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Quantum information in curved spacetime

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Abstract

This is a study text for the "Quantum information in curved spacetime" course taught at Charles University in 2023/24. The text builds on similar courses delivered by Eduardo Martin-Martinez at the Perimeter Institute/University of Waterloo, as well as stems from a number of recent papers.

Basically, we will be touching on some topics studied by Relativistic Quantum Information (RQI), which is a new discipline that has emerged around 2010, as an attempt to merge three fields: *general relativity (GR), quantum field theory (QFT), and quantum information (QI)*. The main idea is to incorporate the relativistic description into QI processing and to study structure of spacetime and nature of gravity from QI perspective.

For example, we would like to tackle the following problems:

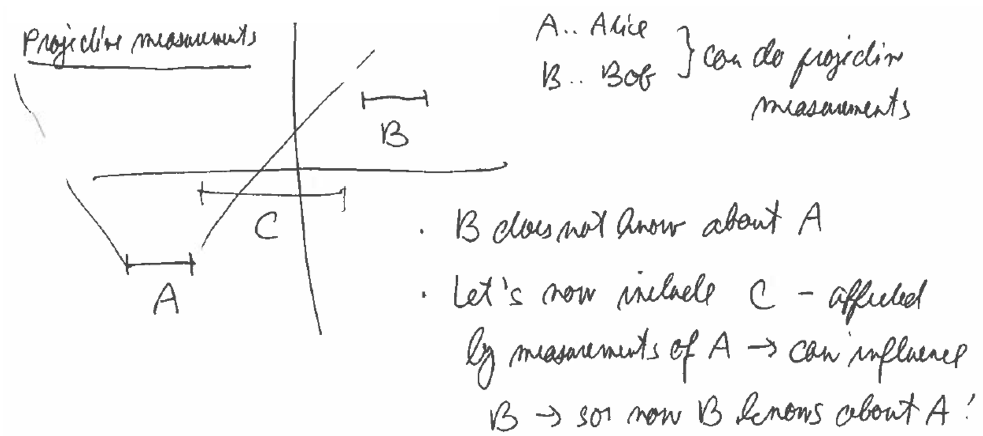
- Early Universe Cosmology – how much info we can get about early Universe?
- Black hole (BH) information loss – do BHs destroy information?
- QFT vacuum information content about given spacetime. Can we use quantum fields for spacetime reconstruction, or even to recast classical Einstein equations in QFT language?
- Thermalization, Unruh effect, ...
- Spacetime engineering – can we create states violating energy conditions, such as warp drives, wormhole, ...? (On average, QFT can violate energy conditions
- Is gravity really quantized? What is a superposition of spacetimes, and can large masses be entangled?
- Can we make a direct connection with available experiments?

Contents

Chapter 1: Measuring quantum fields: Particle detectors

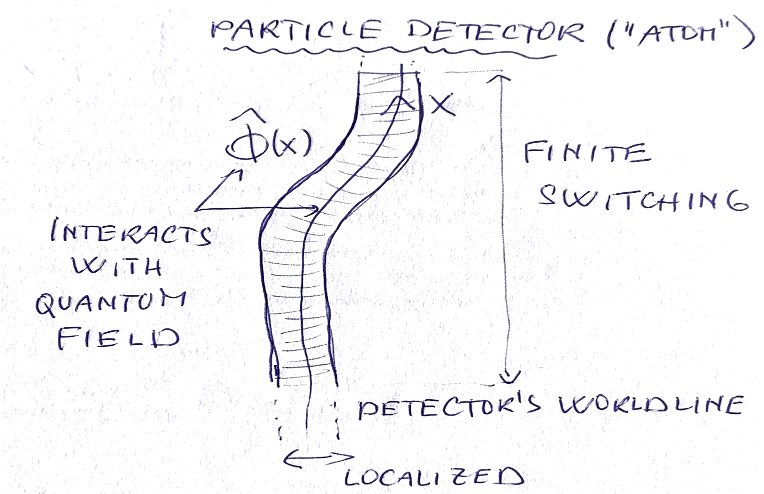
1.1 Motivation

- Projective measurements (at a given instant of time) are not a satisfactory description, as can be seen from the following picture:

  It also treats the detector at a different level than the quantum system.

- Instead use particle detectors: "couple 1st quantized system to the full 2nd quantized system":

  What is a particle detector? It better be i) localized (in time and space) quantum system that is ii) coupled to a quantum field, and is iii) easy to measure (to do projective measurements on), that is, it has a 'clicking quality' and is non-relativistic (1st quantized).

It seems like hydrogen atom could be a good model.

Particle-detector tautology. "A particle is what the particle detector measures; a particle detector is a device that detects particles."

- What do people do in Quantum optics? People typically use the Jaynes–Cummings (J-C) model. This is a '2-level atom' with two energy states $|e\rangle, |g\rangle$;

$$|\psi\rangle = \psi_e|e\rangle + \psi_g|g\rangle = \begin{pmatrix} \psi_e \\ \psi_g \end{pmatrix}, \tag{1.1}$$

separated by energy gap Ω, which couples to a 'mode' of the EM field (described by the harmonic oscillator) via the following interaction Hamiltonian:

$$\hat{H}_I = \lambda\Big(\sigma^+ a e^{i(\Omega-\omega)t} + \sigma^- a^+ e^{-i(\Omega-\omega)t}\Big), \tag{1.2}$$

where $\sigma^\pm$ are the $SU(2)$ ladder operators obeying[1]

$$\sigma^+|g\rangle = |e\rangle\,, \quad \sigma^-|e\rangle = |g\rangle\,, \tag{1.7}$$

that is, $\sigma^+ = |e\rangle\langle g|, \sigma^- = |g\rangle\langle e|$, and ω is the photon's frequency. Such a model has the following intuitive meaning: "annihilation of a photon excites the detector, whereas creation of a photon de-excites it". We would also typically expect $\Omega \approx \omega$ ("conservation of energy").

- The light matter interaction from first principles. Consider a hydrogen atom with an electron

$$\hat{H}_0 = \frac{\hat{\vec{p}}^2}{2m} + eV(\hat{x})\,, \tag{1.8}$$

where m is the effective electron's mass, and the electron can couple to an electromagnetic field. To this purpose we can perform the multipole expansion and

[1] In terms of the standard Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.3}$$

which obey

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma_k\,, \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij}1 \tag{1.4}$$

we have

$$\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y)\,, \quad \sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y)\,. \tag{1.5}$$

This also implies that

$$[\sigma^+, \sigma^-] = \sigma^z\,, \quad [\sigma^z, \sigma^\pm] = \pm 2\sigma^\pm\,. \tag{1.6}$$

restrict to the dipole approximation. In this approximation the interaction Hamiltonian reads

$$\boxed{\hat{H}_I = e\hat{\vec{x}} \cdot \vec{E}(\hat{\vec{x}})\,.} \tag{1.9}$$

Here, field is not yet quantized. Let us see what do we get from this 'definition'.

- Expanding in matrix elements (energy states of the unperturbed Hamiltonian), we have

$$\hat{H}_I = e\hat{\vec{x}} \cdot \vec{E}(\hat{\vec{x}}) = e\sum_{i,j} \langle j|\hat{\vec{x}} \cdot \vec{E}(\hat{\vec{x}})|i\rangle e^{i\Omega_{ij}t}|j\rangle\langle i|\,. \tag{1.10}$$

Here, $e^{i\Omega_{ij}t}$, where $\Omega_{ij} = \Omega_j - \Omega_i$ is the energy between states i and j, comes from the interaction picture.[2] We now insert the identity $\int dx|x\rangle\langle x|$ and employ the standard hydrogen atom wave functions $\psi_i(x) = \langle x|i\rangle$. Thus we find

$$\begin{aligned} \hat{H}_I &= e\int d^3x d^3x' \sum_{i,j} \langle j|x\rangle\langle x|\hat{\vec{x}} \cdot \vec{E}(\hat{\vec{x}})|x'\rangle\langle x'|i\rangle e^{i\Omega_{ij}t}|j\rangle\langle i| \\ &= e\sum_{i,j}\int d^3x \psi_j^*(x)\vec{x}\psi_i(x) \cdot \vec{E}(\vec{x})e^{i\Omega_{ij}t}|j\rangle\langle i|\,. \end{aligned} \tag{1.14}$$

