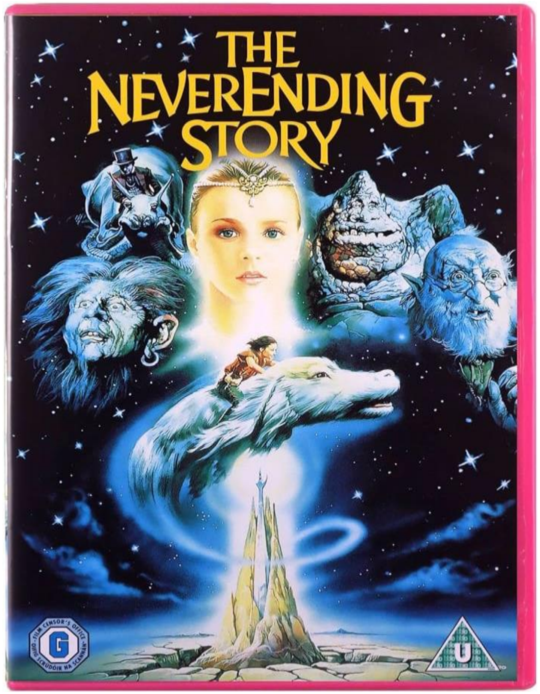
# Never ending adventures with C-metric thermodynamics





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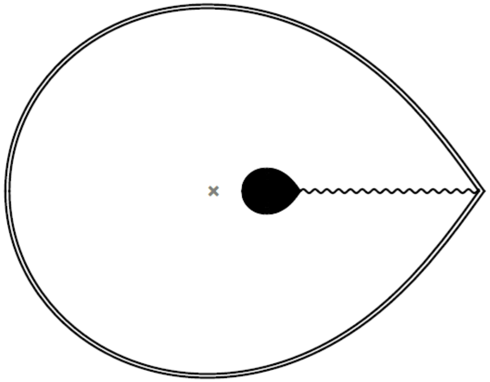
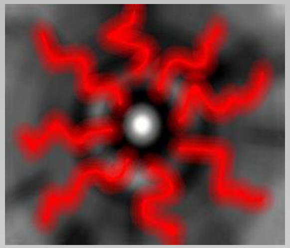


**Black hole thermodynamics: classical and quantum**
Lecture 3
Charles University, 2025

# Black hole thermodynamics



- First law of black hole thermodynamics:

$$\delta M = T\delta S + \sum_i \Omega_i \delta J_i + \Phi \delta Q + V\delta P$$

$$P = -\frac{1}{8\pi}\Lambda$$

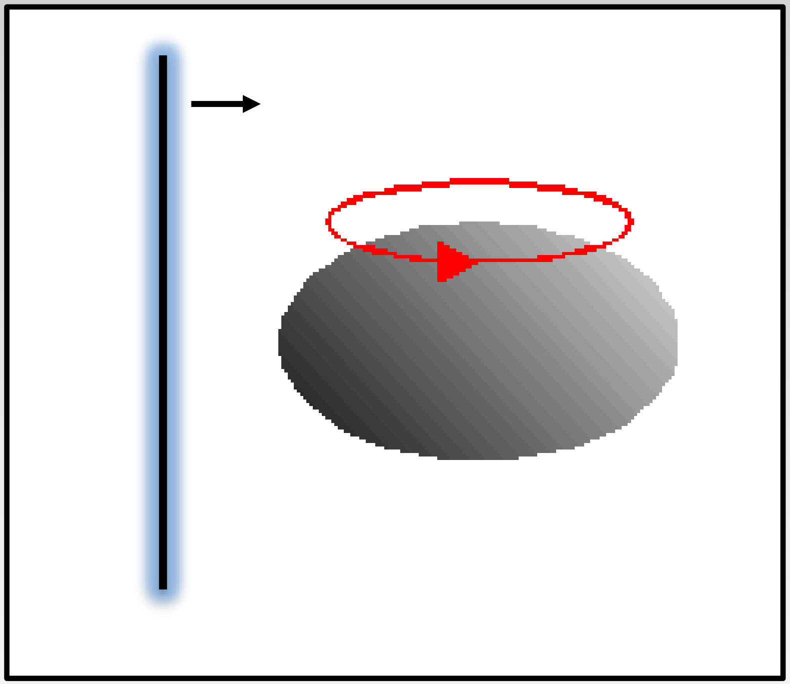
- Smarr-Gibbs-Duhem relation:

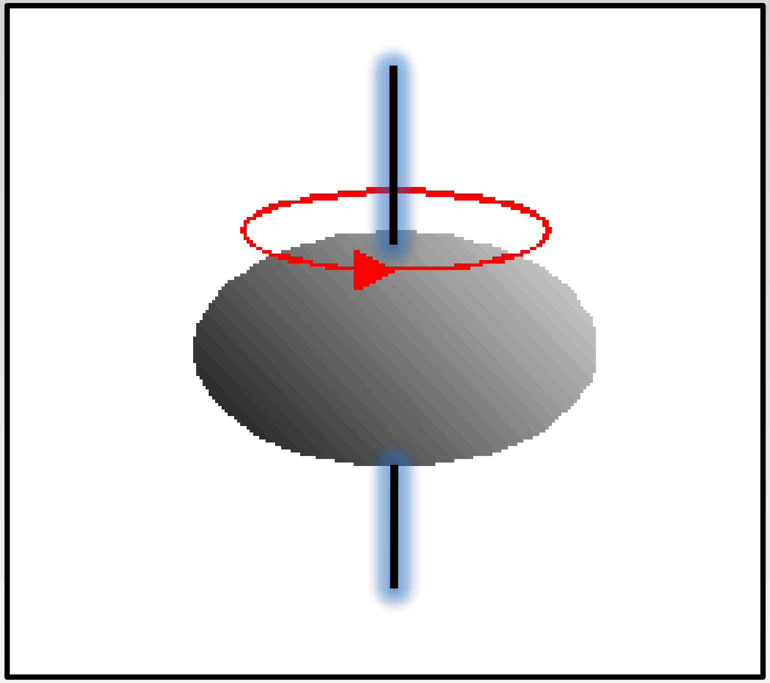
$$\frac{d-3}{d-2}M = TS + \sum_i \Omega_i J_i + \frac{d-3}{d-2}\Phi Q - \frac{2}{d-2}VP$$

