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Quantum information in curved spacetime

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Abstract

This is a study text for the "Quantum information in curved spacetime" course taught at Charles University in 2023/24. The text builds on similar courses delivered by Eduardo Martin-Martinez at the Perimeter Institute/University of Waterloo, as well as stems from a number of recent papers.

Basically, we will be touching on some topics studied by Relativistic Quantum Information (RQI), which is a new discipline that has emerged around 2010, as an attempt to merge three fields: *general relativity (GR), quantum field theory (QFT), and quantum information (QI).* The main idea is to incorporate the relativistic description into QI processing and to study structure of spacetime and nature of gravity from QI perspective.

For example, we would like to tackle the following problems:

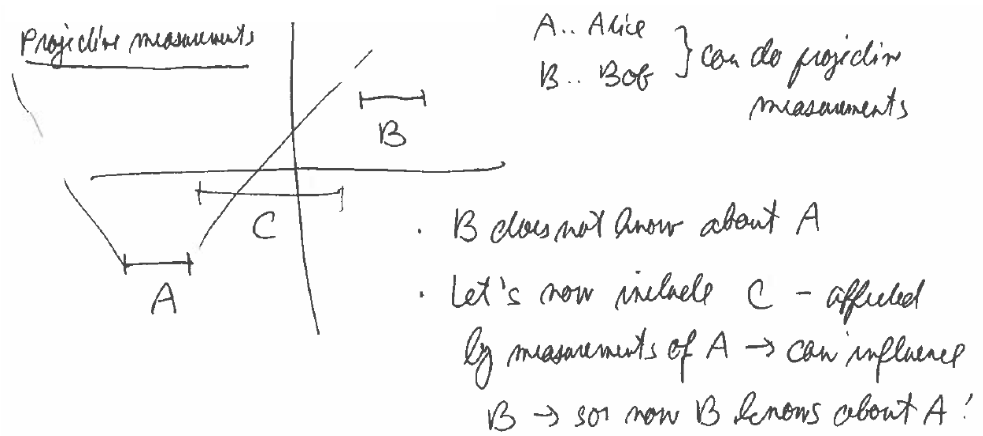
- Early Universe Cosmology – how much info we can get about early Universe?
- Black hole (BH) information loss – do BHs destroy information?
- QFT vacuum information content about given spacetime. Can we use quantum fields for spacetime reconstruction, or even to recast classical Einstein equations in QFT language?
- Thermalization, Unruh effect, ...
- Quantum communication and energy teleportation.
- Spacetime engineering – can we create states violating energy conditions, such as warp drives, wormhole, ...? (On average, QFT can violate energy conditions.)
- Is gravity really quantized? What is a superposition of spacetimes, and can large masses be entangled?
- Can we make a direct connection with available experiments?

Contents

Chapter 1: Measuring quantum fields: Particle detectors

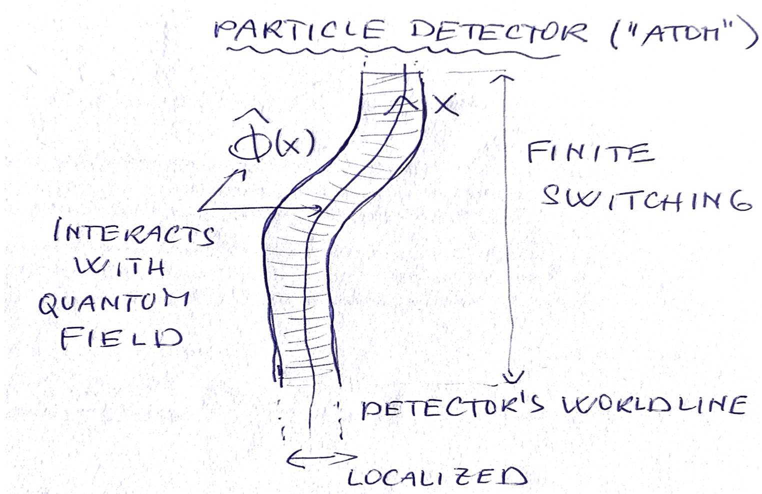
1.1 Motivation

- Projective measurements (at a given instant of time) are not a satisfactory description, as can be seen from the following picture:

  It also treats the detector at a different level than the quantum system.

- Instead use particle detectors: "couple 1st quantized system to the full 2nd quantized system":

  What is a particle detector? It better be i) localized (in time and space) quantum system that is ii) coupled to a quantum field, and is iii) easy to measure (to do projective measurements on), that is, it has a 'clicking quality' and is non-relativistic (1st quantized).

It seems like hydrogen atom could be a good model.

Particle-detector tautology. "A particle is what the particle detector measures; a particle detector is a device that detects particles."

- What do people do in Quantum optics? People typically use the Jaynes–Cummings (J-C) model. This is a '2-level atom' with two energy states $|e\rangle, |g\rangle$;

$$|\psi\rangle = \psi_e|e\rangle + \psi_g|g\rangle = \begin{pmatrix} \psi_e \\ \psi_g \end{pmatrix}, \tag{1.1}$$

separated by energy gap Ω, which couples to a 'mode' of the EM field (described by the harmonic oscillator) via the following interaction Hamiltonian:

$$\hat{H}_I = \lambda\Big(\sigma^+ a e^{i(\Omega-\omega)t} + \sigma^- a^+ e^{-i(\Omega-\omega)t}\Big), \tag{1.2}$$

where $\sigma^\pm$ are the $SU(2)$ ladder operators obeying[1]

$$\sigma^+|g\rangle = |e\rangle, \quad \sigma^-|e\rangle = |g\rangle, \tag{1.7}$$

that is, $\sigma^+ = |e\rangle\langle g|, \sigma^- = |g\rangle\langle e|$, and ω is the photon's frequency. Such a model has the following intuitive meaning: "annihilation of a photon excites the detector, whereas creation of a photon de-excites it". We would also typically expect $\Omega \approx \omega$ ("conservation of energy").

- The light matter interaction from first principles. Consider a hydrogen atom with an electron

$$\hat{H}_0 = \frac{\hat{\vec{p}}^2}{2m} + eV(\hat{x}), \tag{1.8}$$

where m is the effective electron's mass, and the electron can couple to an electromagnetic field. To this purpose we can perform the multipole expansion and

[1] In terms of the standard Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.3}$$

which obey

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma_k, \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij}1 \tag{1.4}$$

we have

$$\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y), \quad \sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y). \tag{1.5}$$

This also implies that

$$[\sigma^+, \sigma^-] = \sigma^z, \quad [\sigma^z, \sigma^\pm] = \pm 2\sigma^\pm. \tag{1.6}$$

restrict to the dipole approximation. In this approximation the interaction Hamiltonian reads

$$\boxed{\hat{H}_I = e\hat{\vec{x}} \cdot \vec{E}(\hat{\vec{x}})\,.} \tag{1.9}$$

Here, field is not yet quantized. Let us see what do we get from this 'definition'.

- Expanding in matrix elements (energy states of the unperturbed Hamiltonian), we have

$$\hat{H}_I = e\hat{\vec{x}} \cdot \vec{E}(\hat{\vec{x}}) = e\sum_{i,j} \langle j|\hat{\vec{x}} \cdot \vec{E}(\hat{\vec{x}})|i\rangle e^{i\Omega_{ij}t}|j\rangle\langle i|\,. \tag{1.10}$$

Here, $e^{i\Omega_{ij}t}$, where $\Omega_{ij} = \Omega_j - \Omega_i$ is the energy between states i and j, comes from the interaction picture.[2] We now insert the identity $\int dx|x\rangle\langle x|$ and employ the standard hydrogen atom wave functions $\psi_i(x) = \langle x|i\rangle$. Thus we find

$$\begin{aligned} \hat{H}_I &= e\int d^3x d^3x' \sum_{i,j} \langle j|x\rangle\langle x|\hat{\vec{x}} \cdot \vec{E}(\hat{\vec{x}})|x'\rangle\langle x'|i\rangle e^{i\Omega_{ij}t}|j\rangle\langle i| \\ &= e\sum_{i,j} \int d^3x \psi_j^*(x)\vec{x}\psi_i(x) \cdot \vec{E}(\vec{x}) e^{i\Omega_{ij}t}|j\rangle\langle i|\,. \end{aligned} \tag{1.14}$$

Of course, many of the matrix elements are (to the lowest order in perturbation theory) zero, using for example selection rules. In what follows we concentrate on a 2 level model, with ground state $|g\rangle$ and excited state $|e\rangle$, separated by energy gap Ω. Let's also denote the ladder operators

$$\sigma^+ = |e\rangle\langle g|\,, \quad \sigma^- = |g\rangle\langle e|\,, \tag{1.15}$$

and $\vec{F}(\vec{x}) = \psi_e^*(\vec{x})\vec{x}\psi_g(\vec{x})$ the corresponding 'smearing function'. With this we have

$$\hat{H}_I = \int d^3x e\Big(\vec{F}(\vec{x})e^{i\Omega t}\sigma^+ + \vec{F}^*(\vec{x})e^{-i\Omega t}\sigma^-\Big) \cdot \vec{E}(\vec{x}) \equiv \int d^3x \hat{\vec{d}}(\vec{x}) \cdot \vec{E}(\vec{x})\,, \tag{1.16}$$

where $\hat{\vec{d}}$ is the dipole operator. We can now proceed and canonically quantize the EM field $\vec{E} \to \hat{\vec{E}}$. However, if we are not interested in exchange of angular momentum, we can consider a simplified scalar model. This is known as:

[2] Splitting the total Hamiltonian, into the 'basic' and 'interaction' parts:

$$H = H_0 + H_I\,, \tag{1.11}$$

the interaction picture operators and states are related to Schrodinger picture operators and states as follows (setting $\hbar = 1$):

$$A_I = e^{iH_0t}A_S e^{-iH_0t}\,, \quad |\psi_I\rangle = e^{iH_0t}|\psi_S\rangle\,. \tag{1.12}$$

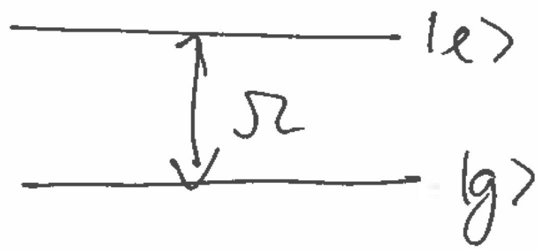
Such operators and states evolve as follows:

$$i\frac{dA_I}{dt} = [A_I, H_0]\,, \quad i\frac{d\psi_I}{dt} = H_I\psi_I\,. \tag{1.13}$$

Of course, we also have $\langle A_I\rangle = \langle A_S\rangle = \mathrm{Tr}(\rho_I A_I)\,.$

1.2 Unruh De Witt (UdW) detector

- Unruh-De Witt (UdW) detector (Unruh 1976 [1], De Witt 1979 [2]). This is a 'scalar version' of the above, namely



$$\boxed{H_I = \lambda\chi(t)\int d^3x\,\hat{\mu}(t,\vec{x})\hat{\phi}(t,\vec{x})\,,} \tag{1.17}$$

where $\hat{\mu}$ is the monopole operator (with smearing function $F(\vec{x})$):

$$\hat{\mu}(t,\vec{x}) = F(\vec{x})\underbrace{\left(\sigma^+e^{i\Omega t}+\sigma^-e^{-i\Omega t}\right)}_{\hat{m}(t)}\,, \tag{1.18}$$

and we have also included the switching function $\chi(t)$ for the detector, governing the duration for which the detector is switched on.

Here, the massless scalar field ϕ (in $d=(n+1)$ dimensions) is quantized:

$$\hat{\phi}(t,\vec{x}) = \int\frac{d^nk}{\sqrt{2(2\pi)^n|\vec{k}|}}\left(a^+_{\vec{k}}e^{-ik\cdot x}+a_{\vec{k}}e^{ik\cdot x}\right). \tag{1.19}$$

- Note that there are extra terms in the UdW detector when compared to the J-C detector. Namely, we schematically have

$$\begin{aligned} &\text{J-C}: && \sigma^+a+\sigma^-a^+\,, \\ &\text{UdW}: && \underbrace{(\sigma^++\sigma^-)}_{\sigma^x}\underbrace{(a+a^+)}_{\hat{x}} \sim \sigma^+a+\sigma^-a^+ + \underbrace{\sigma^+a^++\sigma^-a}_{\text{counter-rotating terms}} \end{aligned} \tag{1.20}$$

The difference is thus the presence of the "counter-rotating terms". Some say that such terms 'do not conserve energy':

$$|e\rangle|0\rangle \xrightarrow{\sigma^-a^-} |g\rangle|1\rangle \qquad \text{OK for } \omega\simeq\Omega$$
$$\hbar\Omega \qquad\qquad \hbar\omega$$

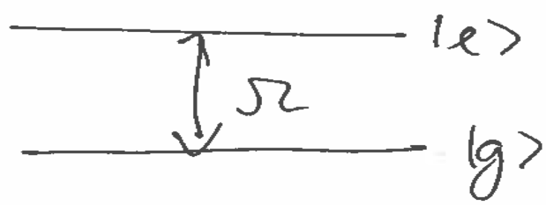
but: $$|g\rangle|0\rangle \xrightarrow{\sigma^+a^+} |e\rangle|1\rangle \qquad \text{DOES NOT CONSERVE ENERGY?}$$
$$\varnothing \qquad\qquad \hbar(\Omega+\omega)$$

However, that is not true for two reasons. First, the interaction Hamiltonian is accompanied by Hamiltonians of detector and field:

$$H_d = \Omega \underbrace{\sigma^+\sigma^-}_{\sim\sigma^z}, \quad H_{\text{field}} = \omega \underbrace{a^+a}_{\hat{n}} \,. \tag{1.21}$$

However, since σ^z and σ^x, and $\hat{n}$ and $\hat{x}$ do not commute, eigenstates of the free Hamiltonian are not the eigenvalues of the total Hamiltonian; so assigning energy to this is false! Second, we do not perform a measurement and the state is that of superposition of all possibilities.

- Vacuum excitation probability of a UdW detector. Let us now calculate the vacuum excitation probability:



$$P^{|0\rangle}_{|g\rangle\to|e\rangle}(\Omega)$$

Once we have calculated that, the opposite process, probability of de-excitation is calculated as:

$$P^{|0\rangle}_{|e\rangle\to|g\rangle}(\Omega) = P^{|0\rangle}_{|g\rangle\to|e\rangle}(-\Omega) \,. \tag{1.22}$$

We have

$$P^{|0\rangle}_{|g\rangle\to|e\rangle}(\Omega) = \sum_{\text{out}} |\langle \text{out}, e|U|g,0\rangle|^2 = \sum_{\text{out}} \langle 0, g|U^+|e, \text{out}\rangle\langle \text{out}, e|U|g,0\rangle \,, \tag{1.23}$$

where we have summed over all final states of the field $|\text{out}\rangle$, and the evolution operator U is given by the time ordered exponential:

$$U = T \exp\Big(-i \int_{-\infty}^{\infty} dt H_I(t)\Big) \,, \tag{1.24}$$

where, w.l.o.g., we can take the limits to be $(-\infty, +\infty)$, as the interaction Hamiltonian H_I already contains the finite switching function χ. To calculate U, we use the perturbation theory, namely Dyson's expansion:

$$U = 1 + U^{(1)} + U^{(2)} + O(\lambda^3) \,, \quad U^{(1)} = -i \int_{-\infty}^{\infty} dt H_I(t) \,. \tag{1.25}$$

Let us calculate the probability to the linear in λ order. Obviously, the first term does not contribute, and we have

$$P^{|0\rangle}_{|e\rangle\to|g\rangle}(\Omega) = \langle 0,g|U^{(1)+}|e\rangle \underbrace{\sum_{\text{out}}|\text{out}\rangle\langle\text{out}|}_{1}\langle e|U^{(1)}|g,0\rangle \tag{1.26}$$

$$= \lambda^2\int dtdt'\,\chi(t)\chi(t')\int d^nxd^nx'F(\vec{x})F(\vec{x}')\underbrace{\langle g|\hat{m}(t)|e\rangle\langle e|\hat{m}(t')|g\rangle}_{\exp\left(i\Omega(t'-t)\right)}W(t,\vec{x},t',\vec{x}')$$

where we for example used that $\langle g|\hat{m}(t)|e\rangle = \langle g|(\sigma^+e^{i\Omega t}+\sigma^-e^{-i\Omega t})|e\rangle = e^{-i\Omega t}$, and defined the Wightman function (2pt. function):

$$\boxed{W(t,\vec{x},t',\vec{x}') = \langle 0|\hat{\phi}(t,\vec{x})\hat{\phi}(t',\vec{x}')|0\rangle\,.} \tag{1.27}$$

That is, we have found that

$$\boxed{P^{|0\rangle}_{|e\rangle\to|g\rangle}(\Omega) = \lambda^2\int dtdt'\chi(t)\chi(t')\int d^nxd^nx'F(\vec{x})F(\vec{x}')W(t,\vec{x},t',\vec{x}')e^{i\Omega(t'-t)}\,.} \tag{1.28}$$

- To proceed further, let us use the flat space Wightman function:

$$W(t,\vec{x},t',\vec{x}') = \frac{i}{(2\pi)^{n+1}}\int d^{n+1}k\frac{e^{ik\cdot(x-x')}}{k^2} = \int\frac{d^nk}{2(2\pi)^n|\vec{k}|}e^{-i\left(|\vec{k}|(t-t')-\vec{k}\cdot(\vec{x}-\vec{x}')\right)}\,. \tag{1.29}$$

Then we can easily perform the integrals over $t,t',\vec{x},\vec{x}'$ – they simply yield the Fourier transform, e.g.[3]

$$\tilde{\chi}(\Omega+|\vec{k}|) = \int dt\chi(t)e^{-i\left(\Omega+|\vec{k}|\right)t}\,. \tag{1.31}$$

We thus recover

$$P^{|0\rangle}_{|g\rangle\to|e\rangle}(\Omega) = \lambda^2\int\frac{d^nk}{2(2\pi)^n|\vec{k}|}\left|\tilde{\chi}(\Omega+|\vec{k}|)\right|^2\left|\tilde{F}(\vec{k})\right|^2\,. \tag{1.32}$$

- Consider now 'long switching' (the detector is switched on forever). Then we have

$$\chi(t) = \text{const.} \quad\Rightarrow\quad \tilde{\chi}(\omega)\propto\delta(\omega)\,. \tag{1.33}$$

[3] Here we adopt the following convention for the Fourier transform:

$$\tilde{f}(\omega) = \int f(x)e^{-i\omega x}dx\,,\quad f(x) = \frac{1}{2\pi}\int\tilde{f}(\omega)e^{i\omega x}d\omega\,. \tag{1.30}$$

Since $\Omega + |\vec{k}| \neq 0$, we then find $P(\Omega) = 0$. This makes sense, while for small enough times we can get excitations ('borrowing energy from vacuum'), when the detector is switched on forever, the detector will not get excited. On the other hand, if our detector is localized in space and time, it will click!