Of course, many of the matrix elements are (to the lowest order in perturbation theory) zero, using for example selection rules. In what follows we concentrate on a 2 level model, with ground state $|g\rangle$ and excited state $|e\rangle$, separated by energy gap Ω. Let's also denote the ladder operators

$$\sigma^+ = |e\rangle\langle g|\,, \quad \sigma^- = |g\rangle\langle e|\,, \tag{1.15}$$

and $\vec{F}(\vec{x}) = \psi_e^*(\vec{x})\vec{x}\psi_g(\vec{x})$ the corresponding 'smearing function'. With this we have

$$\hat{H}_I = \int d^3x e\Big(\vec{F}(\vec{x})e^{i\Omega t}\sigma^+ + \vec{F}^*(\vec{x})e^{-i\Omega t}\sigma^-\Big) \cdot \vec{E}(\vec{x}) \equiv \int d^3x \hat{\vec{d}}(\vec{x}) \cdot \vec{E}(\vec{x})\,, \tag{1.16}$$

where $\hat{\vec{d}}$ is the dipole operator. We can now proceed and canonically quantize the EM field $\vec{E} \to \hat{\vec{E}}$. However, if we are not interested in exchange of angular momentum, we can consider a simplified scalar model. This is known as:

[2]Splitting the total Hamiltonian, into the 'basic' and 'interaction' parts:

$$H = H_0 + H_I\,, \tag{1.11}$$

the interaction picture operators and states are related to Schrodinger picture operators and states as follows (setting $\hbar = 1$):

$$A_I = e^{iH_0t}A_S e^{-iH_0t}\,, \quad |\psi_I\rangle = e^{iH_0t}|\psi_S\rangle\,. \tag{1.12}$$

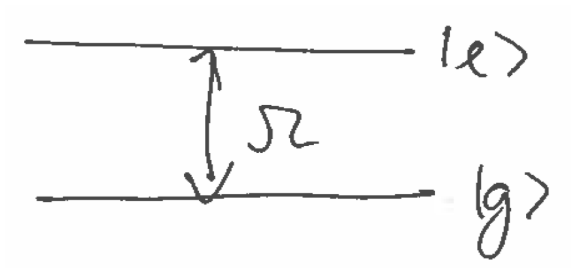
Such operators and states evolve as follows:

$$i\frac{dA_I}{dt} = [A_I, H_0]\,, \quad i\frac{d\psi_I}{dt} = H_I\psi_I\,. \tag{1.13}$$

Of course, we also have $\langle A_I\rangle = \langle A_S\rangle = \mathrm{Tr}(\rho_I A_I)\,.$

1.2 Unruh De Witt (UdW) detector

- Unruh-De Witt (UdW) detector (Unruh 1976 [1], De Witt 1979 [2]). This is a 'scalar version' of the above, namely



$$\boxed{H_I = \lambda\chi(t)\int d^3x\,\hat{\mu}(t,\vec{x})\hat{\phi}(t,\vec{x})\,,} \tag{1.17}$$

where $\hat{\mu}$ is the monopole operator (with smearing function $F(\vec{x})$):

$$\hat{\mu}(t,\vec{x}) = F(\vec{x})\underbrace{\left(\sigma^+ e^{i\Omega t} + \sigma^- e^{-i\Omega t}\right)}_{\hat{m}(t)}\,, \tag{1.18}$$

and we have also included the switching function $\chi(t)$ for the detector, governing the duration for which the detector is switched on.

Here, the massless scalar field ϕ (in $d = (n+1)$ dimensions) is quantized:

$$\hat{\phi}(t,\vec{x}) = \int \frac{d^n k}{\sqrt{2(2\pi)^n|\vec{k}|}}\left(a^+_{\vec{k}} e^{-ik\cdot x} + a_{\vec{k}} e^{ik\cdot x}\right). \tag{1.19}$$

- Note that there are extra terms in the UdW detector when compared to the J-C detector. Namely, we schematically have

$$\begin{aligned} \text{J-C}: &\quad \sigma^+ a + \sigma^- a^+\,,\\ \text{UdW}: &\quad \underbrace{(\sigma^+ + \sigma^-)}_{\sigma^x}\underbrace{(a + a^+)}_{\hat{x}} \sim \sigma^+ a + \sigma^- a^+ + \underbrace{\sigma^+ a^+ + \sigma^- a}_{\text{counter-rotating terms}} \end{aligned} \tag{1.20}$$

The difference is thus the presence of the "counter-rotating terms". Some say that such terms 'do not conserve energy':

$|e\rangle|0\rangle \xrightarrow{\sigma^- a^-} |g\rangle|1\rangle$ OK for
$\hbar\Omega$ $\hbar\omega$ $\omega \simeq \Omega$

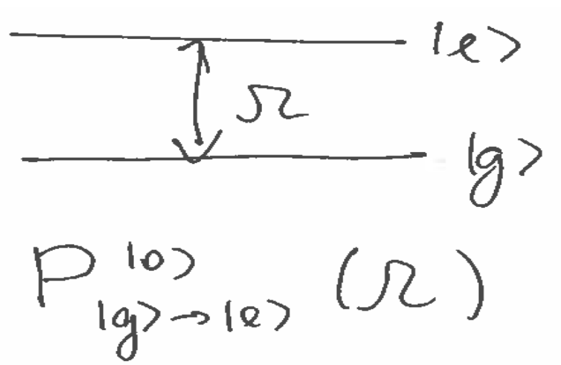
but: $|g\rangle|0\rangle \xrightarrow{\sigma^+ a^+} |e\rangle|1\rangle$ DOES NOT
$\emptyset$ $\hbar(\Omega+\omega)$ CONSERVE ENERGY?

However, that is not true for two reasons. First, the interaction Hamiltonian is accompanied by Hamiltonians of detector and field:

$$H_d = \Omega \underbrace{\sigma^+\sigma^-}_{\sim\sigma^z}, \quad H_{\text{field}} = \omega \underbrace{a^+a}_{\hat{n}} . \tag{1.21}$$

However, since σ^z and σ^x, and $\hat{n}$ and $\hat{x}$ do not commute, eigenstates of the free Hamiltonian are not the eigenvalues of the total Hamiltonian; so assigning energy to this is false! Second, we do not perform a measurement and the state is that of superposition of all possibilities.

- Vacuum excitation probability of a UdW detector. Let us now calculate the vacuum excitation probability:



Once we have calculated that, the opposite process, probability of de-excitation is calculated as:

$$P^{|0\rangle}_{|e\rangle\to|g\rangle}(\Omega) = P^{|0\rangle}_{|g\rangle\to|e\rangle}(-\Omega) . \tag{1.22}$$

We have

$$P^{|0\rangle}_{|g\rangle\to|e\rangle}(\Omega) = \sum_{\text{out}} |\langle \text{out}, e|U|g, 0\rangle|^2 = \sum_{\text{out}} \langle 0, g|U^+|e, \text{out}\rangle\langle \text{out}, e|U|g, 0\rangle , \tag{1.23}$$

where we have summed over all final states of the field $|\text{out}\rangle$, and the evolution operator U is given by the time ordered exponential:

$$U = T \exp\Big(-i \int_{-\infty}^{\infty} dt H_I(t)\Big) , \tag{1.24}$$

where, w.l.o.g., we can take the limits to be $(-\infty, +\infty)$, as the interaction Hamiltonian H_I already contains the finite switching function χ. To calculate U, we use the perturbation theory, namely Dyson's expansion:

$$U = 1 + U^{(1)} + U^{(2)} + O(\lambda^3) , \quad U^{(1)} = -i \int_{-\infty}^{\infty} dt H_I(t) . \tag{1.25}$$

Let us calculate the probability to the linear in λ order. Obviously, the first term does not contribute, and we have

$$\begin{aligned} P^{|0\rangle}_{|e\rangle\to|g\rangle}(\Omega) &= \langle 0,g|U^{(1)+}|e\rangle \underbrace{\sum_{\text{out}}|\text{out}\rangle\langle\text{out}|}_{1}\langle e|U^{(1)}|g,0\rangle \\ &= \lambda^2\int dtdt'\,\chi(t)\chi(t')\int d^nxd^nx'F(\vec x)F(\vec x')\underbrace{\langle g|\hat m(t)|e\rangle\langle e|\hat m(t')|g\rangle}_{\exp\left(i\Omega(t'-t)\right)}W(t,\vec x,t',\vec x') \end{aligned} \tag{1.26}$$

where we for example used that $\langle g|\hat m(t)|e\rangle = \langle g|(\sigma^+e^{i\Omega t}+\sigma^-e^{-i\Omega t})|e\rangle = e^{-i\Omega t}$, and defined the Wightman function (2pt. function):

$$\boxed{W(t,\vec x,t',\vec x') = \langle 0|\hat\phi(t,\vec x)\hat\phi(t',\vec x')|0\rangle\,.} \tag{1.27}$$

That is, we have found that

$$\boxed{P^{|0\rangle}_{|e\rangle\to|g\rangle}(\Omega) = \lambda^2\int dtdt'\chi(t)\chi(t')\int d^nxd^nx'F(\vec x)F(\vec x')W(t,\vec x,t',\vec x')e^{i\Omega(t'-t)}\,.} \tag{1.28}$$

- To proceed further, let us use the flat space Wightman function:

$$W(t,\vec x,t',\vec x') = \frac{i}{(2\pi)^{n+1}}\int d^{n+1}k\frac{e^{ik\cdot(x-x')}}{k^2} = \int\frac{d^nk}{2(2\pi)^n|\vec k|}e^{-i\left(|\vec k|(t-t')-\vec k\cdot(\vec x-\vec x')\right)}\,. \tag{1.29}$$

