+ l^2}}$$

Example: VdW criticality

- DK, R.B. Mann, *P-V criticality of charged AdS black holes*, JHEP 1207 (2012) 033.

Van der Waals fluid

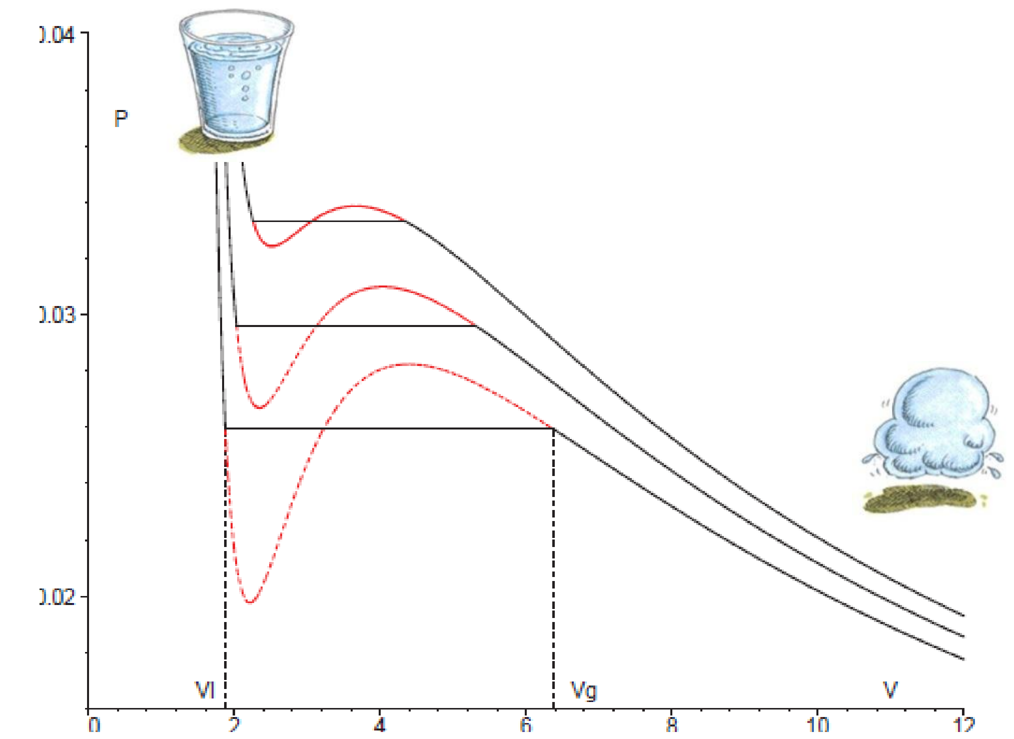


FIG. 2. Maxwell's equal area law. The 'oscillating' (dashed) part of the isotherm $T < T_c$ is replaced by an isobar, such that the areas above and below the isobar are equal one another.

$$\left(P + \frac{a}{v^2}\right)(v - b) = T$$

Parameter a measures the **attraction** between particles (a>0) and b corresponds to "**volume of fluid particles**".

Critical point:

$$\rho_c = \frac{P_c v_c}{T_c} = \frac{3}{8}$$

P-V criticality

- DK, R.B. Mann, *P-V criticality of charged AdS black holes*, JHEP 1207 (2012) 033.