Note also that the non-trivial contribution to P comes from the counter-rotating terms, namely $\sigma^+ a^+$ and $\sigma^- a$. If we adopted the co-rotating wave approximation (RWA) we would not see any excitations.

- Similarly, if we started with the excited detector, the probability of its de-excitation would be $P(-\Omega)$, which gives the condition

$$\Omega = |\vec{k}|\,, \tag{1.34}$$

that is, only 1 mode, with $|\vec{k}| = \Omega$, contributes. For long enough times, we can thus adopt a single mode approximation (SMA). Note also that in this case, the non-trivial contributions come from the co-rotating waves $\sigma^+ a$ and $\sigma^- a^+$.

- When we compare UdW to the quantum optics J-C model, we thus see that the latter adopts i) SMA approximation and ii) RWA approximation. Consequently, J-C will not get excited but will have spontaneous emission via single mode. These approximations are good for long enough times.

More precisely, let T be the support of $\chi(t)$. Then the following approximations are 'valid':

$$\begin{aligned} \text{SMA}: &\qquad T \gg \frac{1}{\Omega - |\vec{k}|}\,, \\ \text{RWA}: &\qquad T \gg \frac{1}{\Omega + |\vec{k}|}\,. \end{aligned} \tag{1.35}$$

We see that if SMA holds, so does RWA.

Often, it happens that $T \gg \Omega^{-1}$. Then both RWA and SMA are OK. While this is the case for optical cavities, for which the J-C model is sufficient, it may not be the case for the effects we shall study in this course – for this reason we are going to use the UdW detector.

1.3 Which frame?

- Which frame? Let us now write down the total Hamiltonian for our system. When doing so, we have two natural frames to use: lab (inertial) frame $(t, \vec{x})$, or the proper detector's frame $(\tau, \vec{\xi})$ (for example associated with the center of mass of the atom).

The detector's free Hamiltonian is most easily written in detector's frame:

$$\boxed{{}^{\tau}H_{0,d} = \Omega\sigma^+\sigma^- \,,} \tag{1.36}$$

where τ is the proper time of the detector, and gap Ω is measured in detector's frame.

Field free Hamiltonian, is most easily written in the Lab frame:

$$\boxed{{}^{t}H_{0,\phi} = \int d^n k |\vec{k}| a^+_{\vec{k}} a_{\vec{k}} \,,} \tag{1.37}$$

where t is the (lab frame) 'quantization time'.

However, the interaction Hamiltonian contains both sets of observables: (detector)$\times$ (field). It is the detector which prescribes the interaction with the field. It is thus natural to write this in the detector's frame:

$$\boxed{{}^{\tau}H_I = \lambda\chi(\tau) \int d^n\xi F(\vec{\xi}) \underbrace{\Big(\sigma^+ e^{i\Omega\tau} + \sigma^- e^{-i\Omega\tau}\Big)}_{\hat{m}(\tau)} \hat{\phi}\big(t(\tau,\vec{\xi}), \vec{x}(\tau,\vec{\xi})\big) \,.} \tag{1.38}$$

Note that when we write $F(\vec{\xi})$, we assume 'rigid atom' along the trajectory – the so called Fermi–Walker rigidity (in the center of mass frame). In other words, our F is not a function of τ (wave functions of the atom are not deformed by motion of the atom – 'atom drags the electrons"). This is okay, for accelerations $a < 10^{17} g$. (A bullet hitting a target has $a \sim 10^{10-11} g$, so it modifies molecules but not atoms.)

However, we can write this in the lab frame as well! To warm up let us start with time reparametrization.

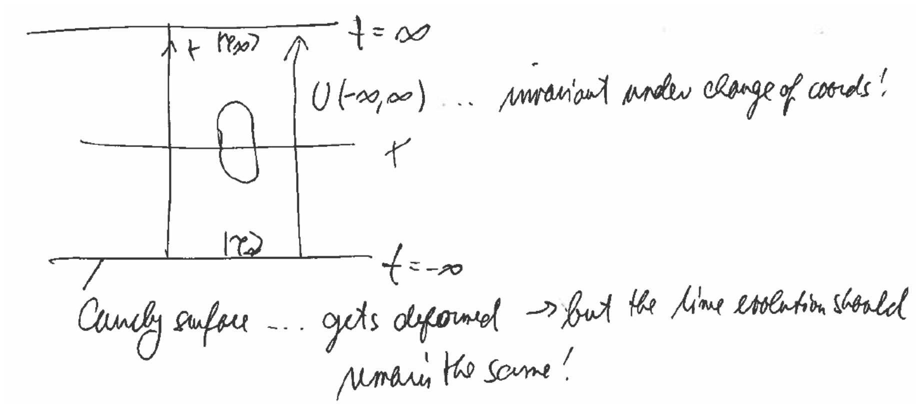
- Time reparametrization. Let ${}^{t}\hat{H}(t)$ be the Hamiltonian of a quantum system generating translations w.r.t. time t. What is ${}^{\tau}\hat{H}(\tau)$ generating translations w.r.t. τ? Under reparametrization $t \to t(\tau)$, we have $\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau}$. Employing the Schrodinger equation, we thus have:

$$i\frac{d}{dt}|\psi\rangle = {}^{t}\hat{H}(t)|\psi\rangle = i\frac{d\tau}{dt}\frac{d}{d\tau}|\psi\rangle \quad \Rightarrow \quad i\frac{d}{d\tau}|\psi\rangle = \underbrace{\frac{dt}{d\tau}{}^{t}\hat{H}(t)}_{{}^{\tau}\hat{H}(\tau)}|\psi\rangle \,, \tag{1.39}$$

that is, we also pick up the 'redshift factor' $\frac{dt}{d\tau}$:

$$\boxed{{}^{\tau}\hat{H}(\tau) = \frac{dt}{d\tau}{}^{t}\hat{H}(t(\tau)) \,.} \tag{1.40}$$

- The same can be seen, for example, from the fact that the time evolution operator must be invariant under time reparametrization, as seen from the following picture:



Thus we have

$$U = T\exp\Big(-i\int dt\,{}^tH(t)\Big) = T\exp\Big(-i\int d\tau\,{}^\tau H(\tau)\Big)\,. \tag{1.41}$$

Using Fubini's theorem we recover (1.40).

- More generally, the evolution operator is invariant under general change of coordinates. We thus have

$$\begin{aligned} U &= T\exp\Big(-i\int_{-\infty}^{\infty} d\tau\,{}^\tau H_I(\tau)\Big) = T\exp\Big(-i\int_{-\infty}^{\infty} d\tau d\vec{\xi}\,{}^\tau h_I(\tau)\Big) \\ &= T\exp\Big(-i\int_{-\infty}^{\infty} dt d\vec{x}\,{}^t h_I(t,\vec{x})\Big)\,, \end{aligned} \tag{1.42}$$

where

$$\begin{aligned} {}^\tau h_I &= \lambda\chi(\tau)F(\vec{\xi})\hat{m}(\tau)\hat{\phi}\big(t(\tau,\vec{\xi}),\vec{x}(\tau,\vec{\xi})\big)\,, \\ {}^t h_I &= \lambda\chi\big(\tau(t,\vec{x})\big)F\big(\vec{\xi}(t,\vec{x})\big)\hat{m}\big(\tau(t,\vec{x})\big)\hat{\phi}(t,\vec{x})\left|\frac{\partial(\tau,\vec{\xi})}{\partial(t,\vec{x})}\right|\,. \end{aligned} \tag{1.43}$$

Note that one cannot really distinguish switching from smearing! Note also that ${}^tH_I = \int d^n\vec{x}\,{}^t h_I$ is pretty non-trivial!

- Often-times one uses the 'point-particle' detector, setting

$$F(\vec{\xi}) = \delta(\vec{\xi})\,. \tag{1.44}$$

This is the standard approximation that is used in many situations. On the other hand, when one uses the extended detector, problems with general covariance and time ordering ambiguity arise, e.g. [3, 4]. Often-times we shall also use the (smooth) Gaussian switching χ, treating it as 'compact support'.

Chapter 2: Unruh Effect

2.1 Standard derivation

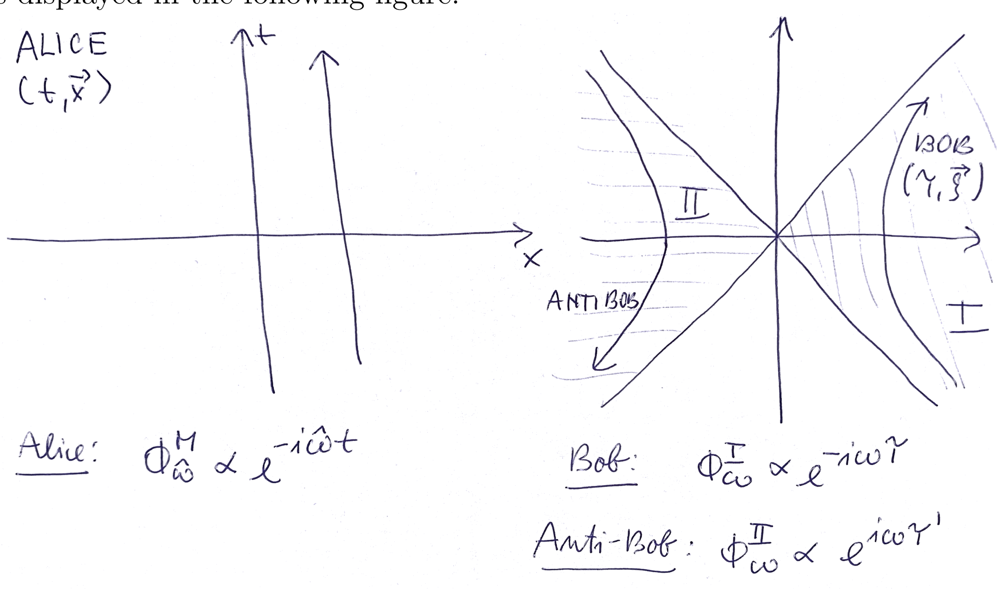
- Key idea: field quantization depends on the observer.
- Let us consider a massless scalar field, obeying

$$\nabla^2 \phi = 0\,, \tag{2.1}$$

as 'perceived' by two different observers (both of whom have a copy of Peskin and Schroeder): inertial Alice who is using Minkowski coordinates (t, x)), and uniformly accelerated Bob – using Rindler coordinates (τ, ξ), related to the Minkowski coordinates by

$$t = \xi \sinh a\tau\,, \quad x = \xi \cosh a\tau\,, \tag{2.2}$$

as displayed in the following figure:



- Note that ϕ^I_ω are not a complete basis of solutions for QFT in the whole spacetime, only in the wedge I. To have a complete set, we also need to consider Anti-Bob in the region II:

$$t = -\xi' \sinh a\tau'\,, \quad = -\xi' \cosh a\tau'\,. \tag{2.3}$$

- $\{\phi^I, \phi^{II}\}$ then form a basis at a given τ, which is a Cauchy surface. So at this Cauchy surface we can expand

$$\begin{aligned}\phi &= \sum_i \Big(a^M_{\hat{\omega}_i} \phi^M_{\hat{\omega}_i} + a^{M+}_{\hat{\omega}_i} \phi^{M*}_{\hat{\omega}_i} \Big) \\ &= \sum_i \Big(a^I_{\omega_i} \phi^I_{\omega_i} + a^{I+}_{\omega_i} \phi^{I*}_{\omega_i} + a^{II}_{\omega_i} \phi^{II}_{\omega_i} + a^{II+}_{\omega_i} \phi^{II*}_{\omega_i} \Big) \end{aligned} \tag{2.4}$$

  These are not unitary equivalent (have different vacua) – a^M's mix with a^I's and a^{I+}'s (a^{II}'s and a^{II+}'s).

- Considering the Minkowski vacuum: $|0\rangle_M$, it can be written as

$$|0\rangle_M = \prod_\omega \frac{1}{\cosh r_\omega} \sum_{n=0}^{\infty} \tanh^n r_\omega |n\rangle_I |n\rangle_{II} \,, \tag{2.5}$$

  where

$$\tanh r_\omega = \exp\Big(-\frac{\pi\omega}{a}\Big) \,. \tag{2.6}$$

  This is a 2-mode squeeze state (entangled) (mixes excitations in I and II regions).

- Bob has only access to region I. Thus uses the following density matrix:

$$\rho_{B\omega} = \mathrm{Tr}_{II}\big(|0\rangle_M \langle 0|_M\big) = \frac{1}{\cosh^2 r_\omega} \sum_n \tanh^{2n} r_\omega |n\rangle_I \langle n|_I \,. \tag{2.7}$$

  Using this to calculate the expectation of the number operator, we arrive at a thermal state

$$\langle N_{\omega,B} \rangle = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1} \,, \tag{2.8}$$

  which is the Bose–Einstein distribution with the Unruh temperature

$$\boxed{T_U = \frac{\hbar a}{2\pi k_B} \,.} \tag{2.9}$$

  So we arrived at a conclusion that Alice's field vacuum corresponds to a thermal bath for Bob at $T_U \propto a$.

- Two physical questions arise: 1) Does Bob need to accelerate forever – what happens for finite time acceleration? 2) Is the calculation above really enough to talk about thermality?

2.2 What is a thermal state?

- Gibbs: *a state that maximizes entropy at constant energy.*
  However, this is ill defined (diverges) for QFT.

- Instead: Let's compute a 2-pt correlator of an observable A of a quantum system in a thermal state (Gibbs). Note that thermal states are stationary (fixed points of time evolution. Since
$$\hat{A}(t) = e^{-iHt} A(0) e^{iHt}\,, \tag{2.10}$$
using stationarity we have to have
$$C(t,t') = \mathrm{Tr}\Big(\rho_\beta A(t) A(t')\Big) = \mathrm{Tr}\Big(\rho_\beta A(\Delta t) A(0)\Big) = C(\Delta t)\,. \tag{2.11}$$
Moreover, using the cyclic property of the trace we have
$$C(\Delta t) = \frac{1}{Z}\mathrm{Tr}\Big(e^{-\beta H} e^{-iH\Delta t} A(0) e^{iH\Delta t} A(0)\Big) = \frac{1}{Z}\mathrm{Tr}\Big(e^{-iH(\Delta t - i\beta)} A(0) e^{iH\Delta t} A(0)\Big)\,. \tag{2.12}$$
At the same time, shifting time, we get
$$\begin{aligned} C(\Delta t + i\beta) &= \frac{1}{Z}\mathrm{Tr}\Big(e^{-iH\Delta t} A(0) e^{iH(\Delta t + i\beta)} A(0)\Big) \\ &= \frac{1}{Z}\mathrm{Tr}\Big(e^{iH(\Delta t + i\beta)} A(0) e^{-iH\Delta t} A(0)\Big) = C(-\Delta t)\,, \end{aligned} \tag{2.13}$$
that is 'complex anti-periodicity'. Thus, all Gibbs states obey Kubo, Martin, Schwinger (KMS) condition
$$\boxed{C(-\Delta t) = C(\Delta t + i\beta)\,.} \tag{2.14}$$
Think about how this is related to the Euclidean trick showing that black holes have a temperature!