Works for black holes of various asymptotic properties, horizon topologies, charges, and rotation parameters. With a remarkable exception of **accelerated black holes**: **C-metric**

# Once upon a time…

Back **in 2013** we have found a solution for a black hole **sporting a cosmic string hair**

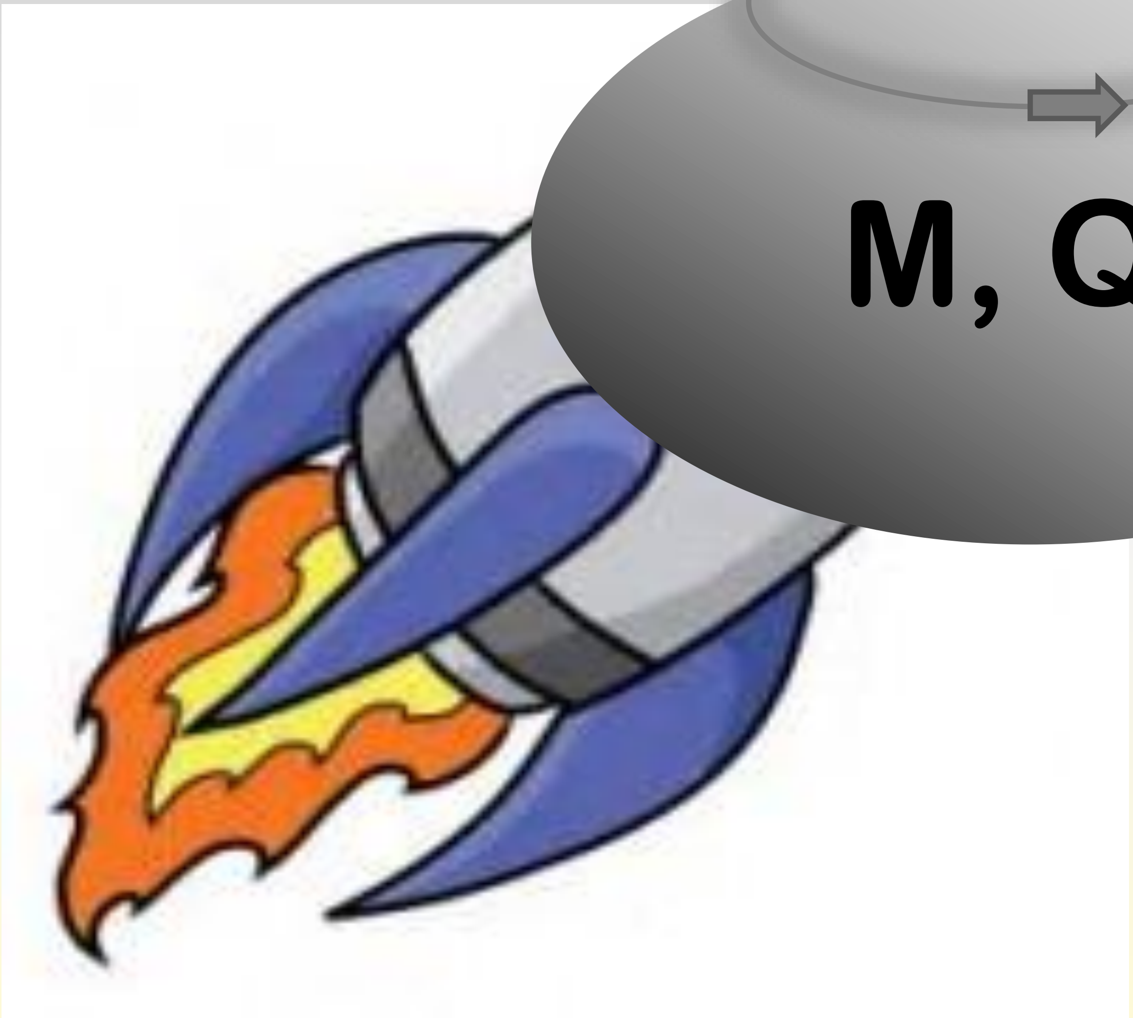


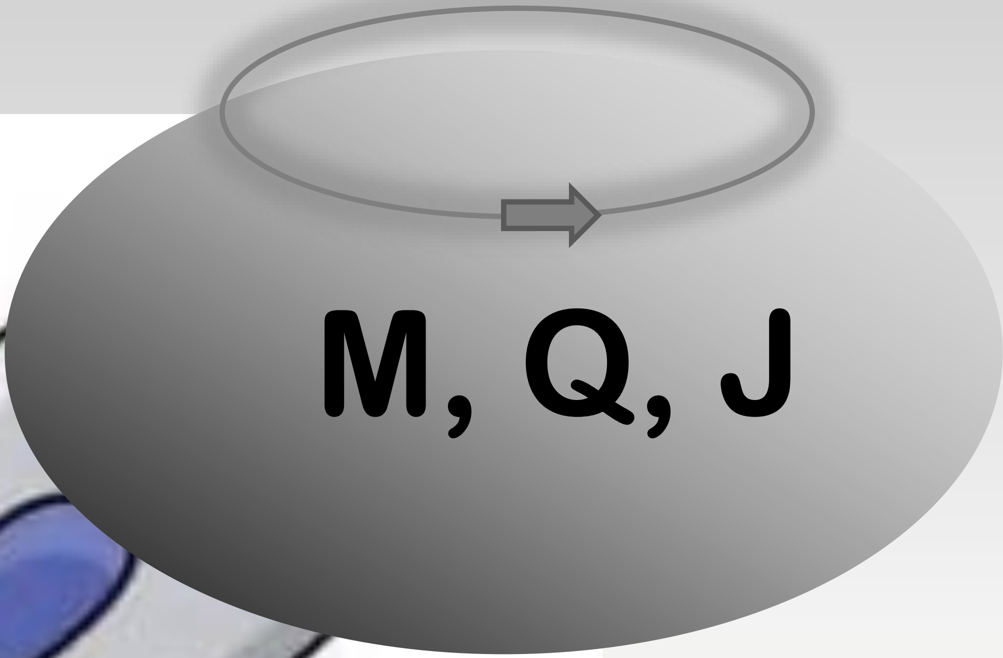