  Then we can easily perform the integrals over $t, t', \vec x, \vec x'$ – they simply yield the Fourier transform, e.g.[3]

$$\tilde\chi(\Omega+|\vec k|) = \int dt\chi(t)e^{-i\left(\Omega+|\vec k|\right)t}\,. \tag{1.31}$$

  We thus recover

$$P^{|0\rangle}_{|g\rangle\to|e\rangle}(\Omega) = \lambda^2\int\frac{d^nk}{2(2\pi)^n|\vec k|}\left|\tilde\chi(\Omega+|\vec k|)\right|^2\left|\tilde F(\vec k)\right|^2\,. \tag{1.32}$$

- Consider now 'long switching' (the detector is switched on forever). Then we have

$$\chi(t) = \text{const.}\quad\Rightarrow\quad\tilde\chi(\omega)\propto\delta(\omega)\,. \tag{1.33}$$

[3] Here we adopt the following convention for the Fourier transform:

$$\tilde f(\omega) = \int f(x)e^{-i\omega x}dx\,,\quad f(x) = \frac{1}{2\pi}\int\tilde f(\omega)e^{i\omega x}d\omega\,. \tag{1.30}$$

Since $\Omega + |\vec{k}| \neq 0$, we then find $P(\Omega) = 0$. This makes sense, while for small enough times we can get excitations ('borrowing energy from vacuum'), when the detector is switched on forever, the detector will not get excited. On the other hand, if our detector is localized in space and time, it will click!

Note also that the non-trivial contribution to P comes from the counter-rotating terms, namely $\sigma^+ a^+$ and $\sigma^- a$. If we adopted the co-rotating wave approximation (RWA) we would not see any excitations.

- Similarly, if we started with the excited detector, the probability of its de-excitation would be $P(-\Omega)$, which gives the condition

$$\Omega = |\vec{k}|\,, \tag{1.34}$$

that is, only 1 mode, with $|\vec{k}| = \Omega$, contributes. For long enough times, we can thus adopt a single mode approximation (SMA). Note also that in this case, the non-trivial contributions come from the co-rotating waves $\sigma^+ a$ and $\sigma^- a^+$.

- When we compare UdW to the quantum optics J-C model, we thus see that the latter adopts i) SMA approximation and ii) RWA approximation. Consequently, J-C will not get excited but will have spontaneous emission via single mode. These approximations are good for long enough times.

More precisely, let T be the support of $\chi(t)$. Then the following approximations are 'valid':

$$\begin{aligned} &\text{SMA}: \qquad && T \gg \frac{1}{\Omega - |\vec{k}|}\,,\\ &\text{RWA}: \qquad && T \gg \frac{1}{\Omega + |\vec{k}|}\,. \end{aligned} \tag{1.35}$$

We see that if SMA holds, so does RWA.

Often, it happens that $T \gg \Omega^{-1}$. Then both RWA and SMA are OK. While this is the case for optical cavities, for which the J-C model is sufficient, it may not be the case for the effects we shall study in this course – for this reason we are going to use the UdW detector.

1.3 Which frame?

- Which frame? Let us now write down the total Hamiltonian for our system. When doing so, we have two natural frames to use: lab (inertial) frame $(t, \vec{x})$, or the proper detector's frame $(\tau, \vec{\xi})$ (for example associated with the center of mass of the atom).

The detector's free Hamiltonian is most easily written in detector's frame:

$$\boxed{{}^{\tau}H_{0,d} = \Omega\sigma^{+}\sigma^{-}\,,} \tag{1.36}$$

where τ is the proper time of the detector, and gap Ω is measured in detector's frame.

Field free Hamiltonian, is most easily written in the Lab frame:

$$\boxed{{}^{t}H_{0,\phi} = \int d^{n}k|\vec{k}|a^{+}_{\vec{k}}a_{\vec{k}}\,,} \tag{1.37}$$

where t is the (lab frame) 'quantization time'.

However, the interaction Hamiltonian contains both sets of observables: (detector) $\times$ (field). It is the detector which prescribes the interaction with the field. It is thus natural to write this in the detector's frame:

$$\boxed{{}^{\tau}H_{I} = \lambda\chi(\tau)\int d^{n}\xi F(\vec{\xi})\underbrace{\left(\sigma^{+}e^{i\Omega\tau} + \sigma^{-}e^{-i\Omega\tau}\right)}_{\hat{m}(\tau)}\hat{\phi}\big(t(\tau,\vec{\xi}),\vec{x}(\tau,\vec{\xi})\big)\,.} \tag{1.38}$$

Note that when we write $F(\vec{\xi})$, we assume 'rigid atom' along the trajectory – the so called Fermi–Walker rigidity (in the center of mass frame). In other words, our F is not a function of τ (wave functions of the atom are not deformed by motion of the atom – 'atom drags the electrons"). This is okay, for accelerations $a < 10^{17}g$. (A bullet hitting a target has $a \sim 10^{10-11}g$, so it modifies molecules but not atoms.)

However, we can write this in the lab frame as well! To warm up let us start with time reparametrization.

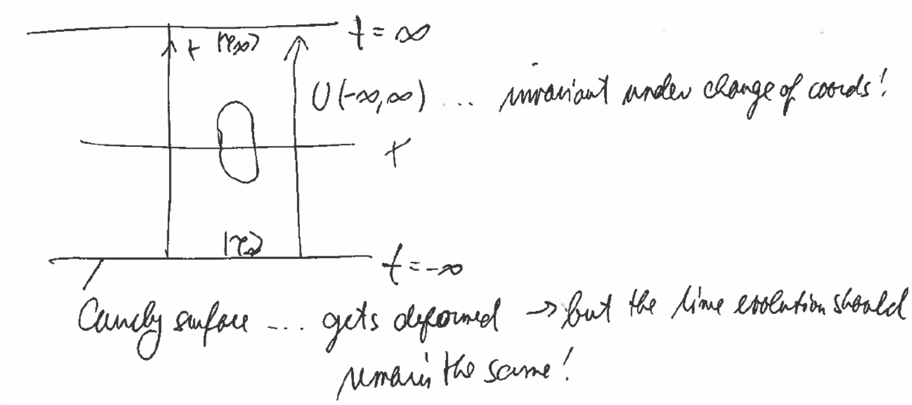
- Time reparametrization. Let ${}^{t}\hat{H}(t)$ be the Hamiltonian of a quantum system generating translations w.r.t. time t. What is ${}^{\tau}\hat{H}(\tau)$ generating translations w.r.t. τ? Under reparametrization $t \to t(\tau)$, we have $\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau}$. Employing the Schrodinger equation, we thus have:

$$i\frac{d}{dt}|\psi\rangle = {}^{t}\hat{H}(t)|\psi\rangle = i\frac{d\tau}{dt}\frac{d}{d\tau}|\psi\rangle \quad \Rightarrow \quad i\frac{d}{d\tau}|\psi\rangle = \underbrace{\frac{dt}{d\tau}{}^{t}\hat{H}(t)}_{{}^{\tau}\hat{H}(\tau)}|\psi\rangle\,, \tag{1.39}$$

that is, we also pick up the 'redshift factor' $\frac{dt}{d\tau}$:

$$\boxed{{}^{\tau}\hat{H}(\tau) = \frac{dt}{d\tau}{}^{t}\hat{H}(t(\tau))\,.} \tag{1.40}$$

- The same can be seen, for example, from the fact that the time evolution operator must be invariant under time reparametrization, as seen from the following picture:



  Thus we have

$$U = T\exp\Big(-i\int dt\,{}^tH(t)\Big) = T\exp\Big(-i\int d\tau\,{}^\tau H(\tau)\Big)\,. \tag{1.41}$$

  Using Fubini's theorem we recover (1.40).

- More generally, the evolution operator is invariant under general change of coordinates. We thus have

$$\begin{aligned} U &= T\exp\Big(-i\int_{-\infty}^{\infty} d\tau\,{}^\tau H_I(\tau)\Big) = T\exp\Big(-i\int_{-\infty}^{\infty} d\tau d\vec{\xi}\,{}^\tau h_I(\tau)\Big) \\ &= T\exp\Big(-i\int_{-\infty}^{\infty} dt d\vec{x}\,{}^t h_I(t,\vec{x})\Big)\,, \end{aligned} \tag{1.42}$$

  where

$$\begin{aligned} {}^\tau h_I &= \lambda\chi(\tau)F(\vec{\xi})\hat{m}(\tau)\hat{\phi}\big(t(\tau,\vec{\xi}),\vec{x}(\tau,\vec{\xi})\big)\,, \\ {}^t h_I &= \lambda\chi\big(\tau(t,\vec{x})\big)F\big(\vec{\xi}(t,\vec{x})\big)\hat{m}\big(\tau(t,\vec{x})\big)\hat{\phi}(t,\vec{x})\left|\frac{\partial(\tau,\vec{\xi})}{\partial(t,\vec{x})}\right|\,. \end{aligned} \tag{1.43}$$

  Note that one cannot really distinguish switching from smearing! Note also that ${}^tH_I = \int d^n\vec{x}\,{}^t h_I$ is pretty non-trivial!