- We have just shown that all Gibbs states are KMS. The converse is not true but 'almost true' :).

  One can also show that KMS states are passive – one cannot extract work from them:
$$\langle E\rangle_{\text{extracted}} = 0\,. \tag{2.15}$$
  Moreover, KMS condition is applicable to QFT, and provides a 'good definition' of thermality for QFTs. We will associate KMS condition with a given observer.

  In particular, since the Wightman function
$$W_\rho(\tau,\tau') = \mathrm{Tr}\Big(\rho\hat{\phi}(\tau)\hat{\phi}(\tau')\Big) \tag{2.16}$$

is a 2-pt function (that knows everything there is to know about the QFT), we can use it to define thermality of the state ρ. Here, and in what follows, we have abbreviated

$$\hat{\phi}(\tau) \equiv \hat{\phi}(t(\tau), \vec{x}(\tau)) \,. \tag{2.17}$$

Namely, we have the following definition:

- KMS states in QFT (better definition of thermality):

  Definition. Let us have a timelike vector ∂_τ, a Hamiltonian ${}^\tau H$, and a field state $\hat{\rho}$. Then $\hat{\rho}$ is a KMS state of KMS temperature

  $$T_{\rm KMS} = \frac{1}{\beta}\,, \tag{2.18}$$

  with respect to ∂_τ if and only if i) it is stationary, that is $W_\rho(\tau, \tau') = W_\rho(\Delta\tau)$ and ii) satisfies KMS condition, that is $W_\rho(\Delta\tau + i\beta) = W_\rho(-\Delta\tau)$.

  Note that different observers can have different KMS temperatures. Namely in the Unruh case, $T_{\rm KMS} = T_A = 0$ with respect to ∂_t and $T_{{\rm K}MS} = T_B = T_U$ w.r.t. ∂_τ.

  Note also, that one can have non-stationary states with complex anti-periodicity. This is of course not enough to have thermality!

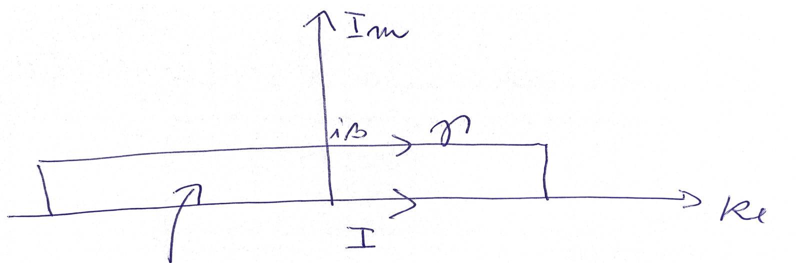
- Detailed balance. Let us have a KMS state, then we have

  $$\begin{aligned}\int_{-\infty}^{\infty} d\Delta\tau W(\Delta\tau + i\beta) e^{i\omega\Delta\tau} &= \int_{-\infty}^{\infty} d\Delta\tau W(-\Delta\tau) e^{i\omega\Delta\tau} \\ &= \int_{\infty}^{-\infty} d(-\Delta\tau) W(\Delta\tau) e^{-i\omega\Delta\tau} \\ &= \int_{-\infty}^{\infty} d\Delta\tau W(\Delta\tau) e^{-i\omega\Delta\tau} = \tilde{W}(-\omega)\,. \end{aligned} \tag{2.19}$$

  Now, let us change the variables, $\Delta\tau' = \Delta\tau + i\beta$. Then we have

  $$\tilde{W}(\omega) = \int_\gamma d\Delta\tau' W(\Delta\tau') e^{i\omega(\Delta\tau' - i\beta)} = e^{\beta\omega} \int_\gamma d\Delta\tau' W(\Delta\tau') e^{i\omega\Delta\tau'} = e^{\beta\omega} \tilde{W}(\omega)\,, \tag{2.20}$$

  where we have effectively done the following:



IF W ANALYTIC HERE (OR REGULARIZED ?)
(CAUCHY INTEGRAL THEOREM)

THEN WE ARE OK.

Thus we have derived 'detailed balance' condition:

$$\boxed{e^{\beta\omega}\tilde{W}(\omega) = \tilde{W}(-\omega)\,.} \tag{2.21}$$

2.3 Thermalization of the detector

- As an experimentalist I carry a thermometer and accelerate – do I see the Unruh temperature $T = a/(2\pi)$? Thermality expectation: detector state evolves to $\rho = \frac{1}{Z}e^{-\beta H_d}$, which implies
$$\frac{P_{\rm ex}(\Omega)}{P_{\rm deex}(\Omega)} = e^{-\beta\Omega}\,. \tag{2.22}$$
Let us show that this is true. We shall do this in several steps.

- 'Experimental setup'. In what follows we shall consider a switching function χ that is strongly supported in a timescale σ (for which the detector is on, with $\chi(\pm 1)$ being the boundary of the support), such that its L^2 norm is equal to one, that is
$$||\chi(\tau/\sigma)||_{L^2} \equiv \int_{-\infty}^{\infty} d\tau |\chi(\tau/\sigma)|^2 = 1\,. \tag{2.23}$$
For such switching we have
$$\chi(\tau/\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \tilde{\chi}(\omega) e^{-i\omega\tau/\sigma} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \tilde{\chi}^*(\omega) e^{i\omega\tau/\sigma} = \chi^*(\tau/\sigma)\,, \tag{2.24}$$
as it is real. Moreover, also we have
$$\begin{aligned}\int_{-\infty}^{\infty} d\omega |\tilde{\chi}(\omega)|^2 &= \int_{-\infty}^{\infty} d\tau d\tau' d\omega \chi(\tau/\sigma)\chi^*(\tau'/\sigma) e^{i\omega(\tau-\tau')}\\ &= 2\pi \int d\tau d\tau' \chi(\tau/\sigma)\chi^*(\tau'/\sigma)\delta(\tau-\tau')\\ &= 2\pi \int_{-\infty}^{\infty} d\tau |\chi(\tau/\sigma)|^2 = 2\pi\,,\end{aligned} \tag{2.25}$$

(a Percival equality).

Moreover, if $\chi(\tau/\sigma)$ strongly supported on sacle σ, $\tilde{\chi}(\omega)$ is strongly supported on a scale $1/\sigma$.

We shall also consider a point-like detector, for which $F(\vec{\xi}) = \delta(\vec{\xi})$, and the UdW interaction Hamiltonian reads

$$H_I = \lambda\chi(\tau)\hat{m}(\tau)\hat{\phi}(\tau)\,, \tag{2.26}$$

and the transition probability (1.28) simplifies.

- Long response. Considering the pointlike detector with finite switching function above, the response function, as per (1.28), reads

$$\begin{aligned}\mathcal{F}(\Omega,\sigma) &\equiv \frac{1}{\lambda^2\sigma}P(\Omega) = \frac{1}{\sigma}\int d\tau d\tau'\chi(\tau/\sigma)\chi(\tau'/\sigma)e^{i\Omega(\tau-\tau')}W(\tau,\tau') \\ &= \frac{1}{4\pi^2\sigma}\int d\tau d\tau' d\omega d\omega'\tilde{\chi}^*(\omega)\tilde{\chi}(\omega')e^{i(\omega\tau/\sigma-\omega'\tau'/\sigma)}e^{i\Omega(\tau-\tau')}W(\tau,\tau')\,.\end{aligned} \tag{2.27}$$

If the field is stationary w.r.t. ∂_τ, $W(\tau,\tau') = W(\tau-\tau')$, it is natural to change variables as

$$u = \tau-\tau'\,,\quad v=\tau+\tau' \quad\Leftrightarrow\quad \tau = \frac{u+v}{2}\,,\quad \tau' = \frac{v-u}{2}\,, \tag{2.28}$$

with the corresponding Jacobian equal to 1/2. We then find

$$\begin{aligned}\mathcal{F}(\Omega,\sigma) &= \frac{1}{8\pi^2\sigma}\int d\omega d\omega' \underbrace{\int dv e^{\frac{i}{2\sigma}(\omega-\omega')v}}_{2\pi\delta\left(\frac{1}{2\sigma}(\omega-\omega')\right)=4\pi\sigma\delta(\omega-\omega')} \int du\tilde{\chi}^*(\omega)\chi(\omega')e^{\frac{i}{2\sigma}(\omega+\omega')u}W(u)e^{i\Omega u} \\ &= \frac{1}{2\pi}\int du d\omega|\tilde{\chi}(\omega)|^2 W(u)e^{i(\Omega+\omega/\sigma)u} \\ &= \frac{1}{2\pi}\int d\omega|\tilde{\chi}(\omega)|^2\tilde{W}(\Omega+\omega/\sigma)\,.\end{aligned} \tag{2.29}$$

Here, in the first line we have used a magic formula for δ functions:

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}\,, \tag{2.30}$$

where x_i are the roots of $f(x)$.

Waiting for a long time corresponds to large σ. More precisely, if $\tilde{\chi}(\omega)$ decays fast enough (‘measure’ $\tilde{\chi}(\omega)$ strongly supported on a s scale $1/\sigma$; we need ‘adiabatic’ (smooth enough switching – not nervous experimentalists)) then we find

$$\lim_{\sigma\to\infty,\ \text{adiab}}\mathcal{F}(\Omega,\sigma) = \frac{1}{2\pi}\int d\omega|\tilde{\chi}(\omega)|^2\tilde{W}(\Omega) = \frac{\tilde{W}(\Omega)}{2\pi}\int d\omega|\tilde{\chi}(\omega)|^2 = \tilde{W}(\Omega)\,. \tag{2.31}$$

That is, a detector acquires information about the Wightman function, after interacting for a long time.

- Response of the detector. Putting things together, the excitation/de-excitation ratio reads

$$R(\Omega,\sigma)=\frac{P_{\mathrm{ex}}(\Omega,\sigma)}{P_{\mathrm{deex}}(\Omega,\sigma)}=\frac{P_{\mathrm{ex}}(\Omega,\sigma)}{P_{\mathrm{ex}}(-\Omega,\sigma)}=\frac{\mathcal{F}(\Omega,\sigma)}{\mathcal{F}(-\Omega,\sigma)}\,. \tag{2.32}$$

We thus have, using the detailed balance condition (2.21)

$$\boxed{\lim_{\sigma\to\infty,\ \mathrm{adiab}} R(\Omega,\sigma)=\frac{\tilde{W}(\Omega)}{\tilde{W}(-\Omega)}=e^{-\beta\Omega}\,.} \tag{2.33}$$

Thus, a detector that interacts with a KMS state (w.r.t. its proper time) of temperature $T_{\mathrm{KMS}}=1/\beta$, as long as it is switched on carefully, thermalizes (catches Boltzmannian population) after intercating with the field for a long time! Moreover, the detector acquires the same temperature as the field – this is the experimental definition of thermality

$$T_{\mathrm{KMS}}=T\,. \tag{2.34}$$

(C.f. zero law of TDs.)

- Intermezzo: Some properties of Wightman functions. First, we may write:

$$\begin{aligned}
W_\rho(\tau,\tau') &= \mathrm{Tr}\Big(\rho\hat{\phi}(\tau)\hat{\phi}(\tau')\Big)\\
&= \frac{1}{2}\mathrm{Tr}\Big(\rho(\hat{\phi}(\tau)\hat{\phi}(\tau')-\hat{\phi}(\tau')\hat{\phi}(\tau))\Big)+\frac{1}{2}\mathrm{Tr}\Big(\rho(\hat{\phi}(\tau)\hat{\phi}(\tau')+\hat{\phi}(\tau')\hat{\phi}(\tau))\Big)\\
&= \underbrace{\frac{1}{2}\langle[\phi(\tau),\phi(\tau')]\rangle_\rho}_{i\mathrm{Im}(W_\rho(\tau,\tau'))}+\underbrace{\frac{1}{2}\langle\{\phi(\tau),\phi(\tau')\}\rangle_\rho}_{\mathrm{Re}(W_\rho(\tau,\tau'))}\,.
\end{aligned} \tag{2.35}$$

Since $[\phi(\tau),\phi(\tau']\propto z1$, the first term, giving $i\mathrm{Im}\big(W(\tau,\tau')\big)$ is state independent; it is the second term that depends on ρ.

Second, we have

$$W^*_\rho(\tau,\tau')=\Big(\mathrm{Tr}(\rho\phi(\tau)\phi(\tau')\Big)^*=\mathrm{Tr}(\phi(\tau')\phi(\tau)\rho)=\mathrm{Tr}(\rho\phi(\tau')\phi(\tau))=W_\rho(\tau',\tau)\,. \tag{2.36}$$

- Consider now the commutator $C(\tau,\tau')$:

$$C(\tau,\tau')=\langle[\phi(\tau),\phi(\tau')]\rangle=2i\mathrm{Im}W(\tau,\tau')\,. \tag{2.37}$$

If W stationary, so is the commutator: $C(\tau,\tau')=C(\Delta\tau)$ (real and imaginary parts do not talk to each other). Moreover,

$$\begin{aligned}
\tilde{C}(\omega) &= \int_{-\infty}^{\infty}d\Delta\tau C(\Delta\tau)e^{i\omega\Delta\tau}=\int_{-\infty}^{\infty}d\Delta\tau\Big(W(\Delta\tau)-\underbrace{W^*(\Delta\tau)}_{W(-\Delta\tau)}\Big)e^{i\omega\Delta\tau}\\
&= \tilde{W}(\omega)-\tilde{W}(-\omega)\,.
\end{aligned} \tag{2.38}$$

Thus, if the state is KMS, we can use the 'detailed balance' (2.21), to obtain

$$\tilde{C}(\omega) = \tilde{W}(\omega) - e^{\beta\omega}\tilde{W}(\omega)\,, \tag{2.39}$$

or by re-arranging:

$$\boxed{\tilde{W}(\omega,\beta) = -\tilde{C}(\omega,\beta)P(\omega,\beta)\,, \quad P(\omega,\beta) = \frac{1}{e^{\beta\omega}-1}\,.} \tag{2.40}$$

Here, $P(\omega,\beta)$ is the Planck's factor. This in particular means that, if the state is KMS, the FT of the Wightman is completely determined by the commutator (commutator contains both real and imaginary parts).

Note that $\tilde{C}(\omega,\beta)$ depends on the pull back to detectors trajectory; this introduces the dependence on β! (Trajectory depends on acceleration, and thence on β!)

2.4 Is Unruh KMS?

- If we can show that $|0\rangle_M$ is KMS w.r.t. ∂_τ of accelerated detector, Unruh effect is as physical as QFT! Let us prove that!
- Wightman function. The trajectory of constant acceleration is given by

$$t(\tau) = \frac{1}{a}\sinh(a\tau)\,, \quad x^1(\tau) = \frac{1}{a}\Big(\cosh(a\tau) - 1\Big)\,, \quad x^2 = x^3 = \cdots = x^d = 0\,. \tag{2.41}$$

The Wightman function for the vacuum state in Minkowski is given by

$$\begin{aligned} W(\tau,\tau') &= \langle 0|_M \phi(\tau)\phi(\tau')|0\rangle_M \\ &= \int \frac{d^d k e^{-\epsilon|\vec{k}|}}{2(2\pi)^d|\vec{k}|} e^{-i\Big(|\vec{k}|\big(t(\tau)-t(\tau')\big) - \vec{k}\cdot\big(\vec{x}(\tau)-\vec{x}(\tau')\big)\Big)} \\ &= \int \frac{d^{d+1}k}{2(2\pi)^d}\Theta(k^0)\delta(k^2)e^{ik\cdot(x-x')}\,. \end{aligned} \tag{2.42}$$

where in the second line we included the regularization ϵ (we shall omit it from now on), and in the third line we have written the covariant expression.