Charged black hole

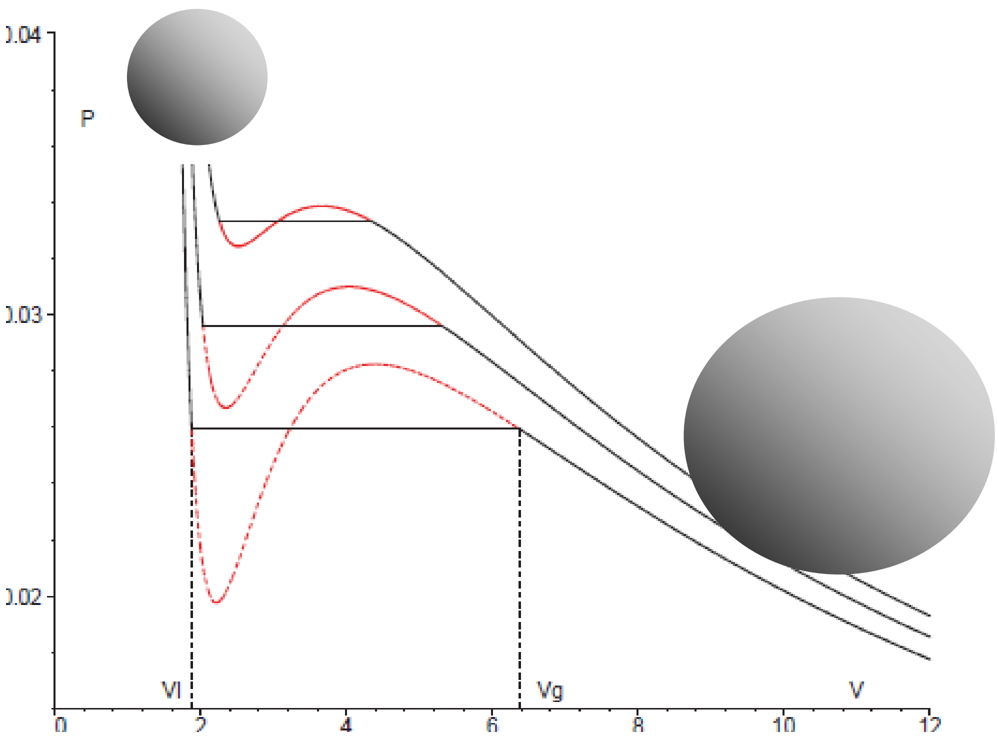


FIG. 2. Maxwell's equal area law. The 'oscillating' (dashed) part of the isotherm $T < T_c$ is replaced by an isobar, such that the areas above and below the isobar are equal one another.

$$\left(P + \frac{a}{v^2}\right)(v - b) = T$$

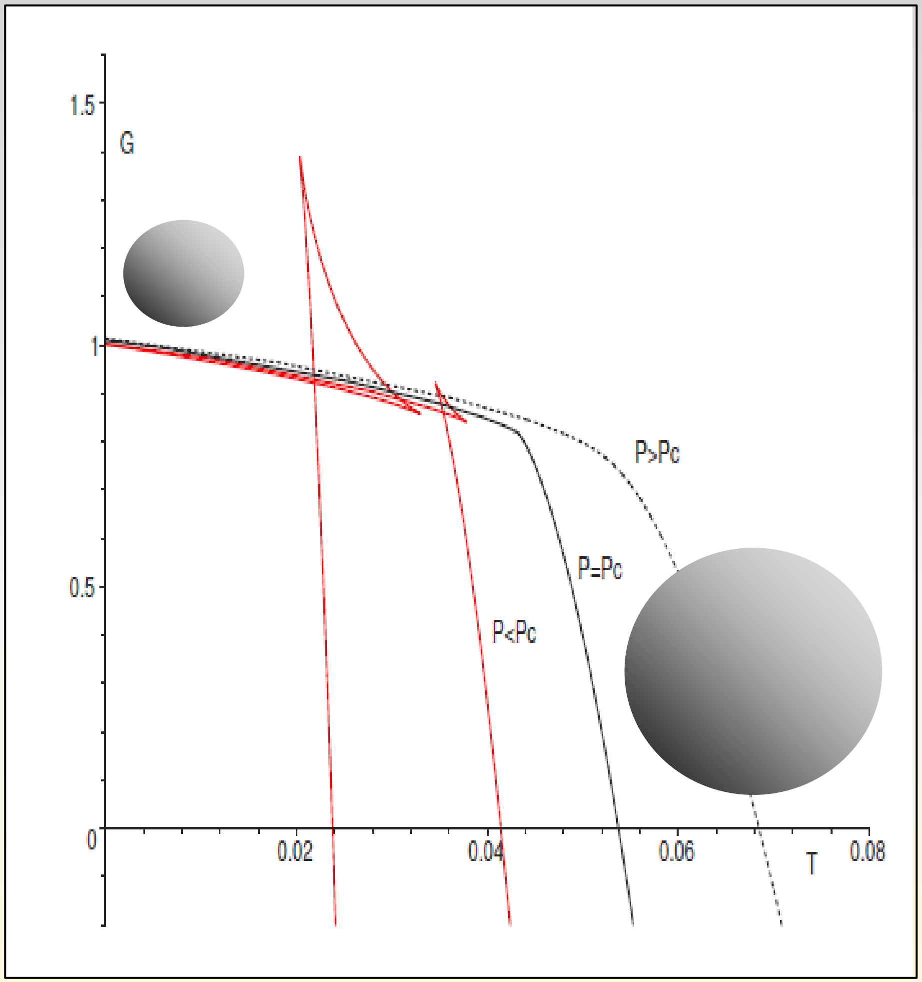
$$P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4}$$