- Often-times one uses the 'point-particle' detector, setting

$$F(\vec{\xi}) = \delta(\vec{\xi})\,. \tag{1.44}$$

  This is the standard approximation that is used in many situations. On the other hand, when one uses the extended detector, problems with general covariance and time ordering ambiguity arise, e.g. [3, 4]. Often-times we shall also use the (smooth) Gaussian switching χ, treating it as 'compact support'.

Chapter 2: Unruh Effect

2.1 Standard derivation

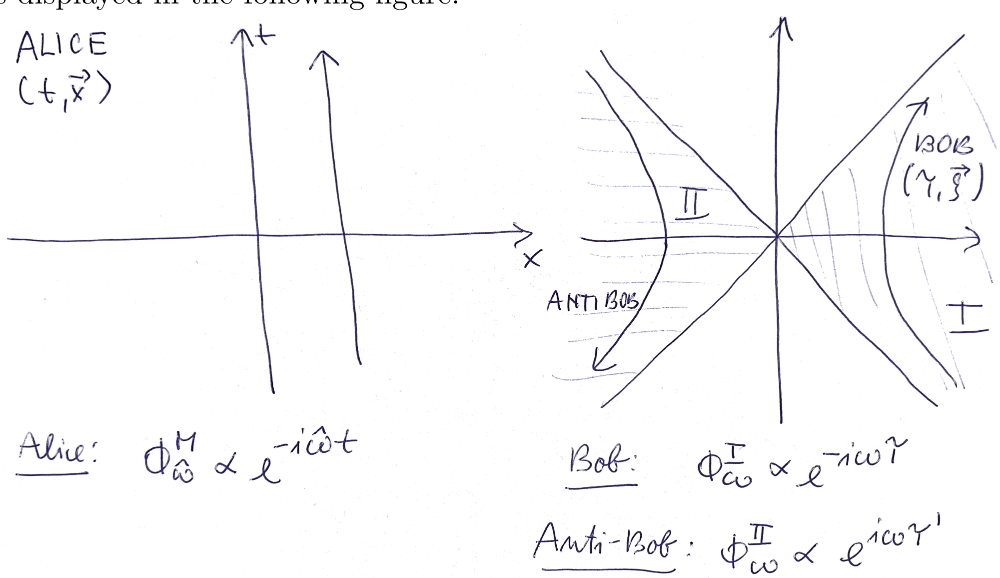
- Key idea: field quantization depends on the observer.
- Let us consider a massless scalar field, obeying

$$\nabla^2 \phi = 0\,, \tag{2.1}$$

as 'perceived' by two different observers (both of whom have a copy of Peskin and Schroeder): inertial Alice who is using Minkowski coordinates (t, x)), and uniformly accelerated Bob – using Rindler coordinates (τ, ξ), related to the Minkowski coordinates by

$$t = \xi \sinh a\tau\,, \quad x = \xi \cosh a\tau\,, \tag{2.2}$$

as displayed in the following figure:



- Note that ϕ^I_ω are not a complete basis of solutions for QFT in the whole spacetime, only in the wedge I. To have a complete set, we also need to consider Anti-Bob in the region II:

$$t = -\xi' \sinh a\tau'\,, \quad = -\xi' \cosh a\tau'\,. \tag{2.3}$$

- $\{\phi^I, \phi^{II}\}$ then form a basis at a given τ, which is a Cauchy surface. So at this Cauchy surface we can expand

$$\begin{aligned}\phi &= \sum_i \Big(a^M_{\hat{\omega}_i}\phi^M_{\hat{\omega}_i} + a^{M+}_{\hat{\omega}_i}\phi^{M*}_{\hat{\omega}_i}\Big) \\ &= \sum_i \Big(a^I_{\omega_i}\phi^I_{\omega_i} + a^{I+}_{\omega_i}\phi^{I*}_{\omega_i} + a^{II}_{\omega_i}\phi^{II}_{\omega_i} + a^{II+}_{\omega_i}\phi^{II*}_{\omega_i}\Big)\end{aligned} \tag{2.4}$$

  These are not unitary equivalent (have different vacua) – a^M's mix with a^I's and a^{I+}'s (a^{II}'s and a^{II+}'s).

- Considering the Minkowski vacuum: $|0\rangle_M$, it can be written as

$$|0\rangle_M = \prod_\omega \frac{1}{\cosh r_\omega} \sum_{n=0}^{\infty} \tanh^n r_\omega |n\rangle_I |n\rangle_{II}\,, \tag{2.5}$$

  where

$$\tanh r_\omega = \exp\Big(-\frac{\pi\omega}{a}\Big)\,. \tag{2.6}$$

  This is a 2-mode squeeze state (entangled) (mixes excitations in I and II regions).

- Bob has only access to region I. Thus uses the following density matrix:

$$\rho_{B\omega} = \mathrm{Tr}_{II}\big(|0\rangle_M\langle 0|_M\big) = \frac{1}{\cosh^2 r_\omega}\sum_n \tanh^{2n} r_\omega |n\rangle_I\langle n|_I\,. \tag{2.7}$$

  Using this to calculate the expectation of the number operator, we arrive at a <u>thermal state</u>

$$\langle N_{\omega,B}\rangle = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1}\,, \tag{2.8}$$

  which is the Bose–Einstein distribution with the <u>Unruh temperature</u>

$$\boxed{T_U = \frac{\hbar a}{2\pi k_B}\,.} \tag{2.9}$$

  So we arrived at a conclusion that Alice's field vacuum corresponds to a thermal bath for Bob at $T_U \propto a$.

- Two physical questions arise: 1) Does Bob need to accelerate forever – what happens for finite time acceleration? 2) Is the calculation above really enough to talk about thermality?

2.2 What is a thermal state?

- Gibbs: *a state that maximizes entropy at constant energy.*
  However, this is ill defined (diverges) for QFT.

- Instead: Let's compute a 2-pt correlator of an observable A of a quantum system in a thermal state (Gibbs). Note that thermal states are stationary (fixed points of time evolution. Since
$$\hat{A}(t) = e^{-iHt} A(0) e^{iHt}\,, \tag{2.10}$$
using stationarity we have to have
$$C(t,t') = \mathrm{Tr}\Big(\rho_\beta A(t) A(t')\Big) = \mathrm{Tr}\Big(\rho_\beta A(\Delta t) A(0)\Big) = C(\Delta t)\,. \tag{2.11}$$
Moreover, using the cyclic property of the trace we have
$$C(\Delta t) = \frac{1}{Z}\mathrm{Tr}\Big(e^{-\beta H} e^{-iH\Delta t} A(0) e^{iH\Delta t} A(0)\Big) = \frac{1}{Z}\mathrm{Tr}\Big(e^{-iH(\Delta t - i\beta)} A(0) e^{iH\Delta t} A(0)\Big)\,. \tag{2.12}$$
At the same time, shifting time, we get
$$\begin{aligned} C(\Delta t + i\beta) &= \frac{1}{Z}\mathrm{Tr}\Big(e^{-iH\Delta t} A(0) e^{iH(\Delta t + i\beta)} A(0)\Big) \\ &= \frac{1}{Z}\mathrm{Tr}\Big(e^{iH(\Delta t + i\beta)} A(0) e^{-iH\Delta t} A(0)\Big) = C(-\Delta t)\,, \end{aligned} \tag{2.13}$$
that is 'complex anti-periodicity'. Thus, all Gibbs states obey Kubo, Martin, Schwinger (KMS) condition
$$\boxed{C(-\Delta t) = C(\Delta t + i\beta)\,.} \tag{2.14}$$
Think about how this is related to the Euclidean trick showing that black holes have a temperature!

- We have just shown that all Gibbs states are KMS. The converse is not true but 'almost true' :).

  One can also show that KMS states are passive – one cannot extract work from them:
$$\langle E\rangle_{\text{extracted}} = 0\,. \tag{2.15}$$
  Moreover, KMS condition is applicable to QFT, and provides a 'good definition' of thermality for QFTs. We will associate KMS condition with a given observer.