Since the trajectory of the detector is timelike, having $(x-x')^2 < 0$, we can define $\Delta = \sqrt{-(x-x')^2}$. Then

$$\begin{aligned} k\cdot(x-x') &= k_0\big(t(\tau)-t(\tau')\big) + k_1\big(x^1(\tau)-x^1(\tau')\big) \\ &= \bar{k}_0\Delta\,\mathrm{sgn}\big(t(\tau)-t(\tau')\big)\,, \end{aligned} \tag{2.43}$$

where

$$\begin{aligned} \bar{k}^0 &= \Delta^{-1}\Big(k^0\big(t(\tau)-t(\tau')\big) + k^1\big(x^1(\tau)-x^1(\tau')\big)\Big)\mathrm{sgn}(t-t')\,, \\ \bar{k}^1 &= \Delta^{-1}\Big(k^1\big(t(\tau)-t(\tau')\big) - k^0\big(x^1(\tau)-x^1(\tau')\big)\Big)\mathrm{sgn}(t-t')\,, \\ \bar{k}^2 &= k^2\,\ldots\,. \end{aligned} \tag{2.44}$$

This is a 'change of coordinates' that corresponds to a Lorentz transformation that aligns k and x. Here, the $\mathrm{sgn}(t-t')$ is important for preserving the volume under ortochronous Lorentz transformations.

We then have (Jacobian is equal to one)

$$d^{d+1}\bar{k}\Theta(\bar{k}^0)\delta(\bar{k}^2) = d^{d+1}k\Theta(k^0)\delta(k^2)\,. \tag{2.45}$$

Thus,

$$\begin{aligned}
W(\tau,\tau') &= \int \frac{d^{d+1}\bar{k}}{2(2\pi)^d}\Theta(\bar{k}^0)\delta(\bar{k}^2)e^{-i\bar{k}^0\Delta\mathrm{sgn}\left(t(\tau-t(\tau')\right)} \\
&= \frac{(4\pi)^{-d/2}}{\Gamma(d/2)}\int_0^\infty d|\vec{k}||\vec{k}|^{d-2}e^{-i|\vec{k}|\Delta\mathrm{sgn}(t(\tau)-t(\tau'))} \\
&= \frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{(d+1)/2}}\left(\Delta\mathrm{sgn}\big(t(\tau)-t(\tau')\big)\right)^{1-d}.
\end{aligned} \tag{2.46}$$

Here, in the second line we have integrated over $\bar{k}^0$ and over the angles; the volume of an n-dimensional sphere is given by

$$\omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\,, \tag{2.47}$$

and the results is valid for any $d>1$. Moreover, since for our accelerated trajectory we have

$$\Delta\mathrm{sgn}\big(t(\tau)-t(\tau')\big) = \frac{2}{a}\sinh\big(\frac{a}{2}(\tau-\tau')\big)\,, \tag{2.48}$$

upon using the trigonometric identities:

$$\sinh x - \sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}\,,\quad \cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}\,. \tag{2.49}$$

So our Wightman function is

$$\boxed{W(\Delta\tau) = \frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{(d+1)/2}}\left[\frac{2}{a}\sinh\big(\frac{a}{2}(\tau-\tau')\big)\right]^{1-d}.} \tag{2.50}$$

Obviously, this is stationary. Moreover, using that $\sinh(x+i\pi) = -\sinh x = \sinh(-x)$, we find

$$W(\Delta\tau + i\underbrace{\frac{2\pi}{a}}_{\beta}) = -W(\Delta\tau) = W(-\Delta\tau)\,. \tag{2.51}$$

So indeed, this Wightman corresponds to a KMS state with $T = a/(2\pi)$.

- Conclusion. A constantly accelerated detector coupled to the vacuum thermalizes to a temperature $T = a/(2\pi)$.
- Remark. Thermal state for inertial observer will not be a thermal state for an accelerated observer! (Initial state will disappear – seen only as a bump transient. This will decay and we will eventually see the normal Unruh temperature, see [].

2.5 Circular Unruh effect

– Circular setup. Following [5] let us now ask what would happen if instead of 'linear Unruh effect' we considered a circular motion, modelling Unruh with *uniform centrifugal acceleration*. As we shall see the corresponding response is not exactly thermal but may be more experimentally viable.

The question is: how does the Minkowski vacuum $|0\rangle_M$ look like to an observer in circular trajectory:

$$x(\tau) = \Big(\gamma\tau, R\cos(\gamma O\tau), R\sin(\gamma O\tau), 0, \dots\Big)\,, \tag{2.52}$$

where R is the radius of the orbit, $O = v/R$ its angular velocity and v is the orbital speed – both with respect to the Minkowksi time t, and $\gamma = 1/\sqrt{1-v^2}$ si the Lorentz factor. The motion is characterized by the proper acceleration

$$a = \sqrt{\ddot{x}^\mu \ddot{x}_\mu} = RO^2\gamma^2 = \frac{v^2}{1-v^2}\frac{1}{R} = \frac{v^2\gamma^2}{R}\,; \tag{2.53}$$

we shall adopt R and v as a pair of independent parameters specifying the trajectory.

The experimental advantage is obvious: i) the system remains within a finite-size laboratory for an arbitrary long time and ii) the Lorentz γ-factor remains constant over the worldline.

Note: if we simply used the (linear) Unruh temperature formula, we would get the following temperature:

$$T_{\text{lin}} = \frac{a}{2\pi} = \frac{v^2\gamma^2}{2\pi R}\,. \tag{2.54}$$

As we shall see, the situation is not so simple.

– Both, the state and the motion are stationary, i.e. we have $W(\tau, \tau') = W(\Delta\tau)$. This means that after interacting with the state for a long time, the detector's response is still given by

$$\lim_{\sigma\to\infty,\text{adiab}} \mathcal{F}(\Omega, \sigma) = \tilde{W}(\Omega)\,. \tag{2.55}$$

(Go through your notes to see that to derive this all we need is stationarity.)

– Let us now define a phenomenological (operational) temperature as

$$R = \frac{\tilde{W}(\Omega)}{\tilde{W}(-\Omega)} = e^{-\beta\Omega}\,. \tag{2.56}$$

For a conventional thermal (KMS) state such $T_{\text{circ}} = 1/\beta$ is independent of Ω (reflecting the detailed balance of KMS states). However, in the case of the circular trajectory, we find a dependence on Ω, that is, the effect depends on the energy scale we probe it – different detectors will see different temperatures. (Apart from acceleration, T_{circ} also depends on v and Ω.)

– As shown in the previous section in the specific case, the pullback of the Wightman function in a $(d+1)$-dimensional Minkowski space to an arbitary trajectory is given by

$$\boxed{W(\tau,\tau') = \frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{(d+1)/2}}\frac{1}{[(x(\tau)-x(\tau'))^2]^{(d-1)/2}}\,.} \tag{2.57}$$

Interestingly, this is not KMS for circular motion.

– In fact, one can show that in $(3+1)$ dimensions, one has

$$\tilde{W}(\Omega) = \underbrace{-\frac{\Omega}{2\pi}\Theta(-\Omega)}_{\tilde{W}^{\text{in}}(\Omega)} + \frac{1}{4\pi^2\gamma v R}\int_0^\infty dz\cos\left(\frac{2\Omega R z}{\gamma v}\right)\left(\frac{\gamma^2 v^2}{z^2} - \frac{1}{z^2/v^2 - \sin^2 z}\right), \tag{2.58}$$

where $\tilde{W}^{\text{in}}(\Omega)$ is the inertial motion response function. (Note that just for inertial motion $R = 0$ which corresponds to $\beta \to \infty$ and thence $T = 0$.)

In particular, in the large gap, $\Omega \to \infty$ and ultrarelativistic, $v \to 1$ limits, one finds that

$$\frac{T_{\text{circ}}}{T_{\text{lin}}} = \frac{\pi}{\sqrt{3}} \sim 1.8\,. \tag{2.59}$$

– Analogue spacetime implementation. Condensed matter systems (Bose–Einstein condensates or superfluid helium) provide an effective Minkowski geometry, where the speed of light is replaced by the speed of sound (phonon-type excitations), giving rise to sonic limit $v = 1$. Then one has the following 'dictionary':

$$\hat{\Omega} = \Omega/\gamma\,, \quad \hat{T} = T/\gamma\,, \quad \hat{a} = a/\gamma^2\,, \tag{2.60}$$

where $\hat{\Omega}$ is the energy gap w.r.t. the laboratory time t (Minkowski time in the effective Minkowski metric). Then one finds that in the near-sonic limit, $v \to 1$, one has (in 3+1 dimensions)

$$\hat{T}_{\text{circ}} \approx \frac{\gamma\hat{a}}{4\sqrt{3}}\,. \tag{2.61}$$

So the effect is enhanced close to the sonic limit.

– Other setups may for example include inertial motion in de Sitter spacetime [6], see e.g. [7] for a proposal for an experimental simulation in an analogue spacetime. In this case, we have a de Sitter spacetime, which in global coordinates takes the spherically symmetric form with

$$f = 1 - \frac{r^2}{\ell^2}\,, \tag{2.62}$$

where ℓ is the cosmological radius, related to the cosmological constant as $\Lambda = 3/\ell^2$. The spacetime admits a cosmological horizon, given by $f(r_c) = 0$, that is $r_c = \ell$. We then have

$$T_{\mathrm{dS}} = \frac{|f'(r_c)|}{4\pi} = \frac{1}{2\pi\ell^2}\,. \tag{2.63}$$

2.6 Hawking effect

– Stellar collapse. Let us now consider a black hole spacetime, formed from a stellar collapse. After everything settles down, we may describe it by a (1-sided) Schwarzschild metric, which we write in the Edington–Finkelstein ingoing coordinates (that cover regions I and II of the Kruskal diagram):

$$ds^2 = -dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 = -f dv^2 + 2dvdr + r^2 d\Omega^2\,. \tag{2.64}$$

with $f = 1 - \frac{2m}{r}$, and we have defined

$$u = t - r^*\,, \quad v = t + r^*\,, \quad r^* = \int \frac{dr}{f} = r + 2m\log\Big|\frac{r}{2m} - 1\Big|\,. \tag{2.65}$$

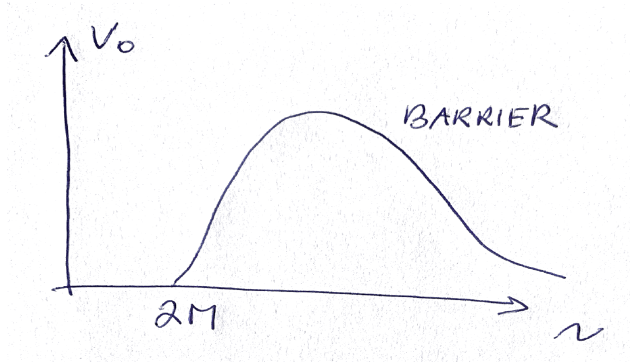
– Expanding further the scalar wave equation $\nabla^2\phi = 0$ in terms of the spherical harmonics,

$$\phi = \frac{1}{r}\psi(r,t)Y_{lm}(\theta,\varphi)\,, \tag{2.66}$$

we arrive at

$$\underbrace{\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial r_*^2}}_{\nabla^2_{(2)}\psi} + \underbrace{f\Big(\frac{l(l+1)}{r^2} + \frac{2m}{r^3}\Big)}_{V_l(r)}\psi = 0\,. \tag{2.67}$$

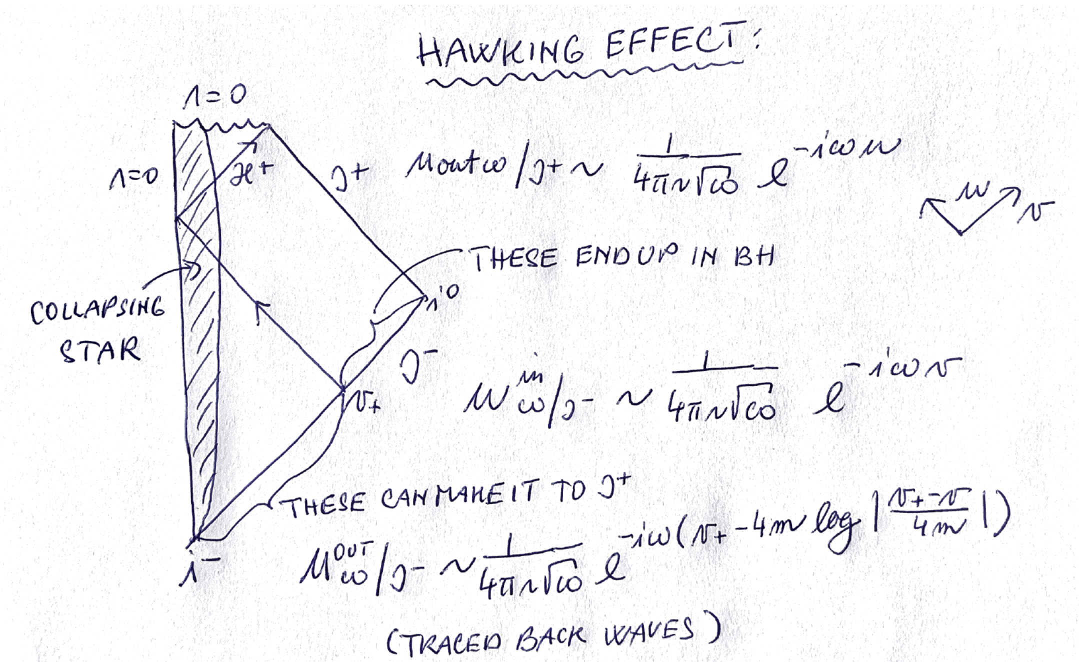
Note that $V_l(r)$ vanishes close to the horizon and thus has ‘nothing to do’ with the particle production close to the horizon – it represents a ‘barrier’ through which each mode has to propagate to reach infinity, e.g. we have:



– Focusing first on the '2-dimensional case' (neglecting for the moment V_l), we can expand in modes, as follows. On $\mathcal{I}^-$ we have the natural in modes:

$$\mathcal{I}^- : \quad u_\omega^{\rm in} \sim \frac{f_\omega^{\rm in}}{r} \sim \frac{1}{4\pi r\sqrt{\omega}} e^{-i\omega v} \,. \tag{2.68}$$

The modes with $v < v_+$ will make it to the future null infinity, whereas those with $v > v_+$ will end up in the black hole.



Consider next the modes that take the standard form on $\mathcal{I}^+$:

$$\mathcal{I}^+ : \quad u_\omega^{\rm out} \propto \frac{1}{4\pi\sqrt{\omega}} e^{-i\omega u} \,. \tag{2.69}$$

These correspond to outgoing modes close to the horizon. When traced back to the past null infinity (upon using the geometric optics approximation, see e.g. [8]) they become:

$$\mathcal{I}^- : \quad u_\omega^{\rm out} \sim \frac{1}{4\pi r\sqrt{\omega}} \exp\Big(-i\omega\big(v_+ - 4m\log\frac{|v_+ - v|}{4m}\big)\Big)\Theta(v_+ - v) \,. \tag{2.70}$$

[Note the exponential redshift due to collapse of the star, resulting in the log term in the exponential.] Similar to the Rindler case, the above modes do not form a complete set, and we need to complete the basis by including modes that go through the horizon. When traced back to $\mathcal{I}^-$ they take the following form:

$$u_\omega^{\text{hor}} \sim \frac{1}{4\pi r\sqrt{\omega}} \exp\Big(i\omega\big(v_+ - 4m\log\frac{|v_+ - v|}{4m}\big)\Big)\Theta(v - v_+)\,. \tag{2.71}$$

Any solution can then be expended either in the basis $\{u^{\text{in}}\}$ or in $\{u^{\text{out}}, u^{\text{hor}}\}$. Similar to the Rindler case, these are not unitary equivalent. Namely, one finds

$$|0\rangle_{\text{in}} = \prod_\omega \frac{1}{\cosh r_\omega} \sum_n (\tanh r_\omega)^n |n_\omega\rangle_{\text{hor}} |n_\omega\rangle_{\text{out}}\,, \tag{2.72}$$

where

$$\tanh r_\omega = \exp\Big(-\frac{\pi\omega}{\kappa}\Big)\,. \tag{2.73}$$

Thus, $|0\rangle_{\text{in}}$ is entangled (vacuum in the past evolves into two mode squeezed state between infalling and outgoing modes in the future).
Here, κ is the surface gravity of the horizon, which is a Killing horizon of the Killing vector field $\xi = \partial_t = \partial_v$, generating the horizon, defined by

$$\xi^c \nabla_c \xi^a \Big|_{r=r_+} = \kappa \xi^a \Big|_{r=r_+}\,. \tag{2.74}$$

Using the ingoing coordinates, we easily find

$$\kappa = \frac{|f'(r_+)|}{2} = \frac{1}{4m}\,. \tag{2.75}$$

– What do we see if we look at the black hole? We perceive the density matrix

$$\rho_{\text{out}} = \text{Tr}_{\text{hor}}\Big(|0\rangle_{\text{in}}\langle 0|_{\text{in}}\Big) = \otimes_\omega \frac{1}{\cosh^2 r_\omega} \sum_n \tanh^{2n} r_\omega |n_\omega\rangle_{\text{out}}\langle n_\omega|_{\text{out}} \tag{2.76}$$

We thus see outgoing radiation:

$$\langle \hat{n}_\omega\rangle = \text{Tr}(\hat{n}_\omega \rho_{\text{out}}) = \frac{1}{e^{\frac{\hbar\omega}{k_B T_H}} - 1}\,, \tag{2.77}$$

where

$$\boxed{T_H = \frac{1}{8\pi G_N m}\frac{\hbar c^3}{k_B} = \frac{\kappa}{2\pi}\frac{\hbar c^3}{k_B G_N}} \tag{2.78}$$

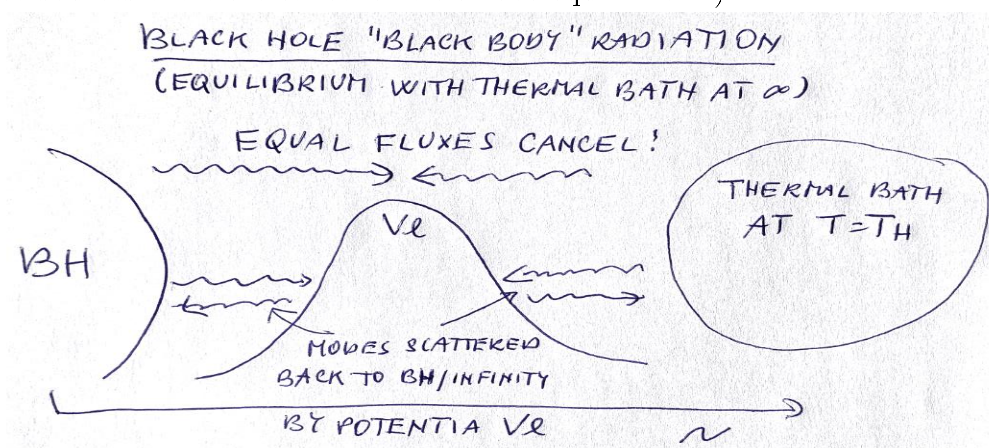
is the Hawking temperature at infinity – Hawking 1975 [9].