Critical point:

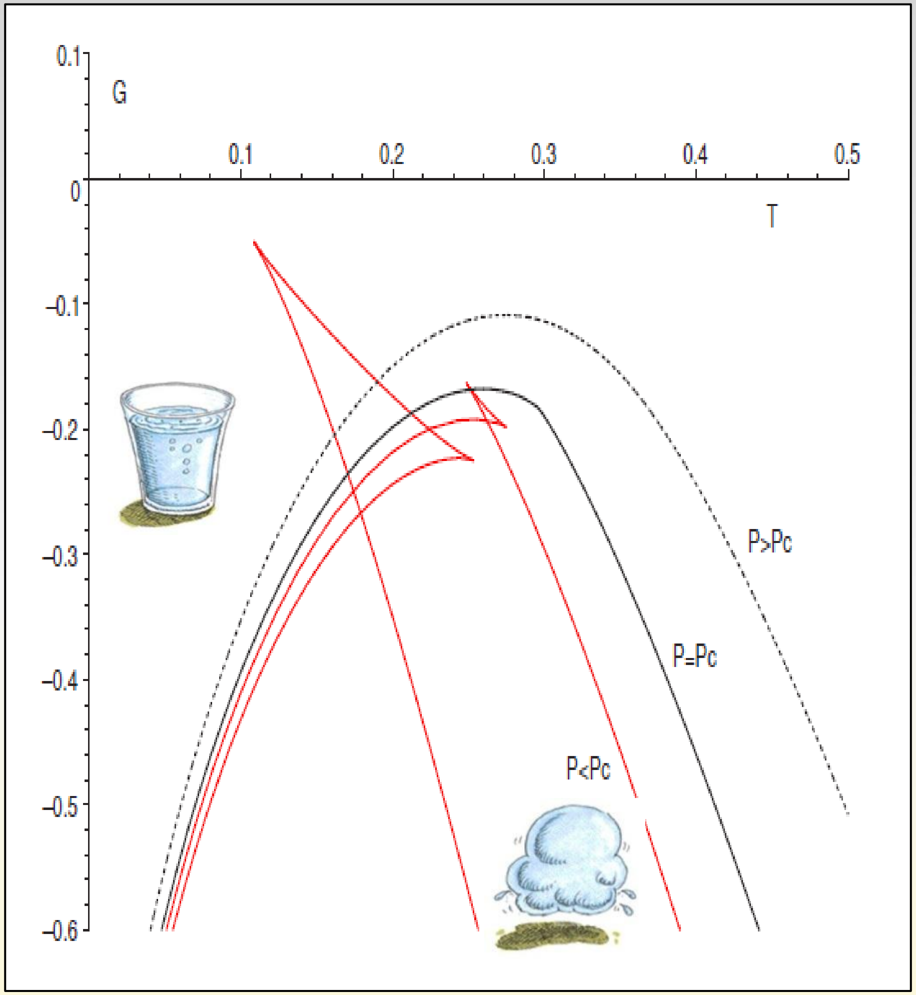
$$\rho_c = \frac{P_c v_c}{T_c} = \frac{3}{8}$$