  In particular, since the Wightman function
$$W_\rho(\tau,\tau') = \mathrm{Tr}\Big(\rho\hat{\phi}(\tau)\hat{\phi}(\tau')\Big) \tag{2.16}$$

is a 2-pt function (that knows everything there is to know about the QFT), we can use it to define thermality of the state ρ. Here, and in what follows, we have abbreviated

$$\hat{\phi}(\tau) \equiv \hat{\phi}(t(\tau), \vec{x}(\tau)) \,. \tag{2.17}$$

Namely, we have the following definition:

- KMS states in QFT (better definition of thermality):

Definition. Let us have a timelike vector ∂_τ, a Hamiltonian ${}^\tau H$, and a field state $\hat{\rho}$. Then $\hat{\rho}$ is a KMS state of KMS temperature

$$T_{\text{KMS}} = \frac{1}{\beta} \,, \tag{2.18}$$

with respect to ∂_τ if and only if i) it is stationary, that is $W_\rho(\tau, \tau') = W_\rho(\Delta\tau)$ and ii) satisfies KMS condition, that is $W_\rho(\Delta\tau + i\beta) = W_\rho(-\Delta\tau)$.

Note that different observers can have different KMS temperatures. Namely in the Unruh case, $T_{\text{KMS}} = T_A = 0$ with respect to ∂_t and $T_{\text{K}MS} = T_B = T_U$ w.r.t. ∂_τ.

Note also, that one can have non-stationary states with complex anti-periodicity. This is of course not enough to have thermality!

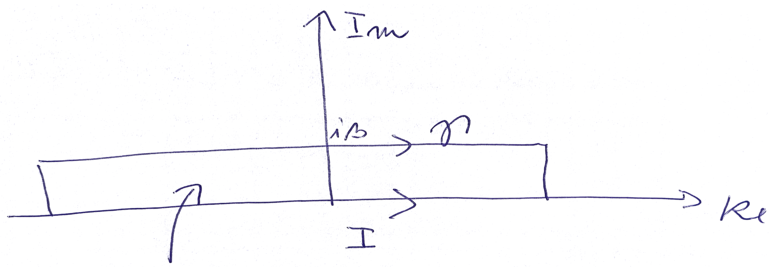
- Detailed balance. Let us have a KMS state, then we have

$$\begin{aligned}
\int_{-\infty}^{\infty} d\Delta\tau W(\Delta\tau + i\beta) e^{i\omega\Delta\tau} &= \int_{-\infty}^{\infty} d\Delta\tau W(-\Delta\tau) e^{i\omega\Delta\tau} \\
&= \int_{\infty}^{-\infty} d(-\Delta\tau) W(\Delta\tau) e^{-i\omega\Delta\tau} \\
&= \int_{-\infty}^{\infty} d\Delta\tau W(\Delta\tau) e^{-i\omega\Delta\tau} = \tilde{W}(-\omega) \,.
\end{aligned} \tag{2.19}$$

Now, let us change the variables, $\Delta\tau' = \Delta\tau + i\beta$. Then we have

$$\tilde{W}(\omega) = \int_\gamma d\Delta\tau' W(\Delta\tau') e^{i\omega(\Delta\tau' - i\beta)} = e^{\beta\omega} \int_\gamma d\Delta\tau' W(\Delta\tau') e^{i\omega\Delta\tau'} = e^{\beta\omega} \tilde{W}(\omega) \,, \tag{2.20}$$

where we have effectively done the following:



IF W ANALYTIC HERE (OR REGULARIZED ?)
(CAUCHY INTEGRAL THEOREM)

THEN WE ARE OK.

Thus we have derived 'detailed balance' condition:

$$\boxed{e^{\beta\omega}\tilde{W}(\omega) = \tilde{W}(-\omega)\,.} \tag{2.21}$$

2.3 Thermalization of the detector

- As an experimentalist I carry a thermometer and accelerate – do I see the Unruh temperature $T = a/(2\pi)$? Thermality expectation: detector state evolves to $\rho = \frac{1}{Z}e^{-\beta H_d}$, which implies
$$\frac{P_{\text{ex}}(\Omega)}{P_{\text{deex}}(\Omega)} = e^{-\beta\Omega}\,. \tag{2.22}$$
Let us show that this is true. We shall do this in several steps.

- 'Experimental setup'. In what follows we shall consider a switching function χ that is strongly supported in a timescale σ (for which the detector is on, with $\chi(\pm 1)$ being the boundary of the support), such that its L^2 norm is equal to one, that is
$$||\chi(\tau/\sigma)||_{L^2} \equiv \int_{-\infty}^{\infty} d\tau |\chi(\tau/\sigma)|^2 = 1\,. \tag{2.23}$$
For such switching we have
$$\chi(\tau/\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\tilde{\chi}(\omega)e^{-i\omega\tau/\sigma} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\tilde{\chi}^*(\omega)e^{i\omega\tau/\sigma} = \chi^*(\tau/\sigma)\,, \tag{2.24}$$
as it is real. Moreover, also we have
$$\begin{aligned}\int_{-\infty}^{\infty} d\omega|\tilde{\chi}(\omega)|^2 &= \int_{-\infty}^{\infty} d\tau d\tau' d\omega \chi(\tau/\sigma)\chi^*(\tau'/\sigma)e^{i\omega(\tau-\tau')} \\ &= 2\pi\int d\tau d\tau' \chi(\tau/\sigma)\chi^*(\tau'/\sigma)\delta(\tau-\tau') \\ &= 2\pi\int_{-\infty}^{\infty} d\tau |\chi(\tau/\sigma)|^2 = 2\pi\,,\end{aligned} \tag{2.25}$$

(a Percival equality).

Moreover, if $\chi(\tau/\sigma)$ strongly supported on sacle σ, $\tilde{\chi}(\omega)$ is strongly supported on a scale $1/\sigma$.

We shall also consider a point-like detector, for which $F(\vec{\xi}) = \delta(\vec{\xi})$, and the UdW interaction Hamiltonian reads

$$H_I = \lambda\chi(\tau)\hat{m}(\tau)\hat{\phi}(\tau)\,, \tag{2.26}$$

and the transition probability (1.28) simplifies.

- Long response. Considering the pointlike detector with finite switching function above, the response function, as per (1.28), reads

$$\begin{aligned}
\mathcal{F}(\Omega,\sigma) &\equiv \frac{1}{\lambda^2\sigma}P(\Omega) = \frac{1}{\sigma}\int d\tau d\tau'\chi(\tau/\sigma)\chi(\tau'/\sigma)e^{i\Omega(\tau-\tau')}W(\tau,\tau') \\
&= \frac{1}{4\pi^2\sigma}\int d\tau d\tau' d\omega d\omega' \tilde{\chi}^*(\omega)\tilde{\chi}(\omega')e^{i(\omega\tau/\sigma-\omega'\tau'/\sigma)}e^{i\Omega(\tau-\tau')}W(\tau,\tau')\,.
\end{aligned} \tag{2.27}$$

If the field is stationary w.r.t. ∂_τ, $W(\tau,\tau') = W(\tau-\tau')$, it is natural to change variables as

$$u = \tau-\tau'\,,\quad v=\tau+\tau' \quad\Leftrightarrow\quad \tau = \frac{u+v}{2}\,,\quad \tau' = \frac{v-u}{2}\,, \tag{2.28}$$

with the corresponding Jacobian equal to $1/2$. We then find

$$\begin{aligned}
\mathcal{F}(\Omega,\sigma) &= \frac{1}{8\pi^2\sigma}\int d\omega d\omega' \underbrace{\int dv e^{\frac{i}{2\sigma}(\omega-\omega')v}}_{2\pi\delta\left(\frac{1}{2\sigma}(\omega-\omega')\right)=4\pi\sigma\delta(\omega-\omega')} \int du\tilde{\chi}^*(\omega)\chi(\omega')e^{\frac{i}{2\sigma}(\omega+\omega')u}W(u)e^{i\Omega u} \\
&= \frac{1}{2\pi}\int du d\omega|\tilde{\chi}(\omega)|^2 W(u)e^{i(\Omega+\omega/\sigma)u} \\
&= \frac{1}{2\pi}\int d\omega|\tilde{\chi}(\omega)|^2\tilde{W}(\Omega+\omega/\sigma)\,.
\end{aligned} \tag{2.29}$$

Here, in the first line we have used a magic formula for δ functions:

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}\,, \tag{2.30}$$

where x_i are the roots of $f(x)$.

Waiting for a long time corresponds to large σ. More precisely, if $\tilde{\chi}(\omega)$ decays fast enough ('measure' $\tilde{\chi}(\omega)$ strongly supported on a s scale $1/\sigma$; we need 'adiabatic' (smooth enough switching – not nervous experimentalists)) then we find

$$\lim_{\sigma\to\infty,\ \text{adiab}}\mathcal{F}(\Omega,\sigma) = \frac{1}{2\pi}\int d\omega|\tilde{\chi}(\omega)|^2\tilde{W}(\Omega) = \frac{\tilde{W}(\Omega)}{2\pi}\int d\omega|\tilde{\chi}(\omega)|^2 = \tilde{W}(\Omega)\,. \tag{2.31}$$

That is, a detector acquires information about the Wightman function, after interacting for a long time.