– One can show that the UdW detector at infinity is KMS and would thermalize to T_H. For other orbits in BH spacetimes (e.g. observers at finite r) the state is not necessarily KMS!

– Grey body factors. So far we have neglected the potential V_l through which each mode escaping the black hole has to propagate. This decreases the intensity of the wave and changes the resulting spectrum by a *greybody spectrum* $\Gamma_l(\omega) < 1$. The flux of particles observed at infinity thus reads:

$$\langle n_\omega \rangle = \frac{\Gamma_l(\omega)}{e^{\omega/T_H} - 1} \, . \tag{2.79}$$

Nevertheless, such a flux remains 'thermal' (black body) in the following sense. It is in thermal equilibrium with the thermal bath at infinity at temperature T_H. (The part of thermal radiation originating from thermal bath that gets reflected by V_l back to infinity equals the part of Hawking radiation that gets reflected by V_l back to black hole. The 'surviving' fluxes from the two sources therefore cancel and we have equilibrium.):



Chapter 3: First look at analogue systems

3.1 GR black hole primer

- Rotating black hole. The unique asymptotically flat rotating black hole solution in GR is given by the Kerr solution:

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\varphi)^2 + \frac{\sin^2\theta}{\Sigma}(adt - (r^2+a^2)d\varphi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2\,, \quad (3.1)$$

  where

$$\Delta = r^2 - 2Mr + a^2\,, \quad \Sigma = r^2 + a^2\cos^2\theta\,. \quad (3.2)$$

  Here, M stands for the mass of the hole (its total energy) and $J = Ma$ is its angular momentum. One of the interesting features associated with rotating black holes, apart from the existence of horizon, is the presence of ergoregion and with it associated *Penrose process* and *superradiance*. Let us briefly recapitulate these phenomena.

- Symmetries. In GR, Noether 'global symmetries' are described by Killing vectors, obeying Killing vector equation

$$\nabla_{(\mu}\xi_{\nu)} = 0\,. \quad (3.3)$$

  By Noether's theorem these give rise to conserved quantities. In particular, for geodesic motion $u^\nu\nabla_\nu u^\mu = 0$, we get the following conserved quantities:

$$C = u^\mu \xi_\mu\,. \quad (3.4)$$

  In particular, Kerr black hole admits two Killing vectors

$$k = \partial_t\,, \quad \eta = \partial_\varphi\,, \quad (3.5)$$

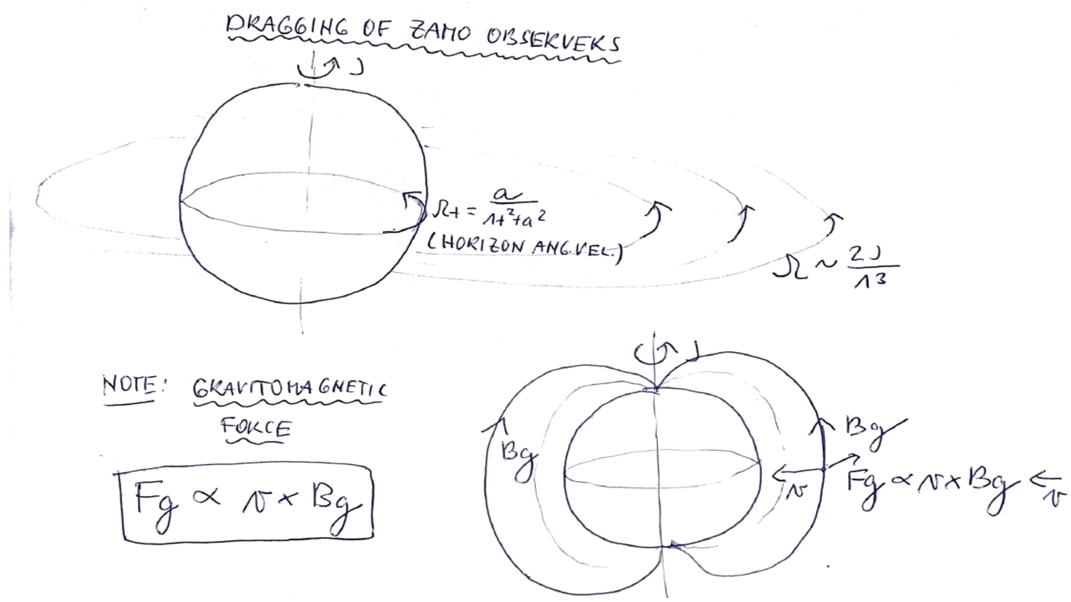
  and they generate two constants of geodesic motion

$$E = -k\cdot u = -g_{tt}\dot{t}\,, \quad L = \eta\cdot u = g_{\varphi t}\dot{t} + g_{\varphi\varphi}\dot{\varphi}\,, \quad (3.6)$$

  interpreted as asymptotic energy and angular momentum of the 'particle'.

- Inertial frame dragging. Consider zero angular momentum observers (ZAMOs), characterized by $L = 0$ (released from infinity at rest). As they approach the black hole they are dragged and will develop a non-trivial angular velocity:

$$L = 0 \quad \Rightarrow \quad \Omega = \frac{d\varphi}{dt} = \frac{\dot{\varphi}}{\dot{t}} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}\,. \tag{3.7}$$



- Horizon. Kerr black hole admits a Killing horizon generated by Killing vector field

$$\xi = \partial_t + \Omega_+ \partial_\varphi\,. \tag{3.8}$$

Thus

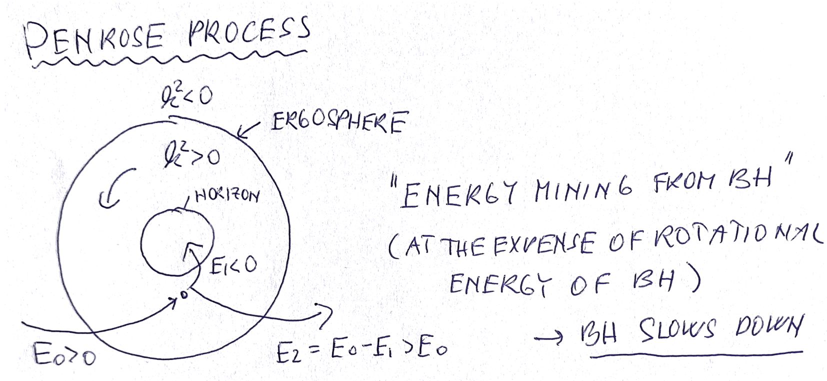
$$\xi^2|_{\mathcal{H}} = 0 \quad \Rightarrow \quad \Delta(r_+) = 0 \quad \Rightarrow \quad r_+ = M + \sqrt{M^2 + a^2}\,. \tag{3.9}$$

- Ergoregion. Static observers have $u \propto k$. However, since k is not timelike everywhere outside the horizon, we have a static 'limit' known as ergosphere, where

$$k^2|_e = g_{tt} = 0 \quad \Rightarrow \quad r_e = M + \sqrt{M^2 - a^2\cos^2\theta}\,. \tag{3.10}$$

The region between horizon and ergosphere is known as ergoregion. Inside ergoregion everything has to co-rotate with the BH (on horizon everything has to have $\Omega = \Omega_+$).

- Penrose process. Since inside the ergoregion vector k is spacelike, $k^2 > 0$, the energy $E = -k \cdot u$ as measured from infinity can become negative. (Obviously, particles with negative E can never leave the ergosphere. Their locally measured energy is positive, though they have negative E as measured at infinity.)



- Laws of BH thermodynamics. Similar to Schwazrschild, Kerr black holes Hawking radiate as 'black body' with temperature T_H. In addition they contain rotational energy. We thus have a generalized first law of black hole thermodynamics:

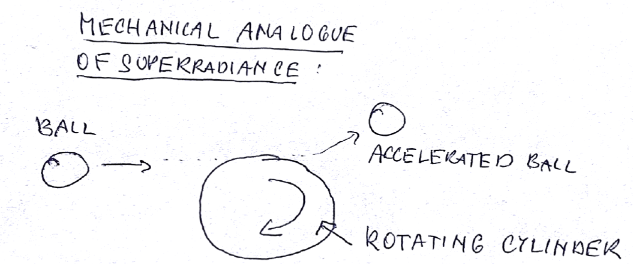
$$\delta M = T_H \delta S + \Omega_+ \delta J\,. \tag{3.11}$$

The black hole entropy S is given by area of the horizon, $S = \text{Area}/4$ (known as Bekenstein's law) and obeys the second law:

$$\delta S \geq 0\,, \tag{3.12}$$

provided some energy conditions are satisfied.

- Superradiance. This is a 'wave analogue' of Penrose process. First studied by Zeldovich for rotating cylinders, this effect has a nice mechanical analogue:



Let us now turn to the black hole case. We send a wave

$$\psi \propto e^{-i\omega t + im\varphi}\,. \tag{3.13}$$

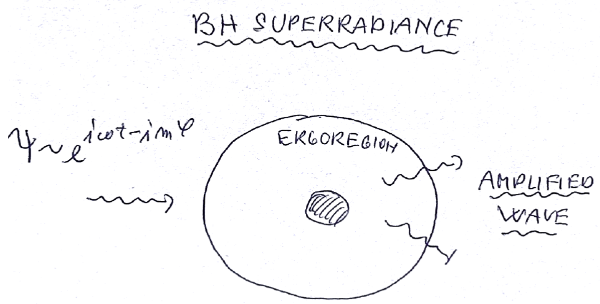
This gets scattered (and partly absorbed) by the black hole. To derive the superradiant condition, let us employ the laws of black hole thermodynamics. Namely, from the first law we have

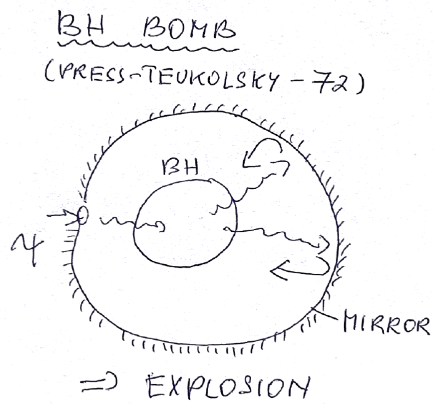
$$\frac{T_H \delta S}{\delta M} = 1 - \frac{\Omega_+ \delta J}{\delta M} = 1 - \frac{\Omega_+ m}{\omega} = \frac{(\omega - \Omega m)\omega}{\omega^2} < 0\,. \tag{3.14}$$

Here, we have $T_H > 0, \omega > 0$. By second law, $\delta S > 0$ and we want to extract black hole energy, that is $\delta M < 0$. Thus we get a superradiant condition:

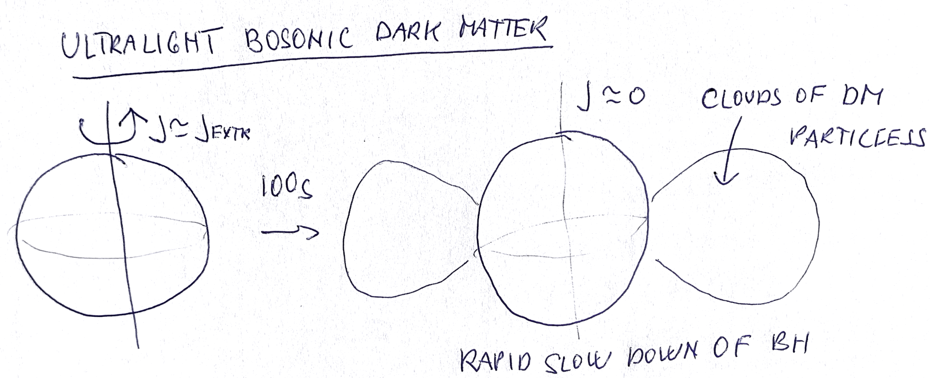
$$\boxed{\omega < \omega_c = \Omega_+ m = \frac{ma}{2Mr_+}\,,} \tag{3.15}$$

for which we get amplification of the waves. This may be used for destructive purposes, namely to build a black hole bomb:





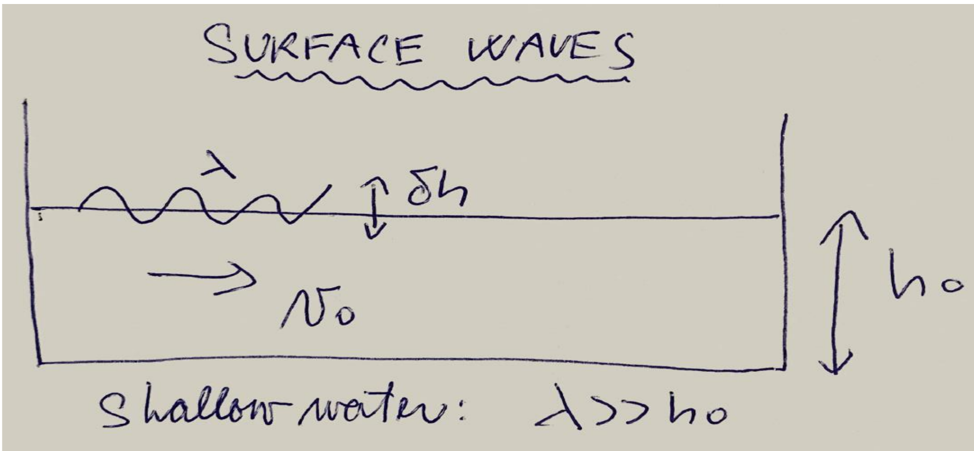
This effect i) is naturally present in AdS black hole spacetimes, where the gravitational pull of AdS (toward the center) provides a natural replacement of the mirror ii) may play the role in discovering dark matter candidates, such as ultralight bosonic particles, in black hole laboratories, e.g. [10], where the mass provides a reflective potential acting as a mirror:



Can we simulate some of these effects in analogue black hole spacetimes?

3.2 One analogue system

- Surface waves analogue. Analogues capture curved spacetime. Consider the water tank as follows:



Let us have two simplifying assumptions: i) *irrotational fluid*, that is *vorticity free*:

$$\nabla \times v = 0 \quad \Rightarrow \quad v = \nabla\Phi\,, \tag{3.16}$$

where Φ is the velocity potential, and ii) *shallow water*, that is $\lambda \gg h$.

The dynamics governed by the Navier-Stokes equations, which in this simplified case are equivalent to i) continuity equation for the water height h and ii) the Bernoulli equation:

$$\partial_t h + \nabla\cdot(hv) = 0\,, \tag{3.17}$$

$$\partial_t \Phi + \frac{1}{2}v^2 + gh = \text{const.}\,. \tag{3.18}$$

We further assume that the background velocity flow is 2-dimensional, $v_0 = v_x\hat{e}_x + v_y\hat{e}_y$ (there is no vertical component of the velocity).