Free energy: demonstrates standard swallow tail behavior

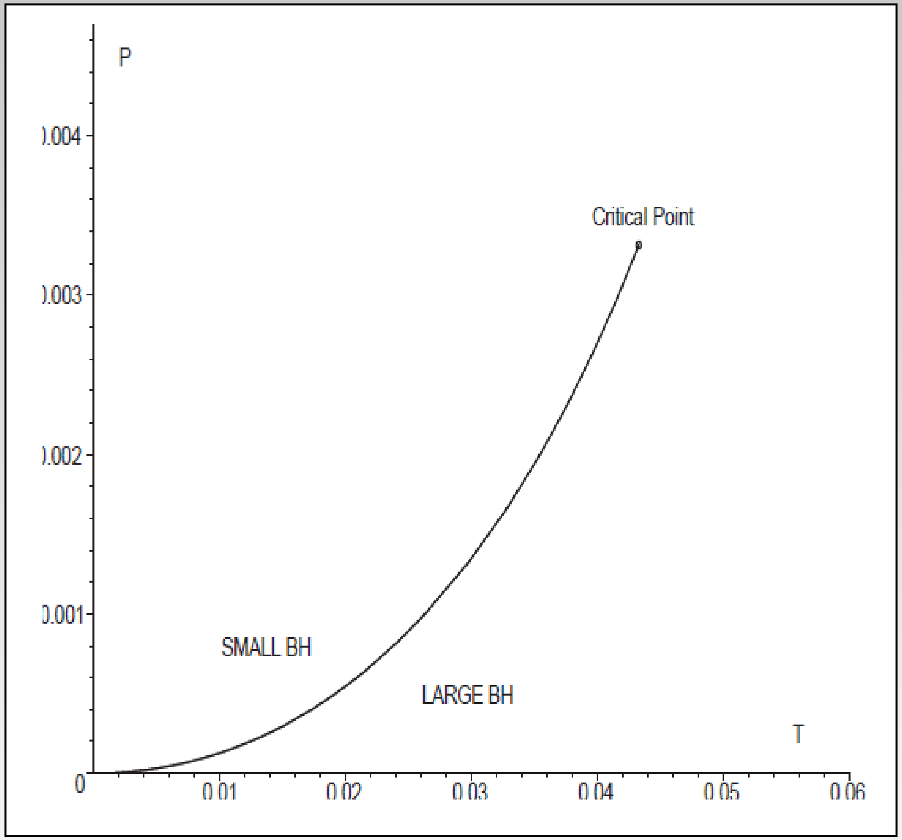
$$F = F(T,P,Q) = \frac{1}{4}\left(r_+ - \frac{8\pi}{3}Pr_+^3 + \frac{3Q^2}{r_+}\right)$$



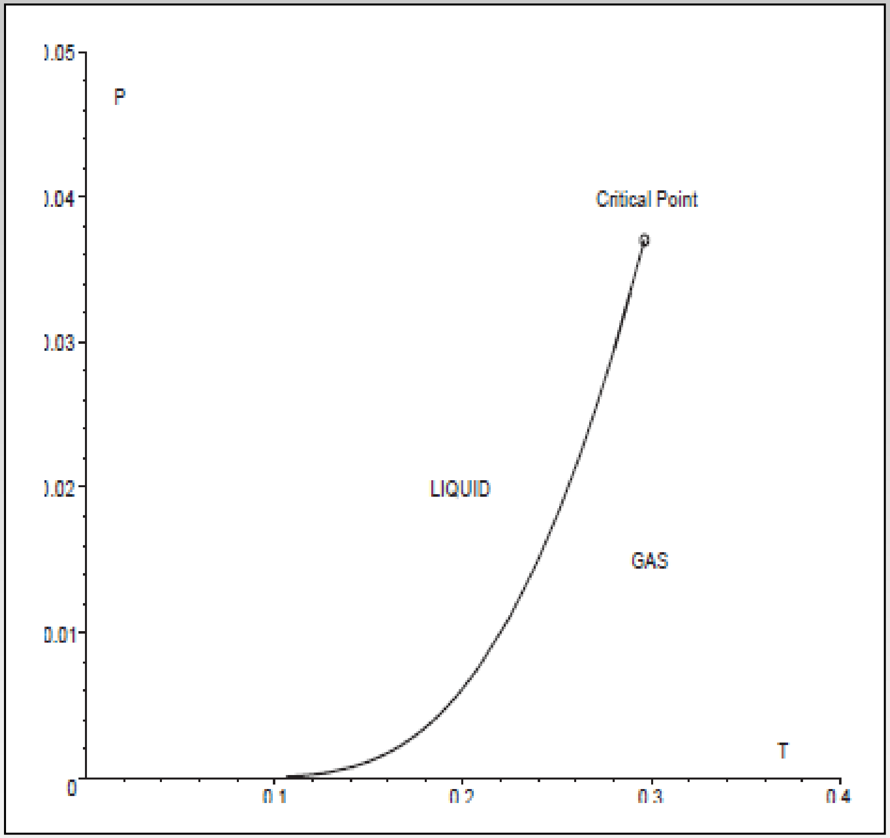
vs.



Phase diagrams: complete analogy



vs.