- Response of the detector. Putting things together, the excitation/de-excitation ratio reads

$$R(\Omega,\sigma)=\frac{P_{\mathrm{ex}}(\Omega,\sigma)}{P_{\mathrm{deex}}(\Omega,\sigma)}=\frac{P_{\mathrm{ex}}(\Omega,\sigma)}{P_{\mathrm{ex}}(-\Omega,\sigma)}=\frac{\mathcal{F}(\Omega,\sigma)}{\mathcal{F}(-\Omega,\sigma)}\,. \tag{2.32}$$

We thus have, using the detailed balance condition (2.21)

$$\boxed{\lim_{\sigma\to\infty,\ \mathrm{adiab}} R(\Omega,\sigma)=\frac{\tilde{W}(\Omega)}{\tilde{W}(-\Omega)}=e^{-\beta\Omega}\,.} \tag{2.33}$$

Thus, a detector that interacts with a KMS state (w.r.t. its proper time) of temperature $T_{\mathrm{KMS}}=1/\beta$, as long as it is switched on carefully, thermalizes (catches Boltzmannian population) after intercating with the field for a long time! Moreover, the detector acquires the same temperature as the field – this is the experimental definition of thermality

$$T_{\mathrm{KMS}}=T\,. \tag{2.34}$$

(C.f. zero law of TDs.)

- Intermezzo: Some properties of Wightman functions. First, we may write:

$$\begin{aligned}
W_\rho(\tau,\tau') &= \mathrm{Tr}\Big(\rho\hat{\phi}(\tau)\hat{\phi}(\tau')\Big)\\
&= \frac{1}{2}\mathrm{Tr}\Big(\rho(\hat{\phi}(\tau)\hat{\phi}(\tau')-\hat{\phi}(\tau')\hat{\phi}(\tau))\Big)+\frac{1}{2}\mathrm{Tr}\Big(\rho(\hat{\phi}(\tau)\hat{\phi}(\tau')+\hat{\phi}(\tau')\hat{\phi}(\tau))\Big)\\
&= \underbrace{\frac{1}{2}\langle[\phi(\tau),\phi(\tau')]\rangle_\rho}_{i\mathrm{Im}(W_\rho(\tau,\tau'))}+\underbrace{\frac{1}{2}\langle\{\phi(\tau),\phi(\tau'\}\rangle_\rho}_{\mathrm{Re}(W_\rho(\tau,\tau'))}\,.
\end{aligned} \tag{2.35}$$

Since $[\phi(\tau),\phi(\tau'] \propto z1$, the first term, giving $i\mathrm{Im}\big(W(\tau,\tau')\big)$ is state independent; it is the second term that depends on ρ.

Second, we have

$$W^*_\rho(\tau,\tau')=\Big(\mathrm{Tr}(\rho\phi(\tau)\phi(\tau')\Big)^*=\mathrm{Tr}(\phi(\tau')\phi(\tau)\rho)=\mathrm{Tr}(\rho\phi(\tau')\phi(\tau))=W_\rho(\tau',\tau)\,. \tag{2.36}$$

- Consider now the commutator $C(\tau,\tau')$:

$$C(\tau,\tau')=\langle[\phi(\tau),\phi(\tau')]\rangle=2i\mathrm{Im}W(\tau,\tau')\,. \tag{2.37}$$

If W stationary, so is the commutator: $C(\tau,\tau')=C(\Delta\tau)$ (real and imaginary parts do not talk to each other). Moreover,

$$\begin{aligned}
\tilde{C}(\omega) &= \int_{-\infty}^{\infty}d\Delta\tau C(\Delta\tau)e^{i\omega\Delta\tau}=\int_{-\infty}^{\infty}d\Delta\tau\Big(W(\Delta\tau)-\underbrace{W^*(\Delta\tau)}_{W(-\Delta\tau)}\Big)e^{i\omega\Delta\tau}\\
&= \tilde{W}(\omega)-\tilde{W}(-\omega)\,.
\end{aligned} \tag{2.38}$$

Thus, if the state is KMS, we can use the 'detailed balance' (2.21), to obtain

$$\tilde{C}(\omega) = \tilde{W}(\omega) - e^{\beta\omega}\tilde{W}(\omega)\,, \tag{2.39}$$

or by re-arranging:

$$\boxed{\tilde{W}(\omega,\beta) = -\tilde{C}(\omega,\beta)P(\omega,\beta)\,, \quad P(\omega,\beta) = \frac{1}{e^{\beta\omega}-1}\,.} \tag{2.40}$$

Here, $P(\omega,\beta)$ is the Planck's factor. This in particular means that, if the state is KMS, the FT of the Wightman is completely determined by the commutator (commutator contains both real and imaginary parts).

Note that $\tilde{C}(\omega,\beta)$ depends on the pull back to detectors trajectory; this introduces the dependence on β! (Trajectory depends on acceleration, and thence on β!)

2.4 Is Unruh KMS?

- If we can show that $|0\rangle_M$ is KMS w.r.t. ∂_τ of accelerated detector, Unruh effect is as physical as QFT! Let us prove that!
- Wightman function. The trajectory of constant acceleration is given by

$$t(\tau) = \frac{1}{a}\sinh(a\tau)\,, \quad x^1(\tau) = \frac{1}{a}\Big(\cosh(a\tau)-1\Big)\,, \quad x^2 = x^3 = \cdots = x^d = 0\,. \tag{2.41}$$

The Wightman function for the vacuum state in Minkowski is given by

$$\begin{aligned} W(\tau,\tau') &= \langle 0|_M\phi(\tau)\phi(\tau')|0\rangle_M \\ &= \int \frac{d^dk e^{-\epsilon|\vec{k}|}}{2(2\pi)^d|\vec{k}|} e^{-i\Big(|\vec{k}|\big(t(\tau)-t(\tau')\big)-\vec{k}\cdot\big(\vec{x}(\tau)-\vec{x}(\tau')\big)\Big)} \\ &= \int \frac{d^{d+1}k}{2(2\pi)^d}\Theta(k^0)\delta(k^2)e^{ik\cdot(x-x')}\,. \end{aligned} \tag{2.42}$$

where in the second line we included the regularization ϵ (we shall omit it from now on), and in the third line we have written the covariant expression.

Since the trajectory of the detector is timelike, having $(x-x')^2<0$, we can define $\Delta = \sqrt{-(x-x')^2}$. Then

$$\begin{aligned} k\cdot(x-x') &= k_0\big(t(\tau)-t(\tau')\big) + k_1\big(x^1(\tau)-x^1(\tau')\big) \\ &= \bar{k}_0\Delta\mathrm{sgn}\big(t(\tau)-t(\tau')\big)\,, \end{aligned} \tag{2.43}$$

where

$$\begin{aligned} \bar{k}^0 &= \Delta^{-1}\Big(k^0\big(t(\tau)-t(\tau')\big) + k^1\big(x^1(\tau)-x^1(\tau')\big)\Big)\mathrm{sgn}(t-t')\,, \\ \bar{k}^1 &= \Delta^{-1}\Big(k^1\big(t(\tau)-t(\tau')\big) - k^0\big(x^1(\tau)-x^1(\tau')\big)\Big)\mathrm{sgn}(t-t')\,, \\ \bar{k}^2 &= k^2\ \ldots\,. \end{aligned} \tag{2.44}$$

This is a 'change of coordinates' that corresponds to a Lorentz transformation that aligns k and x. Here, the $\mathrm{sgn}(t-t')$ is important for preserving the volume under ortochronous Lorentz transformations.

We then have (Jacobian is equal to one)

$$d^{d+1}\bar{k}\Theta(\bar{k}^0)\delta(\bar{k}^2) = d^{d+1}k\Theta(k^0)\delta(k^2)\,. \tag{2.45}$$

Thus,

$$\begin{aligned} W(\tau,\tau') &= \int \frac{d^{d+1}\bar{k}}{2(2\pi)^d}\Theta(\bar{k}^0)\delta(\bar{k}^2)e^{-i\bar{k}^0\Delta\mathrm{sgn}\big(t(\tau-t(\tau')\big)} \\ &= \frac{(4\pi)^{-d/2}}{\Gamma(d/2)}\int_0^\infty d|\vec{k}||\vec{k}|^{d-2}e^{-i|\vec{k}|\Delta\mathrm{sgn}(t(\tau)-t(\tau'))} \\ &= \frac{\Gamma\Big(\frac{d-1}{2}\Big)}{4\pi^{(d+1)/2}}\Big(\Delta\mathrm{sgn}\big(t(\tau)-t(\tau')\big)\Big)^{1-d}\,. \end{aligned} \tag{2.46}$$

Here, in the second line we have integrated over $\bar{k}^0$ and over the angles; the volume of an n-dimensional sphere is given by

$$\omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\Big(\frac{n+1}{2}\Big)}\,, \tag{2.47}$$

and the results is valid for any $d>1$. Moreover, since for our accelerated trajectory we have

$$\Delta\mathrm{sgn}\big(t(\tau)-t(\tau')\big) = \frac{2}{a}\sinh\big(\frac{a}{2}(\tau-\tau')\big)\,, \tag{2.48}$$

upon using the trigonometric identities:

$$\sinh x - \sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}\,,\quad \cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}\,. \tag{2.49}$$

So our Wightman function is

$$\boxed{W(\Delta\tau) = \frac{\Gamma\Big(\frac{d-1}{2}\Big)}{4\pi^{(d+1)/2}}\Big[\frac{2}{a}\sinh\big(\frac{a}{2}(\tau-\tau')\big)\Big]^{1-d}\,.} \tag{2.50}$$

Obviously, this is stationary. Moreover, using that $\sinh(x+i\pi) = -\sinh x = \sinh(-x)$, we find

$$W(\Delta\tau + i\underbrace{\frac{2\pi}{a}}_{\beta}) = -W(\Delta\tau) = W(-\Delta\tau)\,. \tag{2.51}$$

So indeed, this Wightman corresponds to a KMS state with $T = a/(2\pi)$.