$$S = \int d^4x\sqrt{-g}\left[D_\mu\Phi^\dagger D^\mu\Phi - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{4}\lambda(\Phi^\dagger\Phi - \eta^2)^2\right]$$

- Schwarzschild solution known since 1995 (Achucarro, Gregory & Kijken 1995)
- Rotating black hole – R. Gregory, DK & D. Wills 2013

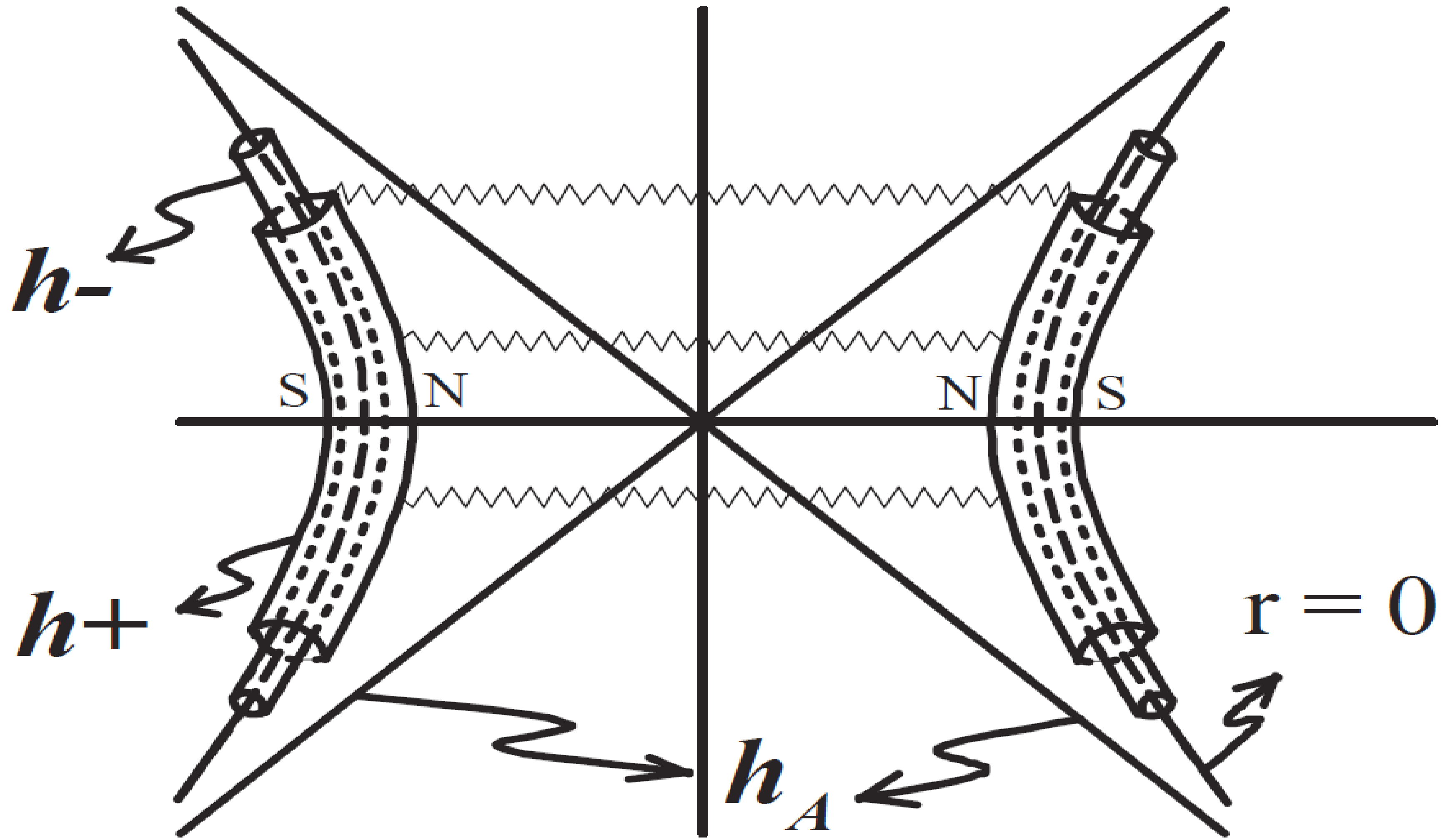
# What is an accelerated black hole?