Now we consider small perturbations of the background flow:

$$\Phi = \Phi_0 + \phi\,, \quad h = h_0 + \delta h\,. \tag{3.19}$$

These obey

$$(\partial_t + \nabla\cdot v_0)\delta h + \nabla\cdot(h_0\nabla\phi) = 0\,, \tag{3.20}$$

$$(\partial_t + v_0\cdot\nabla)\phi + g\delta h = 0\,. \tag{3.21}$$

Eliminating δh from the latter yields

$$(\partial_t + \nabla\cdot v_0)(\partial_t + v_0\cdot\nabla)\phi - g\nabla\cdot(h_0\nabla\phi) = 0\,. \tag{3.22}$$

Introduce now the speed of (shallow) surface waves

$$c = \sqrt{gh_0}\,, \tag{3.23}$$

the latter has a form of the wave equation in curved space:

$$\nabla^2\phi = \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b\phi\right) = 0\,, \tag{3.24}$$

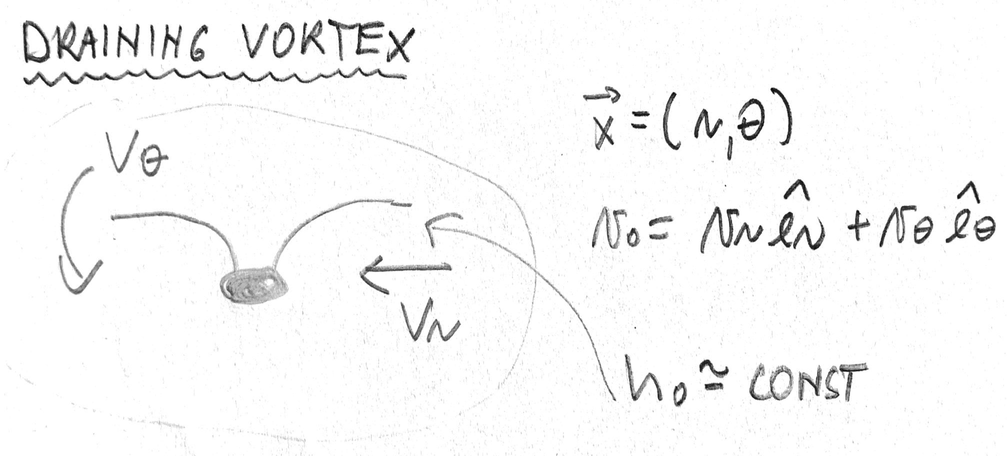
where we have an effective 3d metric

$$g_{\mu\nu} = c^2\begin{pmatrix} -(c^2 - v_0^2) & -v_{0i} \\ -v_{0j} & \delta_{ij} \end{pmatrix}\,. \tag{3.25}$$

Note that such metric is indeed Lorentzian, with signature $(-,+,+)$. Moreover, since $v_0 = v_0(x,y)$ and $h = h(x,y)$, the effective spacetime may be curved.

3.3 Draining vortex

- Draining vortex. Let us now specify to the following (infinite water tank) vortex system (kitchen sink when you pull your plug out):



From the background continuity equation, and since our water tank is infinite, we must have $\partial_t h_0 = 0$. Far away from the sink we also have $h_0 \approx$ const. And thence

$$\nabla \cdot v_0 = 0\,. \tag{3.26}$$

We also have $\nabla \times v_0 = 0$ by our approximation. From here we have

$$v_0 = -\frac{D}{r}\hat{e}_r + \frac{C}{r}\hat{e}_\theta\,. \tag{3.27}$$

Here, constant $D > 0$ is related to draining and constant $C > 0$ (w.l.o.g.) to circulation (we have chosen a direction of rotation).

What about our effective metric? Since h_0 is approximately constant, so will be the speed of waves c; we set $c = 1$. We thus find

$$ds^2 = -dt^2 + \left(dr + \frac{D}{r}dt\right)^2 + \left(rd\theta - \frac{C}{r}dt\right)^2\,. \tag{3.28}$$

This is written in Painlevé–Gullstrand coordinate system.

We can change coordinates (since the wave equation is covariant, we are allowed to change coordinates – have diffeomorphism invariance):

$$dt' = dt + \frac{v_r dr}{c^2 - v_r^2}\,, \quad d\theta' = d\theta + \frac{v_r v_\theta}{r(c^2 - v_r^2)} dr\,, \tag{3.29}$$

to obtain

$$ds^2 = -f dt'^2 + \frac{dr^2}{f} + \frac{1}{r^2}(C dt - r^2 d\theta')^2\,, \quad f = 1 - \frac{D^2}{r^2}\,. \tag{3.30}$$

Let us compare this to the (3+1)-dimensional Kerr metric, by restricting it to the equatorial plane $\theta = \pi/2$:

$$ds^2_{\text{Kerr eq}} = -f dt'^2 + \frac{dr^2}{f} + \frac{1}{r^2}\Big(a dt' - r^2 d\varphi\Big)^2\,, \quad f \equiv \frac{\Delta}{r^2}\,, \tag{3.31}$$

and we introduce new time t' by $dt - a d\varphi = dt'$. Obviously, the above f determines the black hole horizon. By analogy, the analogue system will have an analogue horizon at

$$\boxed{f(r_h) = 0 \quad \Rightarrow \quad r_h = D/c\,.} \tag{3.32}$$

Note that when $D = 0$ we have an analogue of a 'naked singularity'.

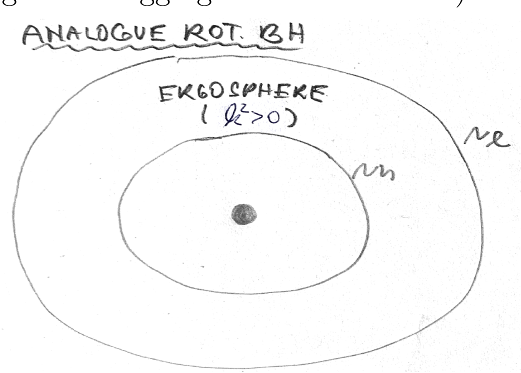
We also observe an ergoregion. Namely, consider a static observer whose trajectory is described by $k = \partial_{t'}$. Calculating its norm we find

$$k^2 = -1 + \frac{C^2 + D^2}{r^2}\,. \tag{3.33}$$

This becomes zero at

$$\boxed{r_e = \frac{\sqrt{C^2 + D^2}}{c}\,,} \tag{3.34}$$

which is known as analogue ergosurface (the boundary where static observers can exist). Inside r_e, everything is swept around by the rotation of the vortex (analogue of dragging of inertial frames).



3.4 Superradiance

- Related to <u>Penrose effect</u> – mining energy from black holes.

  Let us decompose the scalar modes into:

$$\phi = \sum_{\omega m} R_{\omega m}(r) e^{im\theta' - i\omega t'}\,, \tag{3.35}$$

where m is the azimuthal number (integer), and ω the frequency of the wave. Inserting this into our Klein–Gordon equation in the effective metric, we get the following radial equation:

$$\frac{f}{r}\partial_r(rf\partial_r R_{\omega m}) + \left[\left(\omega - \frac{mC}{r^2}\right)^2 - f\frac{m^2}{r^2}\right]R_{\omega m} = 0\,. \tag{3.36}$$

By introducing the 'tortoise coordinate'

$$dr^* = \frac{dr}{f} \quad \Rightarrow \quad r^* = r + \frac{D}{2}\log\left|\frac{r - r_h}{r + r_h}\right| \in (-\infty, \infty)\,, \tag{3.37}$$

where $r^* = -\infty$ correspond to the horizon and $r^* = \infty$ to spatial infinity, together with

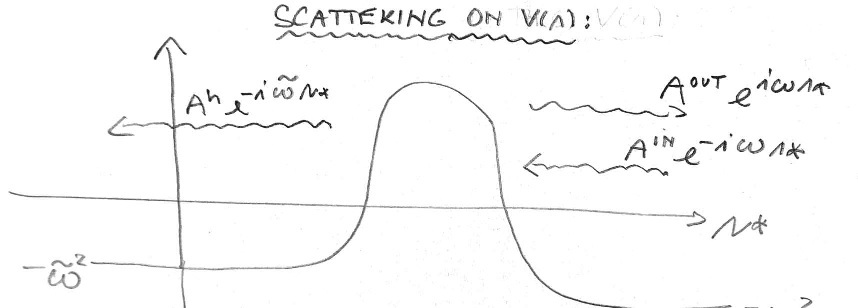
$$R_{\omega m}(r) = \frac{\psi(r, \omega, m)}{\sqrt{r}}\,, \tag{3.38}$$

we can recast this in the Schrodinger-like form:

$$-\frac{d^2\psi}{dr_*^2} + V(r)\psi = 0\,, \quad V(r) = -\left(\omega - \frac{mC}{r^2}\right)^2 + f\left(\frac{m^2 - 1/4}{r^2} + \frac{5D^2}{4r^4}\right). \tag{3.39}$$

Thus we have the following scattering problem:

SCATTERING ON V(r):



$$\tilde{\omega} = \omega - m\Omega_h = \omega - m\frac{C}{r_h^2}$$

ANGULAR VELOCITY OF HORIZON

We want to find the relationship between the amplitude coefficients. Instead of solving the full equation exactly, we use the following 'conserved quantity', known as the Wronskian:

$$W = \psi^* \partial_{r^*}\psi - \psi\partial_{r^*}\psi^*\,. \tag{3.40}$$

Namely, we have

$$\partial_{r^*} W = \psi^* \underbrace{\partial^2_{r^*}\psi}_{V\psi} - \psi \underbrace{\partial^2_{r^*}\psi^*}_{V\psi^*} + 0 = 0\,. \tag{3.41}$$

On the horizon/infinity we have

$$\begin{aligned} W(r_n) &= -2i\tilde{\omega}|A^h|^2 \\ &= -2i\omega|A^{\rm in}|^2 + 2i\omega|A^{\rm out}|^2\,, \end{aligned} \tag{3.42}$$

from where it follows that

$$|A^{\rm in}|^2 = |A^{\rm out}|^2 + \frac{\tilde{\omega}}{\omega}|A^{\rm h}|^2\,. \tag{3.43}$$

Introducing the reflection $R = A^{\rm out}/A^{\rm in}$ and transition $T = A^{\rm h}/A^{\rm in}$ coefficients, the latter rewrites as

$$\boxed{|R|^2 + \frac{\tilde{\omega}}{\omega}|T|^2 = 1\,.} \tag{3.44}$$

Interesting if $\tilde{\omega}/\omega < 0$. Then we have

$$|R| > 1\,, \tag{3.45}$$

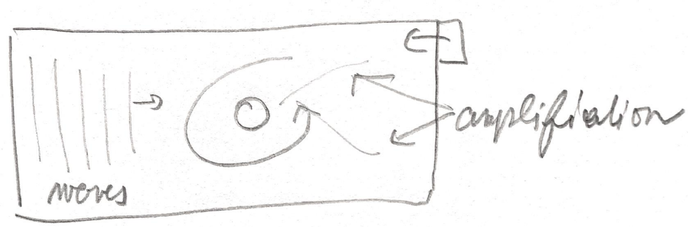
which corresponds to amplification. Since we have $\omega > 0$, for this to happen we must have $\tilde{\omega} < 0$, that is

$$\boxed{\omega < m\Omega_h\,.} \tag{3.46}$$

This is the superradiant condition. Note that in Kerr we have $\Omega_h = a/(r_+^2 + a^2)$. Th energy extraction is due to mining the rotational energy of the vortex/black hole.

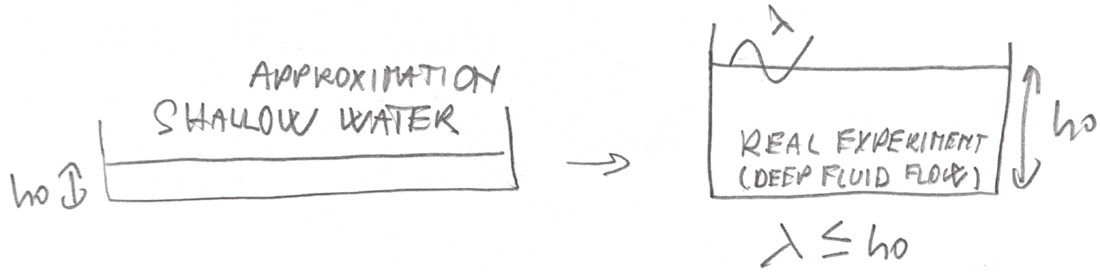
This has been experimentally measured in [11]:

EXP SETUP: SUPERRADIANCE



3.5 Beyond the effective metric approximation

- Real experiments go beyond shallow water approximation:



We now have the following equations:

$$(\partial_t + v_0 \cdot \nabla)\phi + g\delta h - \frac{\sigma}{\rho}\nabla^2 \delta h = 0\,,$$
$$(\partial_t + v_0 \cdot \nabla)\delta h - i\nabla \cdot \tanh(-ih_0\nabla)\phi = 0\,, \tag{3.47}$$

where σ is the surface tension. These break the Lorentz invariance.

Let us have the following ansatz:

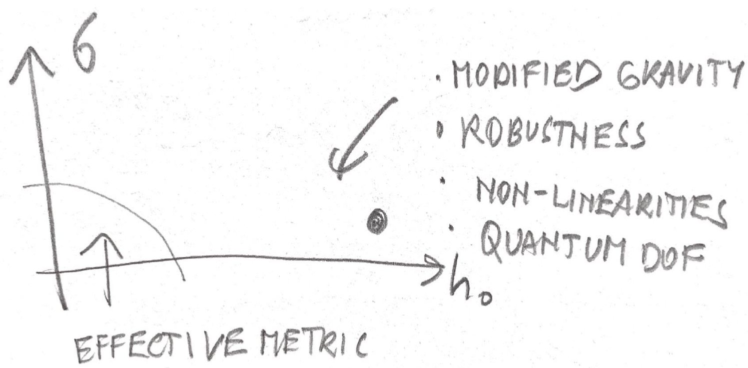
$$v_0 = 0\,, \quad \phi = Ae^{ikx-i\omega t}\,, \quad \delta h = Be^{ikx-i\omega t}\,. \tag{3.48}$$

Then we get the following dispersion relation:

$$\omega^2 = (gk + \frac{\sigma}{\rho}k^3)\tanh(h_0 k)\,, \tag{3.49}$$

which yields relativistic dispersion relation in the limit $k \to 0$. Namely, we have $\omega^2 \approx gh_0 k^2 = c^2 k^2$.

Real experiment was performed far from the effective metric regime:



They could still black hole ringdown and superradiance. Provides access to non-linearities, and perhaps even quantum dof (superfluid He instead of water) – do they imprint on scattering processes.

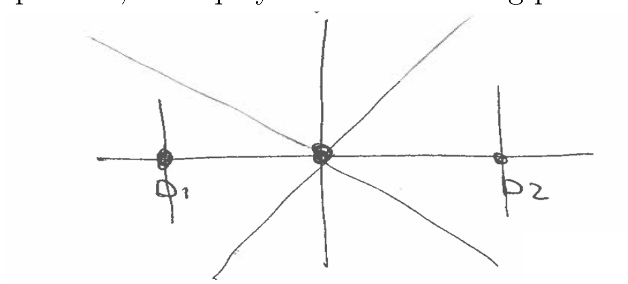
- In any case, if analogue systems caught your attention, please see the review [12].

Chapter 4: Quantum vacuum

Quantum vacuum is not empty. Global ground state (eigenstate of global Hamiltonian, whatever it is) doe snot amount to 'nothing' when we start magnifying – detector would see different stuff when probing vacuum in different scales. Let us explore how we can use it to allow for classically forbidden processes.

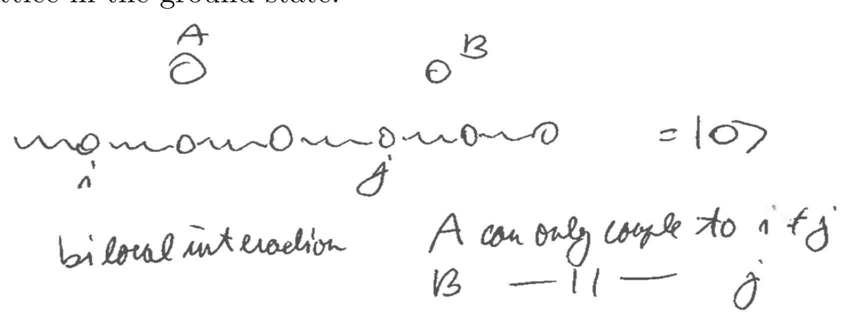
4.1 Entanglement harvesting: why it may work

- Observation. Consider two initially uncorrelated UdW detectors that are spatially separated, as displayed in the following picture:

  Let us switch them on for a short period of time, so that they could not exchange photons (and thence classically, they do not know about each others existence). It turns out that they can nevertheless become correlated. (Does this mean that entanglement is not necessarily a result of direct interaction?)