- Conclusion. A constantly accelerated detector coupled to the vacuum thermalizes to a temperature $T = a/(2\pi)$.
- Remark. Thermal state for inertial observer will not be a thermal state for an accelerated observer! (Initial state will disappear – seen only as a bump transient. This will decay and we will eventually see the normal Unruh temperature, see [].

2.5 Circular Unruh effect

– Circular setup. Following [5] let us now ask what would happen if instead of 'linear Unruh effect' we considered a circular motion, modelling Unruh with *uniform centrifugal acceleration.* As we shall see the corresponding response is not exactly thermal but may be more experimentally viable.

The question is: how does the Minkowski vacuum $|0\rangle_M$ look like to an observer in circular trajectory:

$$x(\tau) = \Big(\gamma\tau, R\cos(\gamma O\tau), R\sin(\gamma O\tau), 0, \dots\Big)\,, \tag{2.52}$$

where R is the radius of the orbit, $O = v/R$ its angular velocity and v is the orbital speed – both with respect to the Minkowksi time t, and $\gamma = 1/\sqrt{1-v^2}$ si the Lorentz factor. The motion is characterized by the proper acceleration

$$a = \sqrt{\ddot{x}^\mu \ddot{x}_\mu} = RO^2\gamma^2 = \frac{v^2}{1-v^2}\frac{1}{R} = \frac{v^2\gamma^2}{R}\,; \tag{2.53}$$

we shall adopt R and v as a pair of independent parameters specifying the trajectory.

The experimental advantage is obvious: i) the system remains within a finite-size laboratory for an arbitrary long time and ii) the Lorentz γ-factor remains constant over the worldline.

Note: if we simply used the (linear) Unruh temperature formula, we would get the following temperature:

$$T_{\rm lin} = \frac{a}{2\pi} = \frac{v^2\gamma^2}{2\pi R}\,. \tag{2.54}$$

As we shall see, the situation is not so simple.

– Both, the state and the motion are stationary, i.e. we have $W(\tau,\tau') = W(\Delta\tau)$. This means that after interacting with the state for a long time, the detector's response is still given by

$$\lim_{\sigma\to\infty,\text{adiab}} \mathcal{F}(\Omega,\sigma) = \tilde{W}(\Omega)\,. \tag{2.55}$$

(Go through your notes to see that to derive this all we need is stationarity.)

– Let us now define a phenomenological (operational) temperature as

$$R = \frac{\tilde{W}(\Omega)}{\tilde{W}(-\Omega)} = e^{-\beta\Omega}\,. \tag{2.56}$$

For a conventional thermal (KMS) state such $T_{\text{circ}} = 1/\beta$ is independent of Ω (reflecting the detailed balance of KMS states). However, in the case of the circular trajectory, we find a dependence on Ω, that is, the effect depends on the energy scale we probe it – different detectors will see different temperatures. (Apart from acceleration, T_{circ} also depends on v and Ω.)

– As shown in the previous section in the specific case, the pullback of the Wightman function in a $(d+1)$-dimensional Minkowski space to an arbitary trajectory is given by

$$\boxed{W(\tau,\tau') = \frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{(d+1)/2}}\frac{1}{[(x(\tau)-x(\tau'))^2]^{(d-1)/2}}\,.} \tag{2.57}$$

Interestingly, this is not KMS for circular motion.

– In fact, one can show that in $(3+1)$ dimensions, one has

$$\tilde{W}(\Omega) = \underbrace{-\frac{\Omega}{2\pi}\Theta(-\Omega)}_{\tilde{W}^{\text{in}}(\Omega)} + \frac{1}{4\pi^2\gamma v R}\int_0^\infty dz\cos\left(\frac{2\Omega R z}{\gamma v}\right)\left(\frac{\gamma^2 v^2}{z^2} - \frac{1}{z^2/v^2 - \sin^2 z}\right), \tag{2.58}$$

where $\tilde{W}^{\text{in}}(\Omega)$ is the inertial motion response function. (Note that just for inertial motion $R = 0$ which corresponds to $\beta \to \infty$ and thence $T = 0$.)

In particular, in the large gap, $\Omega \to \infty$ and ultrarelativistic, $v \to 1$ limits, one finds that

$$\frac{T_{\text{circ}}}{T_{\text{lin}}} = \frac{\pi}{\sqrt{3}} \sim 1.8\,. \tag{2.59}$$

– Analogue spacetime implementation. Condensed matter systems (Bose–Einstein condensates or superfluid helium) provide an effective Minkowski geometry, where the speed of light is replaced by the speed of sound (phonon-type excitations), giving rise to sonic limit $v = 1$. Then one has the following 'dictionary':

$$\hat{\Omega} = \Omega/\gamma\,, \quad \hat{T} = T/\gamma\,, \quad \hat{a} = a/\gamma^2\,, \tag{2.60}$$

where $\hat{\Omega}$ is the energy gap w.r.t. the laboratory time t (Minkowski time in the effective Minkowski metric). Then one finds that in the near-sonic limit, $v \to 1$, one has (in 3+1 dimensions)

$$\hat{T}_{\text{circ}} \approx \frac{\gamma\hat{a}}{4\sqrt{3}}\,. \tag{2.61}$$

So the effect is enhanced close to the sonic limit.

– Other setups may for example include inertial motion in de Sitter spacetime [6], see e.g. [7] for a proposal for an experimental simulation in an analogue spacetime. In this case, we have a de Sitter spacetime, which in global coordinates takes the spherically symmetric form with

$$f = 1 - \frac{r^2}{\ell^2}\,, \tag{2.62}$$

where ℓ is the cosmological radius, related to the cosmological constant as $\Lambda = 3/\ell^2$. The spacetime admits a cosmological horizon, given by $f(r_c) = 0$, that is $r_c = \ell$. We then have

$$T_{\mathrm{dS}} = \frac{|f'(r_c)|}{4\pi} = \frac{1}{2\pi\ell^2}\,. \tag{2.63}$$

2.6 Hawking effect

– Stellar collapse. Let us now consider a black hole spacetime, formed from a stellar collapse. After everything settles down, we may describe it by a (1-sided) Schwarzschild metric, which we write in the Edington–Finkelstein ingoing coordinates (that cover regions I and II of the Kruskal diagram):

$$ds^2 = -dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 = -f dv^2 + 2dvdr + r^2 d\Omega^2\,. \tag{2.64}$$

with $f = 1 - \frac{2m}{r}$, and we have defined

$$u = t - r^*\,, \quad v = t + r^*\,, \quad r^* = \int \frac{dr}{f} = r + 2m\log\Big|\frac{r}{2m} - 1\Big|\,. \tag{2.65}$$

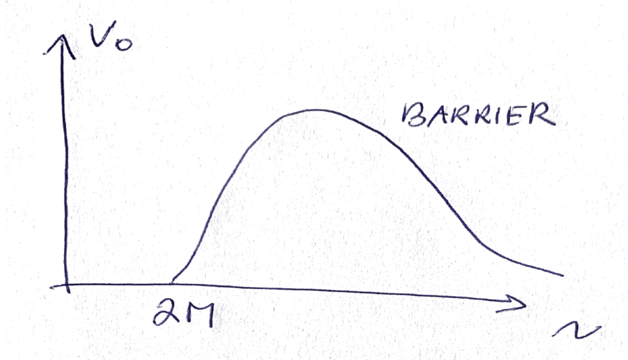
– Expanding further the scalar wave equation $\nabla^2\phi = 0$ in terms of the spherical harmonics,

$$\phi = \frac{1}{r}\psi(r,t)Y_{lm}(\theta,\varphi)\,, \tag{2.66}$$

we arrive at

$$\underbrace{\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial r_*^2}}_{\nabla^2_{(2)}\psi} + \underbrace{f\Big(\frac{l(l+1)}{r^2} + \frac{2m}{r^3}\Big)}_{V_l(r)}\psi = 0\,. \tag{2.67}$$