# Introducing the C-metric

- Exact **boost-rotation symmetric** black hole solution of Einstein equations (with EM field and cosmological constant)



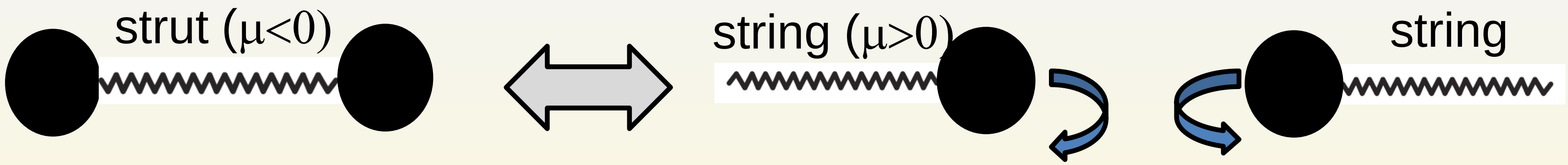

O. Dias and J. Lemos, Phys.Rev. D67 (2003) 064001.

# Introducing the C-metric

- **Early days of GR**: *Weyl (1917), Levi-Civita (1918), Ehlers & Kundt (1963), Kinnersley & Walker (1970),…*

- **Very useful:**
  - Black ring in 5d ("Wick rotated C-metric")
  - AdS/CFT: black funnels and droplets
  - Black hole nucleation: instability of dS space
  - Counter example to "no hair theorem"

- **Setup** we were familiar with: **cosmic strings**




- **Not known**: generalization to higher dimensions & TDs

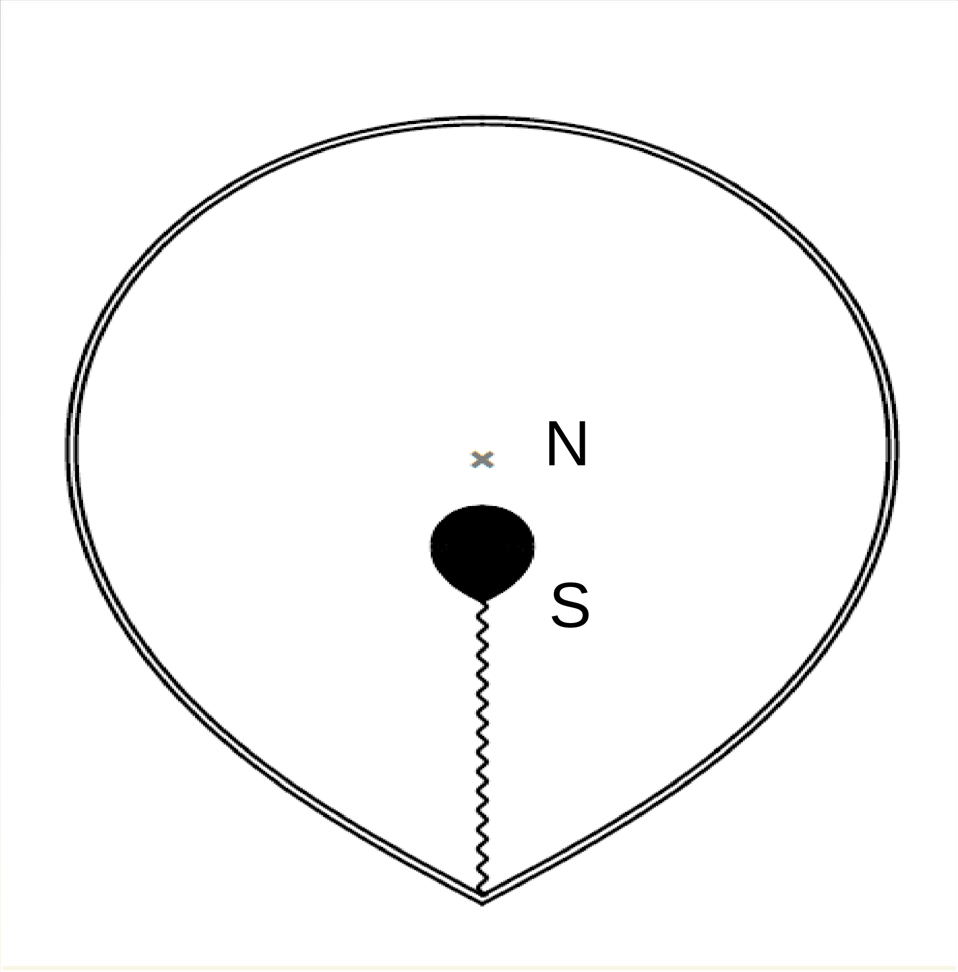
## **Ruth got really intrigued** (and so after a PSI lecture….)

## **TDs of C-metric seems very tough:**

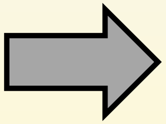
- 2 black holes, cosmic strings, acceleration & cosmic horizons, …
- Exact **radiative spacetime** (Bicak, Krtous, Podolsky, Pravda, Pravdova,…)

# Quiz: Which one best describes flash of genius?


**Ruth's ingenious idea:** consider slowly accelerating black hole in AdS – **simplest setup**




$$A \lesssim 1/l$$



Only single BH, no radiation, no acceleration horizon

# Slowly accelerating AdS C-metric

$$ds^2 = \frac{1}{\Omega^2}\left[-f\,dt^2 + \frac{dr^2}{f} + r^2\Big(\frac{d\theta^2}{h} + h\sin^2\theta\frac{d\phi^2}{K^2}\Big)\right]$$

where:

$$h = 1 + 2mA\cos\theta$$

$$f(r) = (1 - A^2r^2)\left(1 - \frac{2m}{r}\right) + \frac{r^2}{\ell^2}\,.$$

parameters: $\{m, l, A, K\}$

- **Conformal factor**: $$\Omega = 1 + Ar\cos\theta$$

(determines conformal infinity)

# Slowly accelerating AdS C-metric

$$ds^2 = \frac{1}{\Omega^2}\left[-f\,dt^2 + \frac{dr^2}{f} + r^2\left(\frac{d\theta^2}{h} + h\sin^2\theta\frac{d\phi^2}{K^2}\right)\right]$$