- 1d toy model. To motivate that this might be possible, consider 1d harmonic lattice in the ground state:

  The harmonic lattice is an interacting system, described by a Hamiltonian of 'sorts':

  $$H = \sum_i \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} k(x_i - x_j)^2 \,. \tag{4.1}$$

We can imagine two possible mechanism for correlating the two detectors A (Alice) and B (Bob): i) direct communication via exchanging lattice phonons – limited by the speed of sound or ii) take advantage of pre-existing entanglement of the vacuum state.

Let us look at the second possibility more closely. Note that we can have: ‘

$$\begin{aligned} \text{‘non-local basis’:} \quad & \text{normal modes}\{|0\rangle, |1\rangle, |2\rangle, \dots\} \\ \text{‘local basis’:} \quad & \text{individual number states}\{|n_1, \dots, n_i, \dots, n_j, \dots\rangle\}\,. \end{aligned} \tag{4.2}$$

In particular, the ground state

$$|0\rangle \neq \otimes_n |0_n\rangle \tag{4.3}$$

due to interaction ($[p, x] \neq 0$ and thence $[H_0, H_I] \neq 0$). In particular,

$$\rho_{ij} = \mathrm{Tr}_{n\neq i,j}|0\rangle\langle 0| \neq \sum_k p_k \rho_i \otimes \rho_j\,. \tag{4.4}$$

It is non-separable state – it is entangled.[1] Quantum noise in i and j is correlated. By bilocal interactions A with i and B with j, A and B can pick up this entanglement and become correlated. This is not limited by the speed of sound!

Note that to reach a ground state – ‘cooling thermalization’ involves everyone (long distance process)! The above discrete system corresponds to QFT in the continuum limit.

[1] A bipartite pure state

$$|\psi\rangle = \sum_{i,j} c_{ij}|a_i b_j\rangle \tag{4.5}$$

is called separable, if it can be decomposed to a tensor product of the two respective pure states: $|\psi\rangle = |\psi_1\rangle|\psi_2\rangle$. Otherwise it is entangled. For pure state, the entanglement can be measured by entanglement entropy, namely

$$S_1 = \mathrm{Tr}(\rho_1 \log \rho_1)\,, \quad \rho_1 = \mathrm{Tr}_2(|\psi\rangle\langle\psi|)\,. \tag{4.6}$$

A mixed state ρ is separable if there exists $p_k \geq 0$, with $\sum_k p_k = 1$, and mixed states of the respective subsystems $\{\rho_1^k\}$ and $\{\rho_2^k\}$ so that

$$\rho = \sum_k p_k \rho_1^k \otimes \rho_2^k\,. \tag{4.7}$$

For regions of spacetime, entanglement entropy is not a measure of entanglement if the global state is not pure! For example, consider a separable state

$$\rho = \rho_1 \otimes \rho_2\,, \tag{4.8}$$

where ρ_1 and ρ_2 are maximally mixed states. Then the partial trace yields a density matrix and the entanglement entropy is non-trivial (there was already ignorance in the system to begin with!)

4.2 Entanglement harvesting formalism

- Following [13], tet's start with 2 initially uncorrelated particle detectors in the ground state that are coupled to a scalar field in a vacuum state

$$\rho_0 = \underbrace{\rho_{0,AB}}_{\rho_{0,A}\otimes\rho_{0,B}} \otimes \underbrace{\rho_{0,\phi}}_{|0\rangle\langle 0|} . \tag{4.9}$$

Let the detectors are inertial and co-moving (they share the same rest frame). The interaction Hamiltonian is then

$$H_I = \sum_{\nu\in(A,B)} \lambda_\nu \chi_\nu(t) \int d^n x F_\nu(\vec{x}-\vec{x}_\nu)\hat{m}_\nu(t)\hat{\phi}(\vec{x},t)\,, \quad \hat{m}_\nu = \sigma_\nu^+ e^{i\Omega_\nu t} = \sigma_\nu^- e^{-i\Omega_\nu t}\,. \tag{4.10}$$

If the coupling is weak, we have

$$U = T\exp\Big(-i\int_{-\infty}^{\infty} dt H_I(t)\Big) = 1 + U^{(1)} + U^{(2)} + O(\lambda^3)\,, \tag{4.11}$$

where, using Dyson's expansion[2]

$$U^{(1)} = -i\int_{-\infty}^{\infty} dt H_i(t)\,, \quad U^{(2)} = -\int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' H_I(t) H_I(t')\,. \tag{4.12}$$

Then

$$\rho_{T,AB} = \mathrm{Tr}_\phi(U\rho_0 U^+) = \rho_{0,AB} + \rho_{T,AB}^{(1)} + \rho_{T,AB}^{(2)} + O(\lambda^3)\,. \tag{4.13}$$

Here, $\rho_{0,AB} = \mathrm{Tr}_\phi(\rho_0)$,

$$\rho_{T,AB}^{(1)} = \mathrm{Tr}_\phi \rho_T^{(1)}\,, \quad \rho_T^{(1)} = U^{(1)}\rho_0 + \rho_0 U^{(1)+}\,. \tag{4.14}$$

Since $U^{(1)} \sim a + a^+$, we get state for the field like $|0\rangle\langle 1|$ or $|1\rangle\langle 0|$, so

$$\rho_{T,AB}^{(1)} \sim \mathrm{Tr}_\phi(\ ) = 0\,. \tag{4.15}$$

More generally, for any state $\rho_{0,\phi}$ diagonal in the Fock basis, the 1-point function of the field $\mathrm{Tr}_\phi(\rho_{0,\phi}\hat{\phi}) = 0$.

- At second order we get

$$\rho_T^{(2)} = U^{(1)}\rho_0 U^{(1)+} + U^{(2)}\rho_0 + \rho_0 U^{(2)+}\,. \tag{4.16}$$

[2]Dyson's expansion preserves probabilities (traces) at every-order of expansion and is given in terms of powers of the Hamiltonian. However, this is not the only possibility. For example, one alternative, known as the Magnus expansion is designed to preserve symplectic form (phase space volume), and is given in terms of the commutators.

This yields

$$\rho^{(2)}_{T,AB} = \sum_{\nu,\eta}\lambda_\nu\lambda_\eta\Big[\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'\chi_\nu(t')\chi_\eta(t)\hat{m}_\nu(t')\rho_{0,AB}\hat{m}_\eta(t)\mathcal{W}(\vec{x}_\eta,t,\vec{x}_\nu,t')$$
$$-\int_{-\infty}^{\infty}dt\int_{-\infty}^{t}dt'\chi_\nu(t)\chi_\eta(t')\Big(\hat{m}_\nu(t)\hat{m}_\eta(t')\rho_{0,AB}\mathcal{W}(\vec{x}_\nu,t,\vec{x}_\eta,t')$$
$$+\rho_{0,AB}\hat{m}_\eta(t')\hat{m}_\nu(t)\mathcal{W}(\vec{x}_\eta,t',\vec{x}_\nu,t)\Big)\Big]\,, \tag{4.17}$$

where $\mathcal{W}$ is the pullback of the Wightman to smeared trajectory:

$$\mathcal{W}(\vec{x}_\nu,t,\vec{x}_\eta,t') = \int d^nxd^nx'F(\vec{x}-\vec{x}_\nu)F(\vec{x}'-x_\eta)\underbrace{W(t,\vec{x},t',\vec{x}')}_{\mathrm{Tr}_\phi\left(\phi(t,\vec{x})\phi(t',\vec{x}')\rho_{\phi,0}\right)}\quad. \tag{4.18}$$

In particular, assuming $\rho_{0,\phi}=|0\rangle\langle 0|$ and $\rho_{0,AB}=|g_A\rangle\langle g_A|\otimes|g_B\rangle\langle g_B|$, and considering the basis:

$$\{|g_A\rangle\otimes|g_B\rangle, |e_A\rangle\otimes|g_B\rangle, |g_A\rangle\otimes|e_B\rangle, |e_A\rangle\otimes|e_B\rangle\}\,, \tag{4.19}$$

we find

$$\rho_{T,AB} = \begin{pmatrix} 1-L_{AA}-L_{BB} & 0 & 0 & M \\ 0 & L_{AA} & L_{AB} & 0 \\ 0 & L^*_{AB} & L_{BB} & 0 \\ M^* & 0 & 0 & 0 \end{pmatrix} + O(\lambda^4)\,. \tag{4.20}$$

Here, $1-L_{AA}-L_{BB}$ is a probability of remaining in the ground state, L_{AA} is probability of getting A excited, L_{BB} a probability of getting B excited, the rest are entanglement correlations.

- How do we know the system is entangled? Entanglement entropy will not work as this is not a pure state. Negativity will work – it corresponds to negative eigenvalues of the partial-transposed matrix.[3]

  For identical detectors (the same shape of switching, smearing, and the same Ω), the partial-transposed density matrix ρ^{pT}_{AB} has only a (simple) negative eigenvalue. The negativity of $\rho_{T,AB}$ is

$$\boxed{N(\rho_{T,AB}) = \max(0,|M|-L_{AA})+O(\lambda^4)\,.} \tag{4.23}$$

  This is very intuitive: $M-L_{AA}$ measures competition between correlations (M – come from $U^{(2)}$ contributions) and local noise (L_{AA} – comes from $U^{(1)}U^{(1)}$).

[3]Given the matrix of two systems

$$\rho = \rho_{ij,kl}|i\rangle\langle k|\otimes|j\rangle\langle l| = \rho_{ij,kl}|i,j\rangle\langle k,l|\,, \tag{4.21}$$

the partial-transposed matrix w.r.t. B system is given by

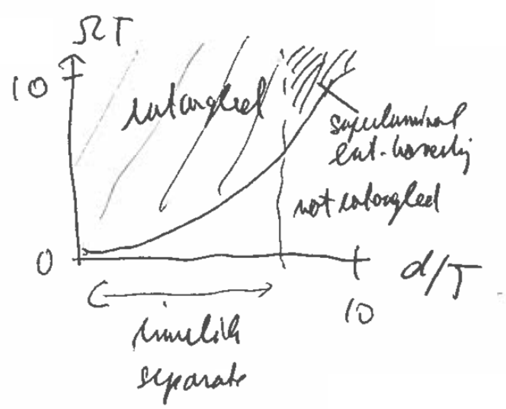
$$\rho^{pT,B} = \rho_{ij,kl}|i\rangle\langle k|\otimes|l\rangle\langle j|\,. \tag{4.22}$$

4.3 Structure of spacetime

As we shall see from here we can learn about topology & curvature of the spacetime! This can be done 'faster' than having to wait clasically! (Groundstate of the field had to 'thermalize' with the given spacetime and its topology – we can now read this information.) Original references on harvesting are [14, 15]; see e.g. [16, 17] for experimental proposals.

Harvesting in flat space

Consider two detctors in flat space. Then have the following entanglement harvesting (negativity) picture [13]:



ΩT .. detector gap
d/T detector separation
Δ/T=3 .. detector time delay
(T .. coupling time)

Here, the switching functions were chosen Gaussian with standard deviations $T/\sqrt{2}$:

$$\chi_\nu(t) = e^{(t-t_\nu)^2/T^2}\,, \quad \text{and} \quad \Delta = t_A - t_B\,. \tag{4.24}$$

There is a lightcone interaction for separations $d \le 8$.

As we see the gap Ω protects us from local noise; we can go as far as we want (all the way to Canada), but the amount of entanglement harvested is exponential decreasing.

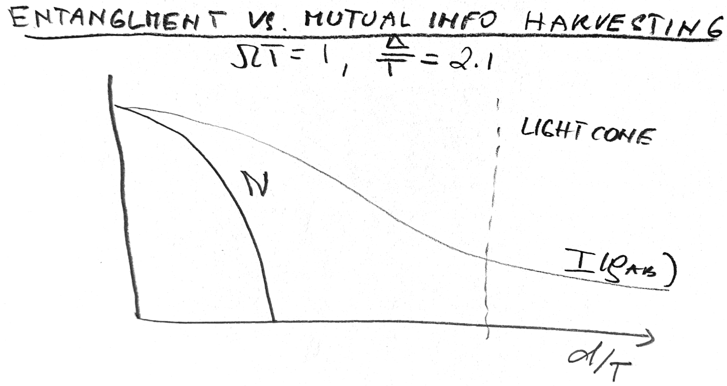
Note: mutual information harvesting, that is harvesting of general correlations (entanglement plus classical correlations), is much easier. Namely, using the following measure:

$$\begin{aligned} I(\rho_{AB}) &= S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \\ &= \mathcal{L}_+ \log \mathcal{L}_+ + \mathcal{L}_- \log \mathcal{L}_- - \mathcal{L}_{AA} \log \mathcal{L}_{AA} - \mathcal{L}_{BB} \log \mathcal{L}_{BB}\,, \end{aligned} \tag{4.25}$$

where, as usual $\rho_A = \mathrm{Tr}_B \rho_{AB}$, $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$, and

$$\mathcal{L}_+ = \frac{1}{2}\Big(\mathcal{L}_{AA} + \mathcal{L}_{BB} \pm \sqrt{(\mathcal{L}_{AA} - \mathcal{L}_{BB})^2 + 4|\mathcal{L}_{AB}|^2}\Big)\,. \tag{4.26}$$

See the following figure for comparison:



dS conformal vaccum vs. Minkowski thermal state

- Consider dS spacetime:

$$ds^2 = -dt^2 + e^{2\kappa t}dx_i^2 = a^2(\eta)(-d\eta^2 + dx_i^2)\,. \tag{4.27}$$

Here t is the 'co-moving' time and η the conformal time (in the latter form, the space is conformal to Minkowski). Consider next the conformally coupled scalar field

$$(\Box + \frac{1}{6}R)\phi = 0\,, \tag{4.28}$$

where $R = 12\kappa^2$, in conformal vacuum state.[4]

As shown by Gibbons and Hawking [6] it is a KMS state w.r.t. 'comoving (geodesic) observers' following ∂_t trajectories – detector's response is thermal, with

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi\ell}\,, \tag{4.30}$$

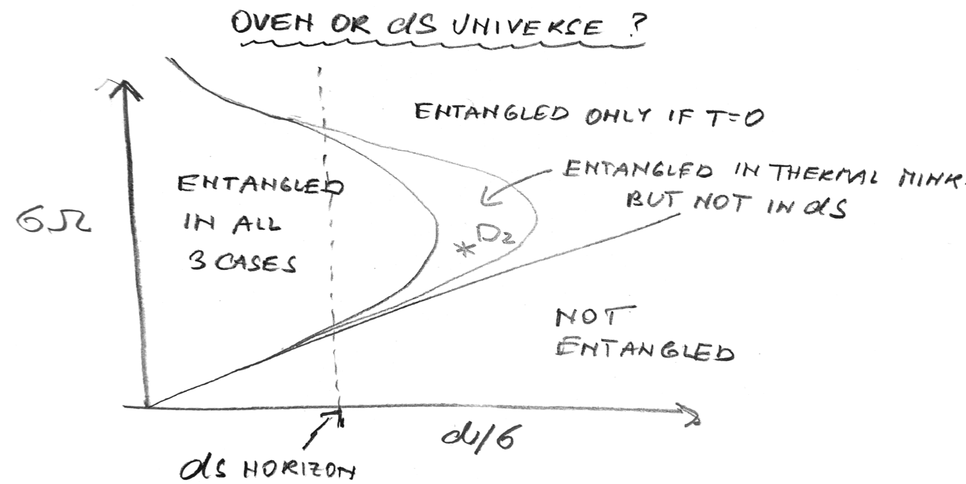
as discussed above. How does this compare to a situation where we have a thermal bath in Minkowski with the same temperature?

The responses of a single detector are identical. However, as shown by Steeg and Menucucci (2007) [18], if we switch on another detector, we find:

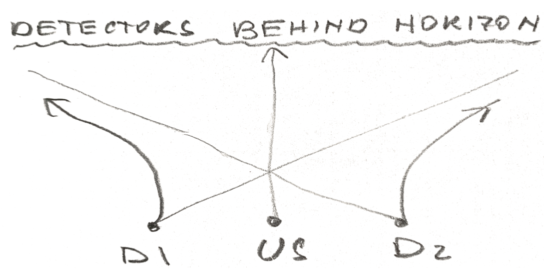
[4]For conformally coupled field, if we start in conformal vacuum in the past, we will find no particles in the future. Note also, that the 2-point function of conformal fields, in Minkowski and conformal Minkowski: $ds^2 = \Omega^2(x)ds_M^2$, are related by

$$W(x,x') = \Omega(x)^{\frac{2-d}{2}} W_M(x,x')\Omega(x)^{\frac{2-d}{2}}\,. \tag{4.29}$$

In particular, this means that the a's and a^+'s in the two cases do not mix and the fields share the 'same vacuum'.