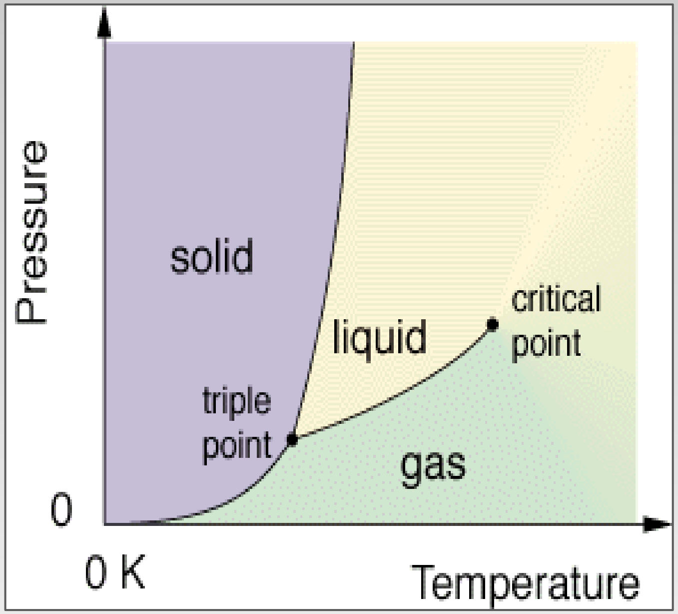
- Coexistence & critical point described by **Clausius-Clapeyron** and **Ehrenfest** equations

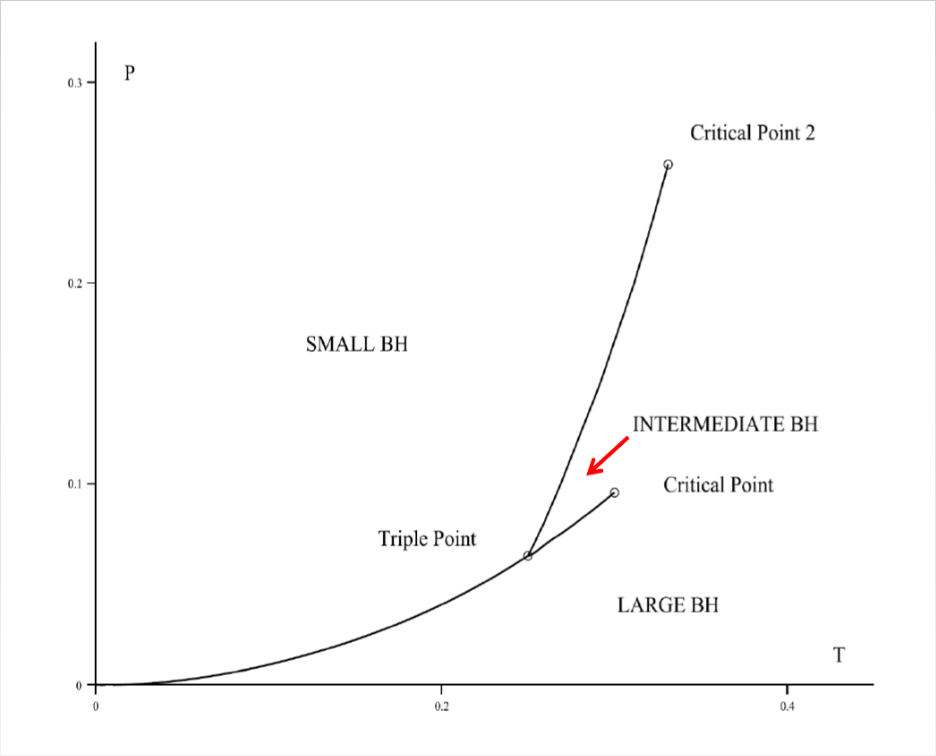
- **MFT critical exponents**

$$\alpha = 0\,, \quad \beta = \frac{1}{2}\,, \quad \gamma = 1\,, \quad \delta = 3$$