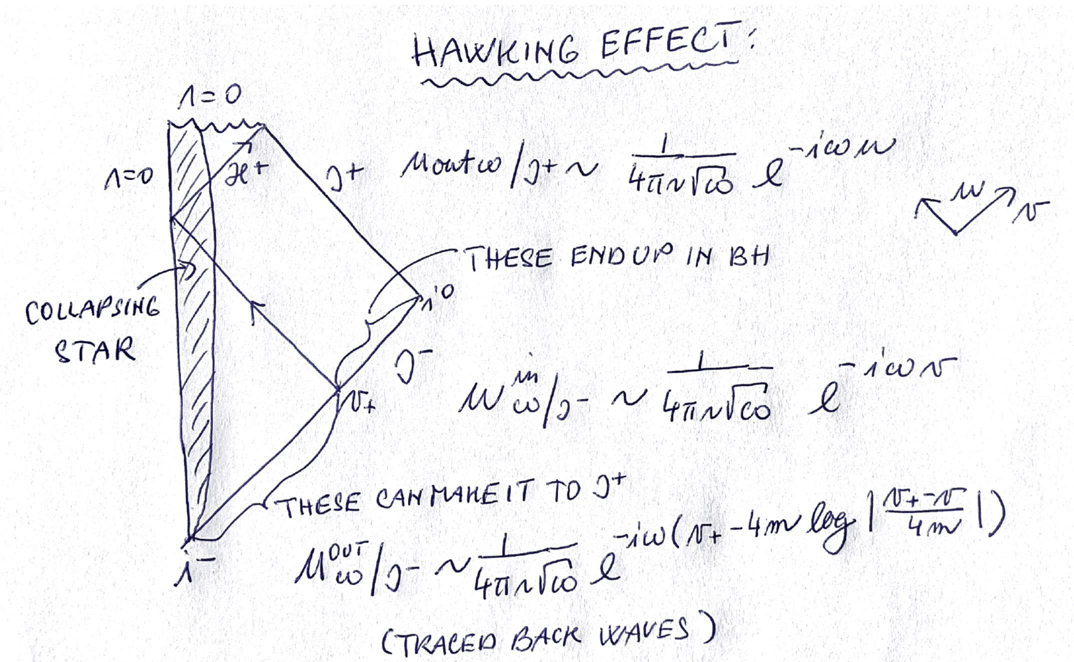
Note that $V_l(r)$ vanishes close to the horizon and thus has 'nothing to do' with the particle production close to the horizon – it represents a 'barrier' through which each mode has to propagate to reach infinity, e.g. we have:



– Focusing first on the '2-dimensional case' (neglecting for the moment V_l), we can expand in modes, as follows. On $\mathcal{I}^-$ we have the natural in modes:

$$\mathcal{I}^- : \quad u_\omega^{\rm in} \sim \frac{f_\omega^{\rm in}}{r} \sim \frac{1}{4\pi r\sqrt{\omega}} e^{-i\omega v} \,. \tag{2.68}$$

The modes with $v < v_+$ will make it to the future null infinity, whereas those with $v > v_+$ will end up in the black hole.



Consider next the modes that take the standard form on $\mathcal{I}^+$:

$$\mathcal{I}^+ : \quad u_\omega^{\rm out} \propto \frac{1}{4\pi\sqrt{\omega}} e^{-i\omega u} \,. \tag{2.69}$$

These correspond to outgoing modes close to the horizon. When traced back to the past null infinity (upon using the geometric optics approximation, see e.g. [8]) they become:

$$\mathcal{I}^- : \quad u_\omega^{\rm out} \sim \frac{1}{4\pi r\sqrt{\omega}} \exp\Big(-i\omega\big(v_+ - 4m \log\frac{|v_+ - v|}{4m}\big)\Big)\Theta(v_+ - v) \,. \tag{2.70}$$

[Note the exponential redshift due to collapse of the star, resulting in the log term in the exponential.] Similar to the Rindler case, the above modes do not form a complete set, and we need to complete the basis by including modes that go through the horizon. When traced back to $\mathcal{I}^-$ they take the following form:

$$u_\omega^{\text{hor}} \sim \frac{1}{4\pi r\sqrt{\omega}} \exp\Big(i\omega\big(v_+ - 4m \log \frac{|v_+ - v|}{4m}\big)\Big)\Theta(v - v_+)\,. \tag{2.71}$$

Any solution can then be expended either in the basis $\{u^{\text{in}}\}$ or in $\{u^{\text{out}}, u^{\text{hor}}\}$. Similar to the Rindler case, these are not unitary equivalent. Namely, one finds

$$|0\rangle_{\text{in}} = \prod_\omega \frac{1}{\cosh r_\omega} \sum_n (\tanh r_\omega)^n |n_\omega\rangle_{\text{hor}} |n_\omega\rangle_{\text{out}}\,, \tag{2.72}$$

where

$$\tanh r_\omega = \exp\Big(-\frac{\pi\omega}{\kappa}\Big)\,. \tag{2.73}$$

Thus, $|0\rangle_{\text{in}}$ is entangled (vacuum in the past evolves into two mode squeezed state between infalling and outgoing modes in the future).

Here, κ is the surface gravity of the horizon, which is a Killing horizon of the Killing vector field $\xi = \partial_t = \partial_v$, generating the horizon, defined by

$$\xi^c \nabla_c \xi^a \Big|_{r=r_+} = \kappa \xi^a \Big|_{r=r_+}\,. \tag{2.74}$$

Using the ingoing coordinates, we easily find

$$\kappa = \frac{|f'(r_+)|}{2} = \frac{1}{4m}\,. \tag{2.75}$$

– What do we see if we look at the black hole? We perceive the density matrix

$$\rho_{\text{out}} = \text{Tr}_{\text{hor}}\Big(|0\rangle_{\text{in}}\langle 0|_{\text{in}}\Big) = \otimes_\omega \frac{1}{\cosh^2 r_\omega} \sum_n \tanh^{2n} r_\omega |n_\omega\rangle_{\text{out}} \langle n_\omega|_{\text{out}} \tag{2.76}$$

We thus see outgoing radiation:

$$\langle \hat{n}_\omega \rangle = \text{Tr}(\hat{n}_\omega \rho_{\text{out}}) = \frac{1}{e^{\frac{\hbar\omega}{k_B T_H}} - 1}\,, \tag{2.77}$$

where

$$\boxed{T_H = \frac{1}{8\pi G_N m}\frac{\hbar c^3}{k_B} = \frac{\kappa}{2\pi}\frac{\hbar c^3}{k_B G_N}} \tag{2.78}$$

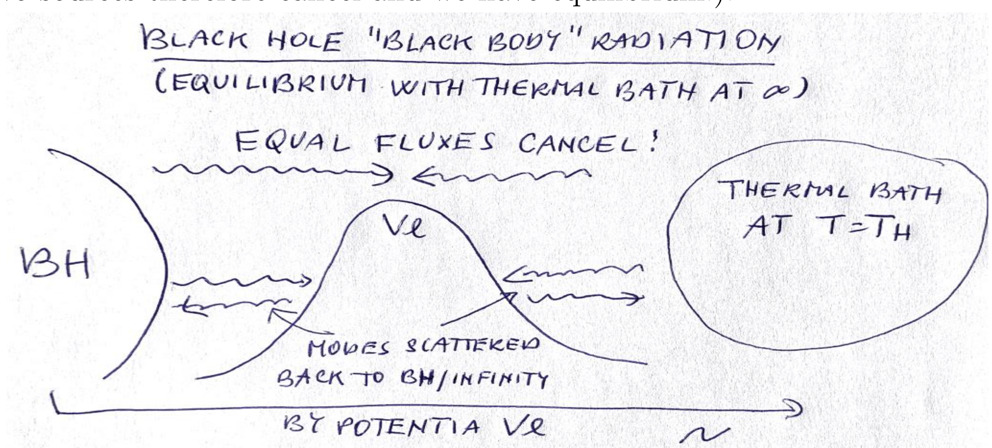
is the Hawking temperature at infinity – Hawking 1975 [9].

– One can show that the UdW detector at infinity is KMS and would thermalize to T_H. For other orbits in BH spacetimes (e.g. observers at finite r) the state is not necessarily KMS!

– Grey body factors. So far we have neglected the potential V_l through which each mode escaping the black hole has to propagate. This decreases the intensity of the wave and changes the resulting spectrum by a *greybody spectrum* $\Gamma_l(\omega) < 1$. The flux of particles observed at infinity thus reads:

$$\langle n_\omega \rangle = \frac{\Gamma_l(\omega)}{e^{\omega/T_H} - 1} \, . \tag{2.79}$$

Nevertheless, such a flux remains 'thermal' (black body) in the following sense. It is in thermal equilibrium with the thermal bath at infinity at temperature T_H. (The part of thermal radiation originating from thermal bath that gets reflected by V_l back to infinity equals the part of Hawking radiation that gets reflected by V_l back to black hole. The 'surviving' fluxes from the two sources therefore cancel and we have equilibrium.):



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