# Conical deficits and cosmic strings

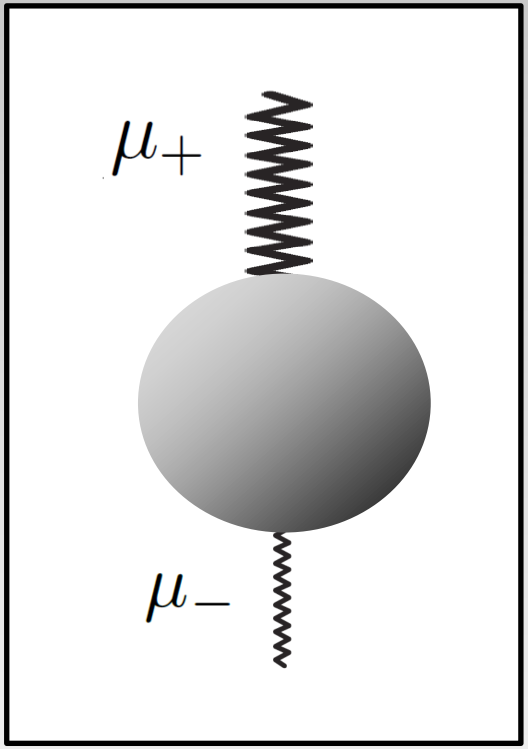
- determined from the behavior of function h at the poles

$$\delta_\pm = 2\pi\left(1 - \frac{h(\theta_\pm)}{K}\right) = 2\pi\left(1 - \frac{1 \pm 2mA}{K}\right)$$

- Correspond to **cosmic strings** with the following **tensions**:

$$\mu_\pm = \frac{\delta_\pm}{8\pi}$$

# Cosmic string tensions




$$\mu_\pm = \frac{1}{4}\Big(1 - \frac{1 \pm 2mA}{K}\Big)$$

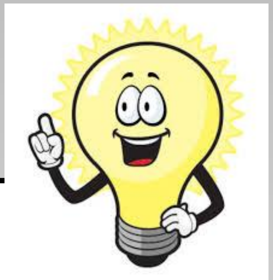
$$\mu_+ + \mu_- = \frac{1}{2}\frac{K-1}{K}\,,$$

$$\mu_- - \mu_+ = \frac{mA}{K}\,.$$

- K – overall tension
- A – due to difference in string tensions

- Can be **dynamical** (Rob Myers' example):

# We expect an extended thermodynamics

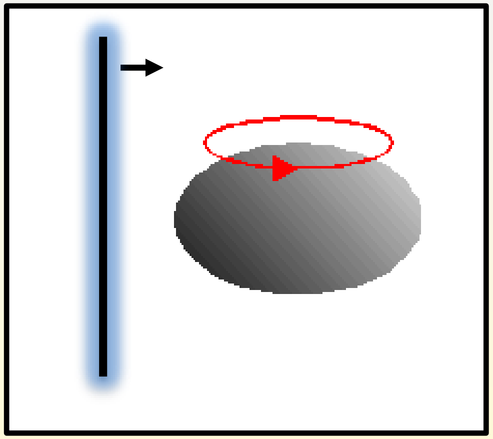


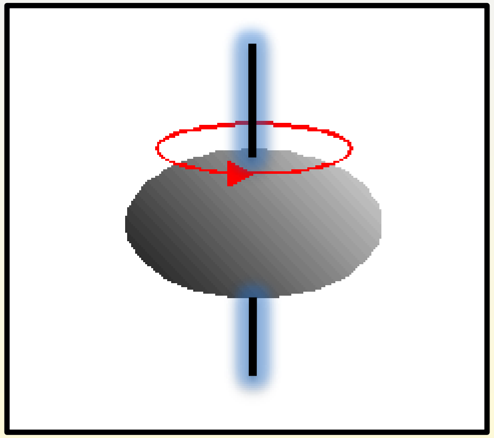
$$\delta M = T\delta S - \lambda_+ \delta\mu_+ - \lambda_- \delta\mu_- + V\delta P + \ldots$$
$$M = 2TS - 2VP + \ldots$$

- Such first law is of “full cohomogeneity”
- It reduces to **standard 1st law** upon fixing the tensions:

$$\delta\mu_+ = 0 = \delta\mu_-$$

- In particular describes processes as:





# Needed some help


# Technical difficulty: normalization of boost KV

Let the correct time is $$\tau = \alpha t$$

$$ds^2 = \frac{1}{\Omega^2}\left[-f\,dt^2 + \frac{dr^2}{f} + r^2\left(\frac{d\theta^2}{h} + h\sin^2\theta\frac{d\phi^2}{K^2}\right)\right]$$

$$\alpha = \alpha(m, A, K, \ell, \dots)$$

$$ds^2 = \frac{1}{\Omega^2}\left[-\frac{f\,d\tau^2}{\alpha^2} + \frac{dr^2}{f} + r^2\left(\frac{d\theta^2}{h} + h\sin^2\theta\frac{d\phi^2}{K^2}\right)\right]$$

… important for **thermodynamic charges**

# Technical difficulty: normalization of boost KV

Let the correct time is $$\tau = \alpha t$$

**Method 1**: set m=0 and do a coordinate transf. to AdS space

$$1 + \frac{R^2}{\ell^2} = \frac{1 + (1 - A^2\ell^2) r^2/\ell^2}{(1 - A^2\ell^2)\Omega^2}\,, \qquad R\sin\vartheta = \frac{r\sin\theta}{\Omega}$$

….recovers AdS in global coordinates provided we set

$$\alpha = \sqrt{1 - A^2\ell^2}$$

**Method 2**: boundary metric (m=0) must be AdS with a round sphere

**Method 3**: the action variation has to vanish

$$\delta I = \int_{\partial M} \sqrt{-\gamma}\,\tau_{ab}\,\delta\gamma^{ab} d^3x = 0$$