More generally: black hole chemistry

- **Triple point** and solid/liquid/gas analogue**:**





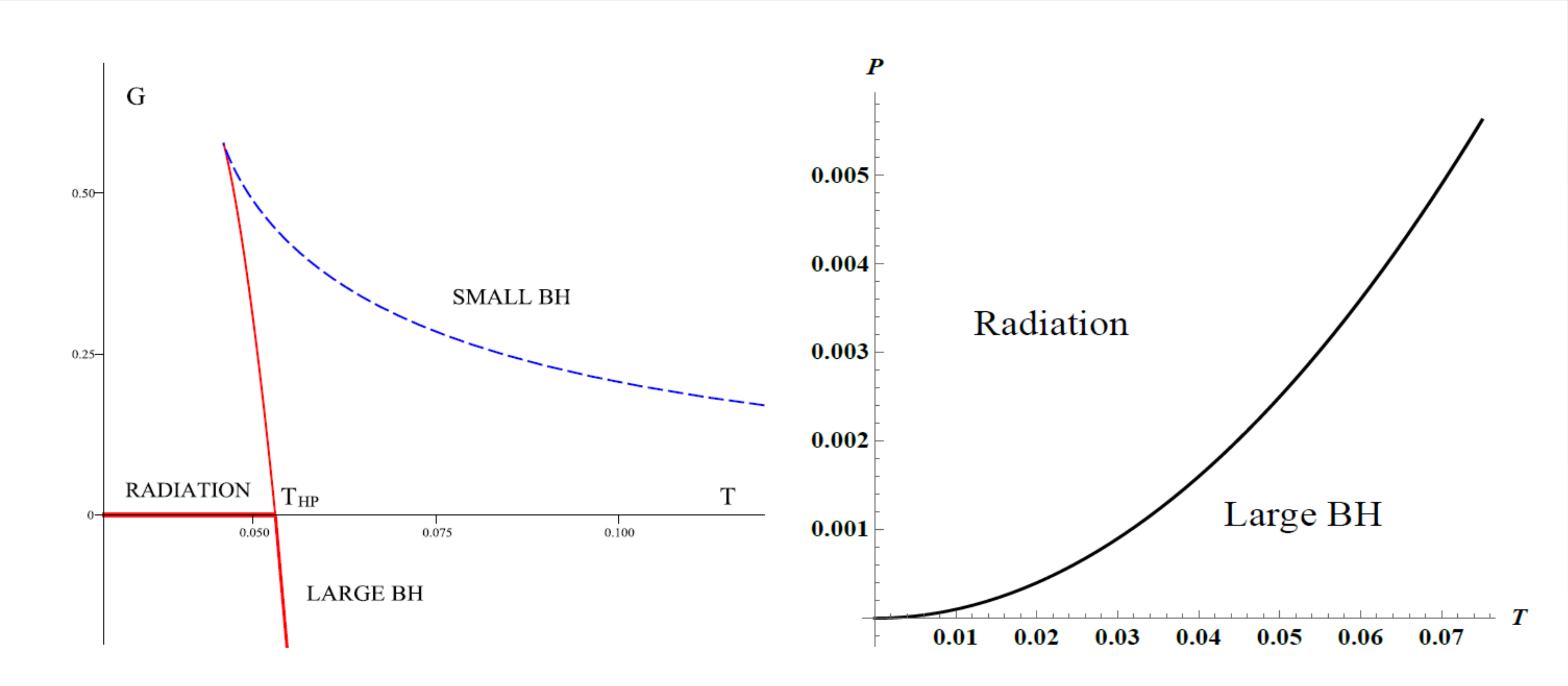
(can have n-tuple points)

- **Many other**:
  - Hawking-Page PT
  - Isolated critical point
  - Reentrant PT
  - superfluid PT

- DK, Mann, Teo, *Black hole chemistry: thermodynamics with Lambda*, CQG 34 (2017) 063001, Arxiv:1608.0614.

a) Hawking-Page transition

S.W. Hawking & D.N. Page, *Thermodynamics of black holes in anti-de-Sitter space*, Commun. Math. Phys. 87, 577 (1983).



1st-order radiation/large black hole phase transition

(dual to **confinement/deconfinement** PT of QGP)

b) Reentrant phase transition

A system undergoes an RPT if a **monotonic** variation of any thermodynamic quantity results in two (or more) phase transitions such that the **final state is macroscopically similar** to the initial state.