# Thermodynamics of AdS C-metric

- **thermodynamic mass** = conformal mass = holographic mass

$$M = \frac{m}{K}\frac{1 - A^2 l^2}{\alpha} = \frac{m\alpha}{K}$$

- other thermodynamic quantities

$$T = \frac{f'(r_+)}{4\pi\alpha}$$ (Wick)

$$S = \frac{\mathcal{A}}{4} = \frac{\pi r_+^2}{K(1 - A^2 r_+^2)}$$ (area law)

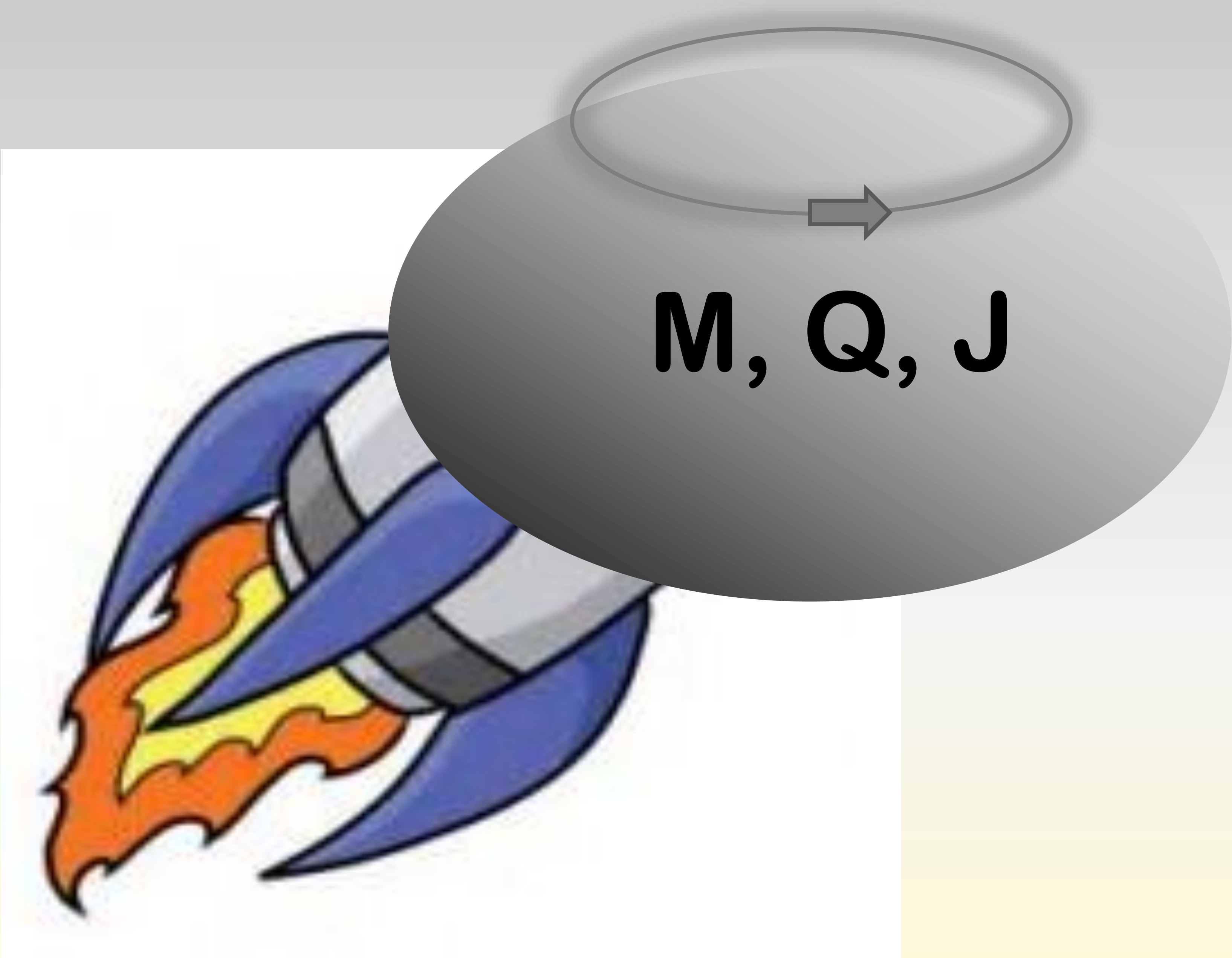
we also have

$$\mu_\pm = \frac{1}{4}\Big(1 - \frac{1 \pm 2mA}{K}\Big)$$

Consistent **first law** plus **Smarr relation** (2018)

$$\delta M = T\delta S - \lambda_+ \delta\mu_+ - \lambda_- \delta\mu_- + V\delta P + \dots$$
$$M = 2TS - 2VP + \dots$$

# Can add charge and spin!



# Did we celebrate too early?

# Recent paper: self-consistency of α?

H. Kim, N. Kim, Y Lee, A. Poole, *Thermodynamics of accelerating AdS4 black holes from the covariant phase space*, *EPJC (2023)*.

- "**Well posedness**" of the variational principle

$$\delta S_{\rm ren} \approx \int_{\mathscr{I}} d^3x \sqrt{-g_{(0)}} \left( \frac{1}{2} T^{ij} \delta g^{(0)}_{ij} + j^i \delta A^{(0)}_i \right) = 0,$$

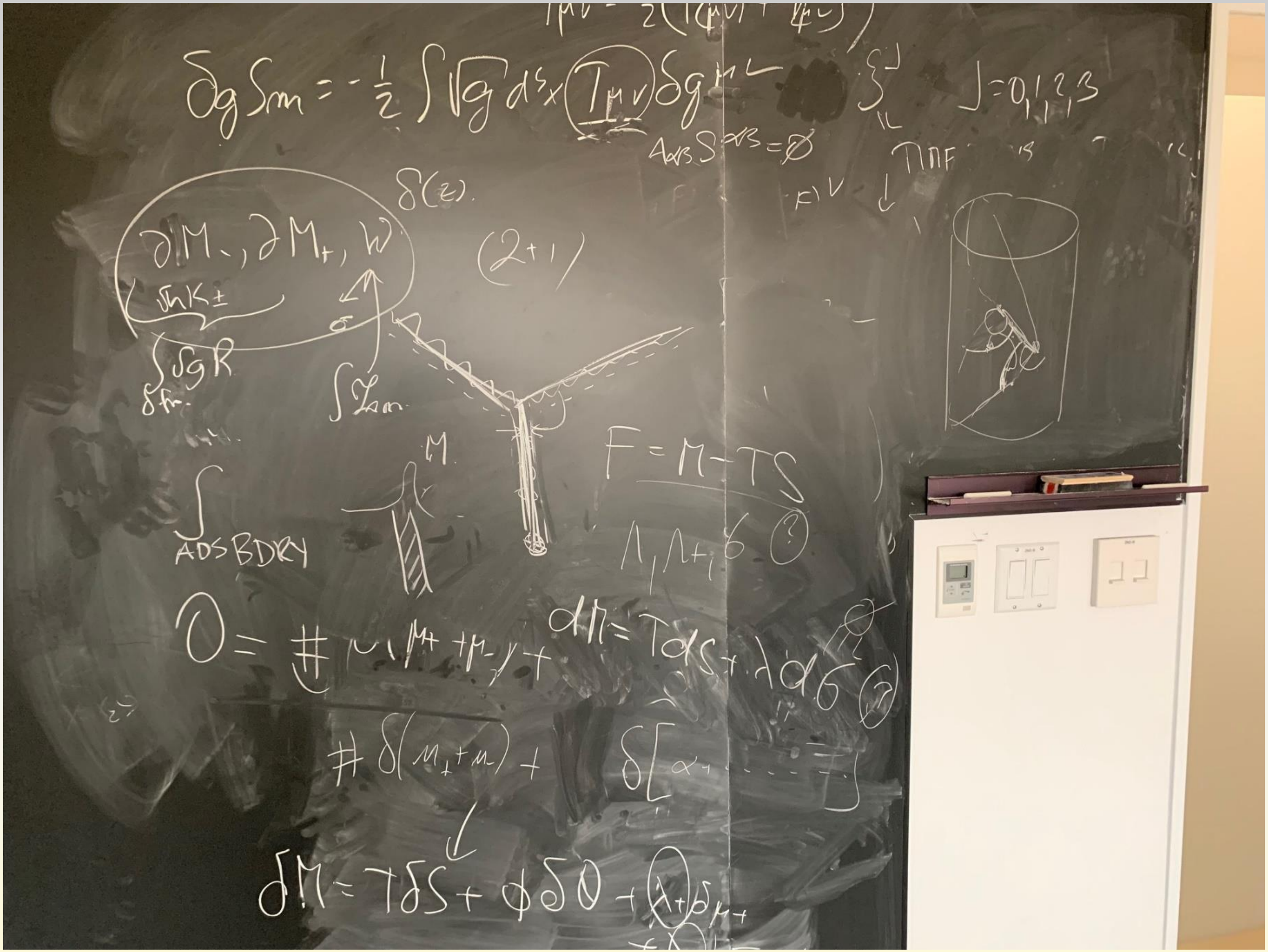
- **Master formula** for constraining the admissible variations in the first law?!?

$$a(\alpha, A, K, e)\delta\alpha + b(\alpha, A, K, e)\delta A + \delta(\alpha, A, K, e)\delta K = 0$$

- Not integrable, unless

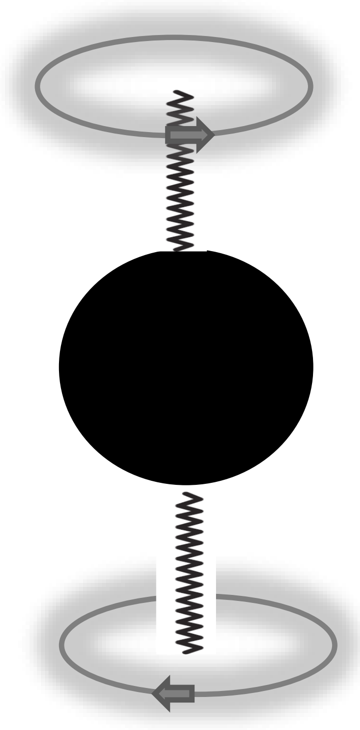
$$\delta(\mu_+ + \mu_-) = 0 \quad \Rightarrow \quad \alpha = \alpha(A, e)$$

# The story continues…



# Yet another "exotic" black hole spacetimes

**Taub-NUT solution**



# Basic properties and brief history

$$ds^2 = -f\left(dt + 2n\cos\theta d\phi\right)^2 + \frac{dr^2}{f} + (r^2+n^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$f = \frac{r^2 - 2mr - n^2}{r^2 + n^2}$$

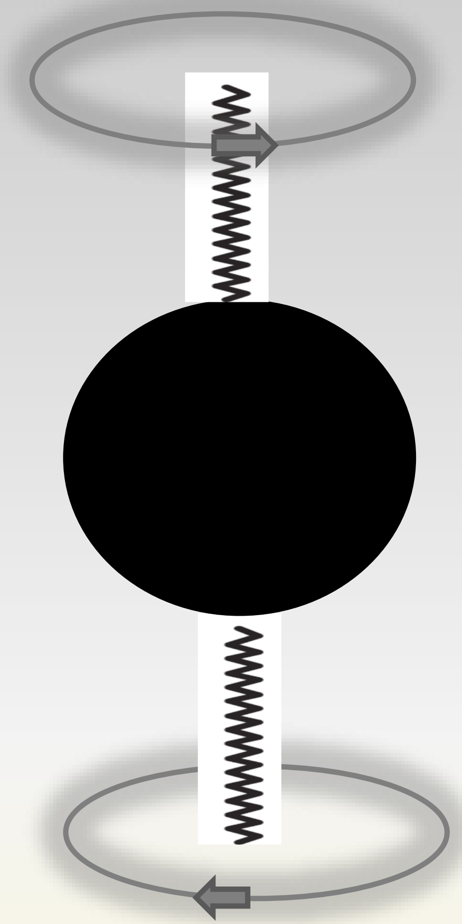
- **Vacuum solution** of Einstein equations
- Features **gravitational magnetic** (NUT) charge n
- Possesses **no curvature singularity**