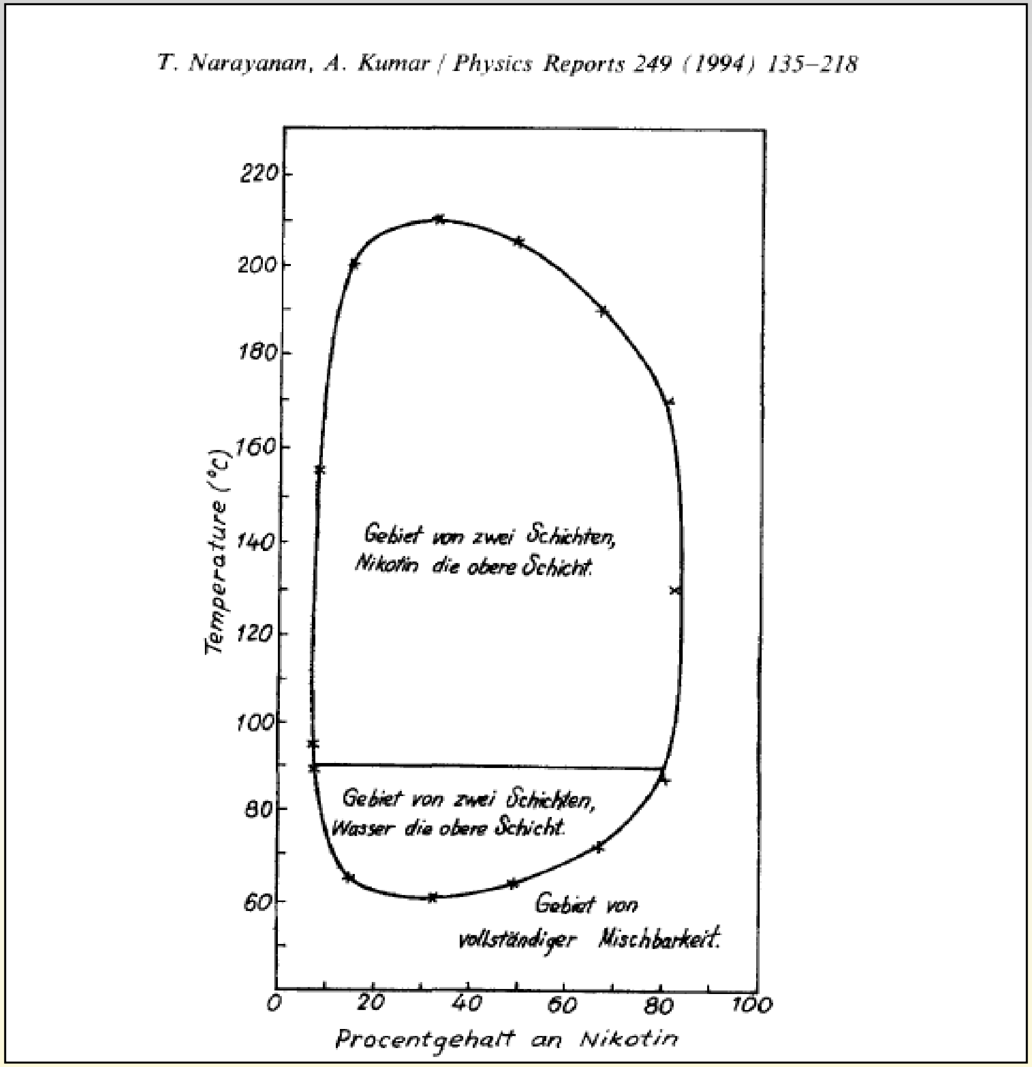
First observed by Hudson (1904) in a nicotine/water mixture

Z. Phys. Chem. 47 (1904) 113.

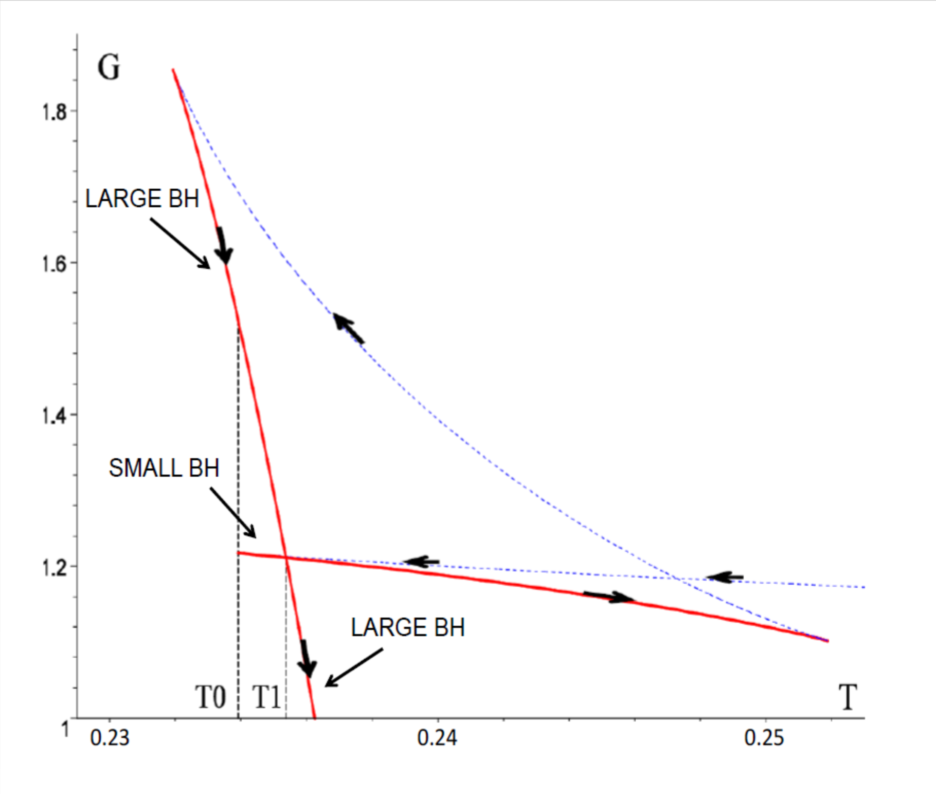
Since then:

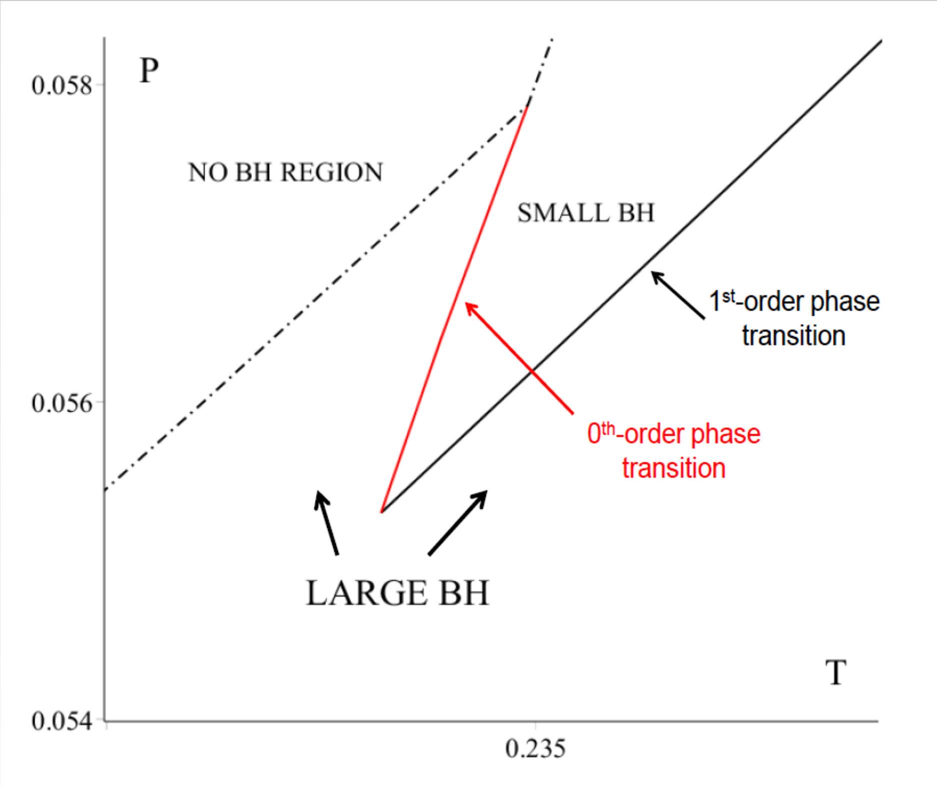
multicomponent fluid systems, gels, ferroelectrics, liquid crystals, and binary gases

T. Narayanan and A. Kumar, Reentrant phase transitions in multicomponent liquid mixtures, Physics Reports 249 (1994) 135–218.



AdS analogue: large/small/large black hole phase transition in Kerr-AdS BH in 6 dimensions (Altamirano et al 2013)





Low T	Medium T	High T
mixed	water/nicotine	mixed
large BH	small BH	large BH

Summary

1) **AdS black holes** are an interesting generalization of their asymptotically flat cousins.

2) **Black hole chemistry** (TDs with variable Lambda) provides an interesting framework for AdS black hole thermodynamics:

   - **Extended first law** consistent with Smarr
   - Black hole mass is **enthalpy**
   - Definition of **black hole volume**
   - Uncovers various **phase transitions** & similarities with TDs of ordinary systems:

   e.g. **Van der Waals criticality** of charged BHs

3) All these **phase transitions** have a **dual description** via the AdS/CFT duality. Can they be **observed** on the CFT side?