- Discovered by Taub in 1951 as a homogeneous but **anisotropic cosmological model**
- Rediscovered as a candidate for **black hole solution** in 1963 by Newman-Tamburino-Unti: 2 BH horizons

# Problems with such an interpretation

- Remains finite at infinity (not AF)

$$g_{t\phi} \sim -2nf \cos\theta$$



- Spacetime contains rotating string-like singularities: **Misner strings**
- Associated regions with closed timelike curves (**CTCs)** in their vicinity
- Misner strings can be moved around by the following "**large coordinate transformation**":

$$t \to t + 2s\phi$$

# Misner's 1963 proposal

- r=const. surfaces are topologically 3-spheres, regularity requires

$$t \sim t + 8\pi n$$

(Dirac's argument)

- This renders **Misner strings unobservable**

- At the expense that:

  - Introduced CTCs everywhere
  - No "good analytic continuation" through BH horizon

# Euclidean Taub-NUT solution

- Since Misner's work Taub-NUT predominantly studied upon **Wick rotation**

$$t \to i\tau \qquad n \to i\nu$$

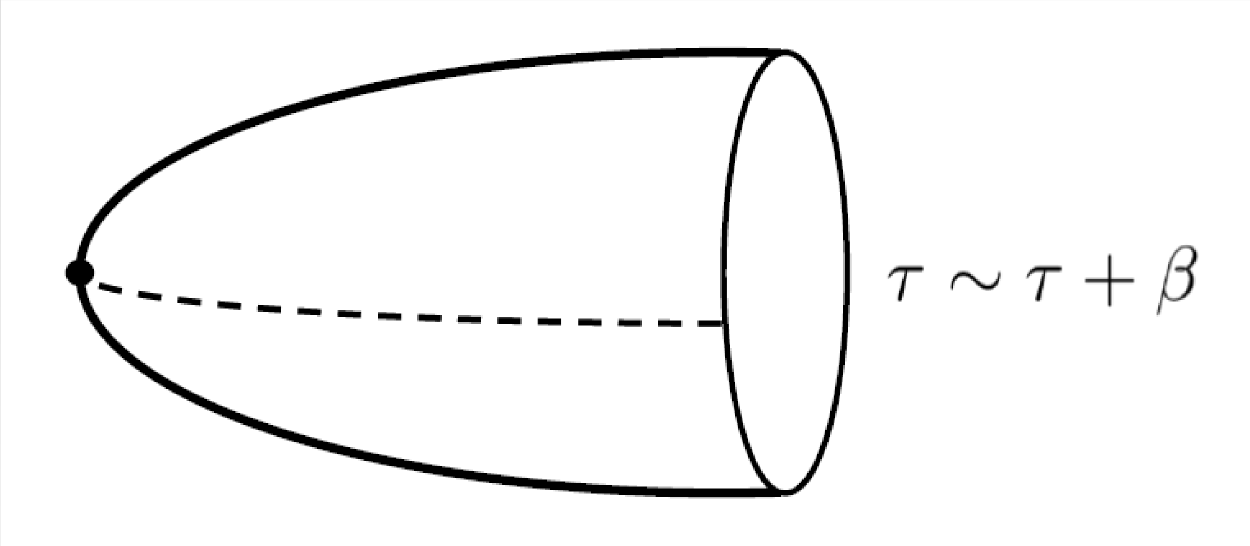
- Upon which it provides an example of **special Riemannian manifold**
    - Don Page, *Taub-NUT istanton with an horizon*, PLB 78 (1978) 249; PLB 79 (1978) 235.
- "**Supersymmetry**" and 4 KY tensors
    - Gibbons, Ruback, *The Hidden Symmeties of Taub-NUT and Monopole Scattering*, PLB 188 (1987) 226.
    - Holten, *Supersymmetry and the geometry of Taub-NUT*, PLB 342 (1995) 47.
- **Celestial holography**
    - Crawley et al., *Self-dual black holes in celestial holography*, JHEP 09 (2023) 109; Arxiv:2302.06661.

# Euclidean thermodynamics: NUTs and Bolts

Hawking & Hunter, *Gravitational entropy and global structure*, PRD59 (1999) 044025.

$$t \to i\tau$$

$$n \to i\nu$$




1) **Regularity** of Euclidean solution requires

$$T_{\rm BH} = \frac{f'(r_+)}{4\pi} \quad = \quad T_{\rm S} = \frac{1}{8\pi\nu} \quad \Rightarrow \quad \nu = \nu(r_+)$$

2) **Euclidean action** then yields

$$S = \frac{A}{4} + \text{stuff}$$

Misner string contribution!

# Summary

1) **C-metric** is one of the oldest and most surprising exact solutions of GR -- many applications and interesting properties.

2) Almost **consistent thermodynamics** for **slowly accelerating** AdS black holes:

$$\delta M = T\delta S - \lambda_+ \delta\mu_+ - \lambda_- \delta\mu_- + V\delta P + \dots$$

$$M = 2TS - 2VP + \dots$$

(works for rotation and charge as well)

3) Can this be extended to **more general settings** of AF and regular C-metrics? What about the AdS/CFT interpretation?

4) There is yet another exotic BH spacetime: **Taub-NUT** "**a counter example to almost everything**"