Note that this does not support the idea that 'curvature generates entanglement'. Note also that to effectively know, the two detectors have to be behind each others horizon. Such detectors could still send us signals:



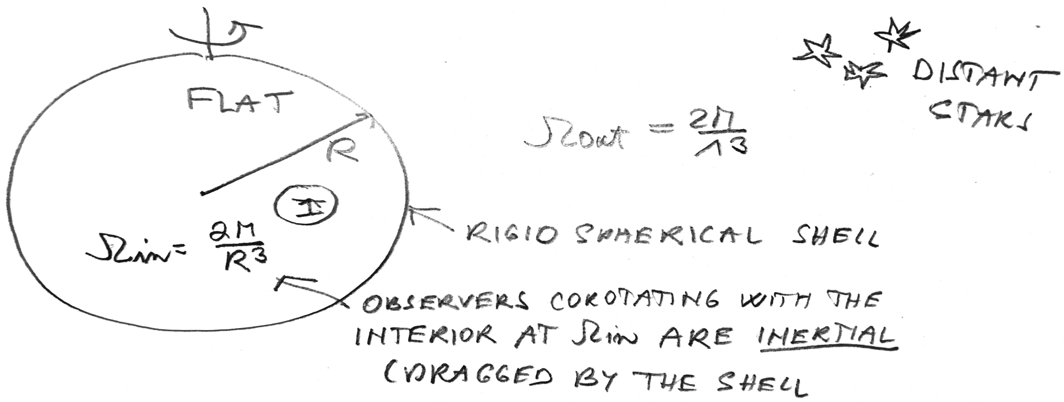
- Robb: Are we in an oven of an alien, or in an expanding Universe? In the end we can's have everything – either we are eaten for lunch or die of starvation.

Do we live inside a (rotating) shell?

A single UdW detector can determine whether we live in the Minkowski space or inside a shell; being inertial, are we static or rotating w.r.t. distant stars [19, 20]:[5]

[5] This is related to Mach's principle: "local physical laws are determined by large-scale structure of our Universe." For example, interaction with distant stars is the origin of mass inertia. See [21] for a recent modern formulation.

QUANTUM DETECTION OF FRAME DRAGGING



Other spacetime features

One can also detect spacetime topology, e.g. identified Minkowksi or Mobius strip topology [22] (this is derction-dpendent and so can recognize which direction is idnetified), or the presence of conical deficits [23].

4.4 Entanglement farming

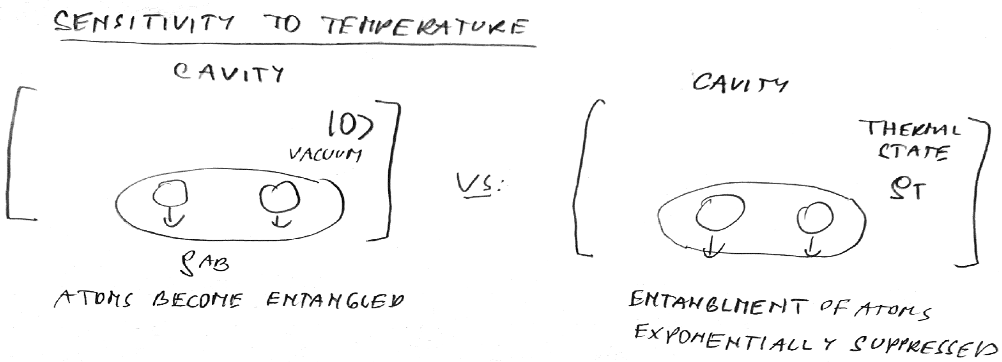
- What are the limits? Can we repeat the process cyclically? Is the vacuum entanglement in a cavity replenishable? Is there a 'Carnot-like' optimal extraction cycle?

  Can we harvest entanglement in a cavity sustainably and reliably? No, with the swapping mechanism alone: Entanglement resources get exhausted: entropy increases, heating, mixedness, ... Yes, by combining swapping and communication [24]. Need to go beyond perturbative approach (killing people with a spoon).

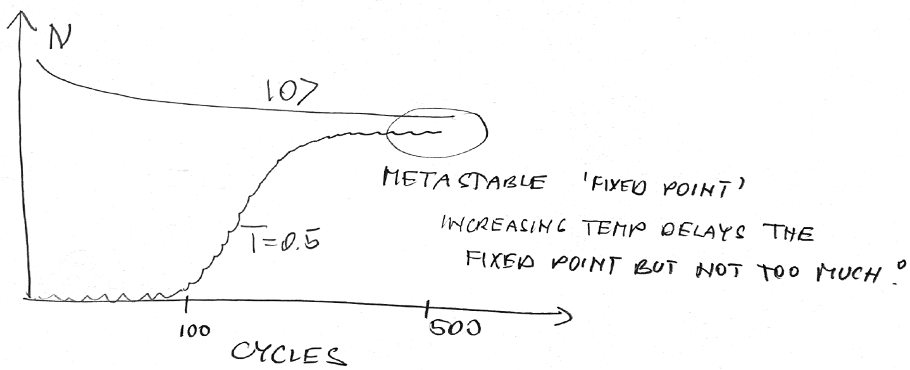
- Gaussian quantum mechanics (classical mechanics of Gaussian distributions - 2 forms instead of scalars) is a useful tool [25] – it is non-perturbative, finite-dimensional, uses symplectic form, trivial partial tracing. (Gaussian states - thermal, coherent, squeezed,...; Gaussianity preserved under quadratic Hamiltonians; this is in particular the case of a harmonic oscillator model of UdW detectors; can have entanglement in this context! See Edu's discussion on classical vs. quantum.)

- Entanglement farming. Following [24], let us imagine we have a finite optical cavity (reservoir).

  Problem: The process of 'entangling' the atoms is not robust under finite temperature – amount of entanglement extracted exponentially vanishes as the temperature increases – tough luck for experimentalists!



Protocol. Prepare many pairs of atoms initialized in the ground state and use them to 'purify the cavity state' as follows. Let the initial cavity state by arbitrary. Send the first pair and let it interact shortly. Remove the atoms, send a fresh pair, and iterate the process. In this way, we drive the state of the cavity to a (metastable) fixed point. From this fixed point we can extract the entanglement:



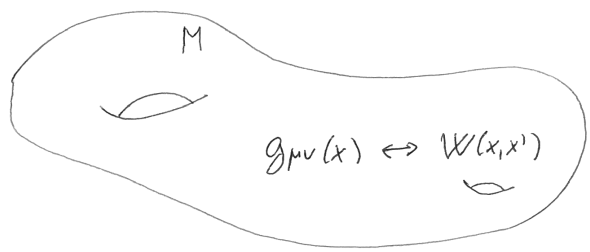
Note: Fixed point only metastable (after 10^6 cycles, we go down again – atoms first extract energy from the cavity, slow build up of high frequency modes; fixed points exist if we introduce a cut-off – the built up of energy eventually kicks you). Note also that realistic optical cavities leak out high energy modes – naturally implement UV cutoff – so we can grow entanglement. This is not really a process of cooling down (atoms do not spend enough time to thermalize with the system) – instead, we are sort of purifying the system.[6]

4.5 Spacetime from quantum fields

- So far, we recovered some spacetime features by performing a local measurement with a quantum probe. Full geometry of spacetime can be reconstructed from local measurements of quantum particle detectors.

[6]Achim Kempf's cold hammer – heats up (high frequency modes) as well as extracts energy due to thermal contact (low energy - resonant modes).

- Idea.



  - Penetrate spacetime with an array of quantum probes.
  - Measure their correlations and recover from here the Wightman function [26, 27].
  - The metric then given by the coincidence limit [28, 29]:

$$\boxed{g_{\mu\nu}(x) = \lim_{x\to x'} -\frac{1}{2}\Big(\frac{\Gamma(D/2-1)}{4\pi^{D/2}}\Big)^{2/(D-2)}\big[\partial_\mu\partial_{\nu'}(W(x,x'))^{2/(2-D)}\big]\,.} \tag{4.31}$$

    (One can show that this is independent of field state.)
  - This means that one can replace rulers and clocks of Einstein's relativity with quantum probes – in scales where the former no longer make sense.
  - Is it possible to reformulate Einstein equations in terms of the dynamics of quantum fields? Ask Rick, when he visits.

- Note. One has to make sure that
  - Using other fields (beyond scalar) one recovers the same geometry $g_{\mu\nu}$. That is, the above has to be universal (has this been checked?).
  - In non-linear classical theories (e.g. theories of non-linear electrodynamics) it may happen that their fundamental degrees of freedom propagate along different than background geometry. (For example, in the presence of strong electromagnetic fields, photons interact with other photons, and we may observe extra photon horizons, and so on.) While these are pure non-linear effects, such theories are often motivated by a desire to capture quantum field theory effects by classical description, e.g. Heisenberg–Euler theory [30]. How do we go around this problem? Is it enough to restrict to vacuum configurations, or due to quantum effects, this is important even therein?

Chapter 5: Quantum communication

5.1 Quantum collect calling

- Let us return back to two detectors interacting with the massless scalar field via

$$H_I = \sum_{\nu\in(A,B)} \lambda_\nu \chi_\nu(t) \int d^n x F_\nu(\vec{x}-\vec{x}_\nu)\hat{m}_\nu(t)\hat{\phi}(\vec{x},t)\,, \quad \hat{m}_\nu = \sigma_\nu^+ e^{i\Omega_\nu t} + \sigma_\nu^- e^{-i\Omega_\nu t}\,. \tag{5.1}$$

where $\hat{m}_\nu = \sigma_\nu^+ e^{i\Omega_\nu t} + \sigma_\nu^- e^{-i\Omega_\nu t}$. Assume uncorrelated state to start with:

$$\rho_0 = \rho_{0,AB} \otimes \rho_{0,\phi}\,, \tag{5.2}$$

where $\rho_{0,\phi}$ is an arbitrary state. Then, using the Dyson's expansion, we have

$$\rho_T = \rho_0 + \rho^{(1)} + \rho^{(2)} + O(\lambda^3)\,. \tag{5.3}$$

At $\rho^{(1)}$, 1 detector interacts with the field – no communication! Thus, leading order terms for signaling are in $\rho^{(2)}$:

$$\begin{aligned}\rho^{(2)}_{T,AB} \;=\; & \mathrm{Tr}_\phi(\rho^{(2)}_T) = \sum_{\nu,\eta}\lambda_\nu\lambda_\eta\Big[\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \chi_\nu(t')\chi_\eta(t)\hat{m}_\nu(t')\rho_{0,AB}\hat{m}_\eta(t)\mathcal{W}(\vec{x}_\eta,t,\vec{x}_\nu,t') \\ & -\int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' \chi_\nu(t)\chi_\eta(t')\Big(\hat{m}_\nu(t)\hat{m}_\eta(t')\rho_{0,AB}\mathcal{W}(\vec{x}_\nu,t,\vec{x}_\eta,t') \\ & +\rho_{0,AB}\hat{m}_\eta(t')\hat{m}_\nu(t)\mathcal{W}(\vec{x}_\eta,t',\vec{x}_\nu,t)\Big)\Big]\,.\end{aligned} \tag{5.4}$$

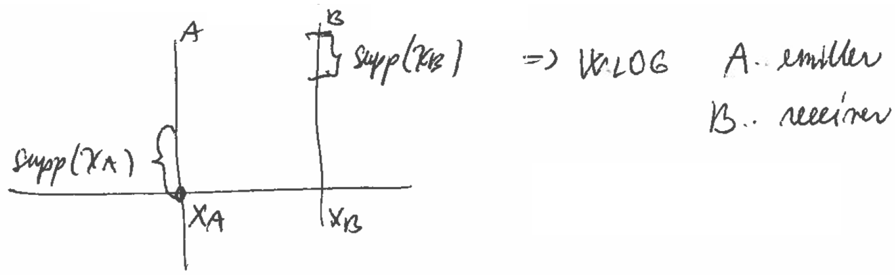
Note that the limits of the second integral (which comes from $U^{(2)}$ terms) are different than those of the first integral (which comes from $U^{(1)}$ term). Here,

$$\mathcal{W}(\vec{x}_\nu,t,\vec{x}_\eta,t') = \int d^n x d^n x' F(\vec{x}-\vec{x}_\nu)F(\vec{x}'-x_\eta)\;\underbrace{W_{\rho_{\phi,0}}(t,\vec{x},t',\vec{x}')}_{\mathrm{Tr}_\phi\left(\phi(t,\vec{x})\phi(t',\vec{x}')\rho_{\phi,0}\right)}\;. \tag{5.5}$$

- Terms involved in communication are proportional to $\lambda_A\lambda_B$. So, separate (local) noise from signal:

$$\rho^{(2)}_{T,AB} = \lambda_A\lambda_B\rho^{(2)}_{\text{pre-signal}} + \sum_\nu \lambda_\nu^2\rho^{(2)}_{\nu,\text{noise}}\,. \tag{5.6}$$

- Let's assume for simplicity that A switches on earlier than B, and that their switchings do not overlap in time:



In consequence, terms from $U^{(2)}$ with $m_A(t')$ vanish. We thus find

$$\begin{aligned}\rho^{(2)}_{\text{pre-signal}} &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \chi_A(t)\chi_B(t')\Big[\mathcal{W}(x_B,t',x_A,t)\Big(\hat{m}_A(t)\rho_{0,AB}\hat{m}_B(t') \\ &\quad -\hat{m}_B(t')m_A(t)\rho_{0,AB}\Big) + \mathcal{W}(x_A,t,x_B,t')\Big(\hat{m}_B(t')\rho_{0,AB}m_A(t) \\ &\quad -\rho_{0,AB}m_A(t)m_B(t')\Big)\Big]\,. \end{aligned} \tag{5.7}$$

This also contains harvesting (non-trivial for spacelike separations that has nothing to do with signaling)! Note also that both integrals over time can now go from $-\infty$ to ∞ by our assumption on detectors' switchings.

- Introduce

$$\rho^{(2)}_{B,\text{signal}} \equiv \mathrm{Tr}_A(\rho^{(2)}_{\text{pre-signal}})\,. \tag{5.8}$$

Contains information about A and is <u>locally accessible to B</u>! Let's assume A and B uncorrelated at the beginning:

$$\rho_{0,AB} = \rho_{0,A}\otimes\rho_{0,B}\,. \tag{5.9}$$

Then

$$\begin{aligned}\rho^{(2)}_{B,\text{signal}} &= \int_{-\infty}^{\infty} dtdt'\chi_A(t)\chi_B(t')\mathrm{Tr}_A(\hat{m}_A\rho_{0,A})2i\mathrm{Im}[\mathcal{W}(x_A,t,x_B,t')] \\ &\quad\times\big(\hat{m}_B(t')\rho_{0,B}-\rho_{0,B}\hat{m}_B(t')\big)\,, \end{aligned} \tag{5.10}$$

where we used that $W^*(t,x,t',x') = W(t',x',t,x)$, and that $c - c^* = 2i\mathrm{Im}(c)$.

Using further the matrix representation

$$|g\rangle = \begin{pmatrix}1\\0\end{pmatrix}\,,\quad |e\rangle = \begin{pmatrix}0\\1\end{pmatrix}\,,\quad m_\nu(t) = \begin{pmatrix}0 & e^{-i\Omega_\nu t}\\ e^{i\Omega_\nu t} & 0\end{pmatrix}\,, \tag{5.11}$$

and

$$\rho_{0,AB} = \rho_{0,A}\otimes\rho_{0,B} = \begin{pmatrix}a_A & b_A\\ b_A^* & 1-a_A\end{pmatrix}\otimes\begin{pmatrix}a_B & b_B\\ b_B^* & 1-a_B\end{pmatrix}\,, \tag{5.12}$$

we find $\mathrm{Tr}_A(\hat{m}_A(t)\rho_{0,A}) = 2\mathrm{Re}(b_A e^{i\Omega_A t})$, and

$$[\hat{m}_B(t'), \rho_{0,B}] = \begin{pmatrix} -2i\mathrm{Im}(b_B e^{i\Omega_B t'}) & e^{-i\Omega_B t'}(1-2a_B) \\ (\ )* & 2i\mathrm{Im}(b_B e^{i\Omega_B t'}) \end{pmatrix} . \tag{5.13}$$

Remember that the imaginary part of $\mathcal{W}$ is the smeared commutator:

$$\mathcal{C}(t,t') = i\int d^n x d^n x' F(x-x_A)F(x-x_B) \underbrace{\langle[\phi(t,x),\phi(t',x')]\rangle_{\rho_{\phi,0}}}_{C(t,x,t',x')} , \tag{5.14}$$

where $\langle[\ ,\ ]\rangle$ is a c-number (independent of the initial state of the field). We thus arrive at a simple conclusion that

$$\boxed{\rho^{(2)}_{B,\mathrm{signal}} = 2\int_{-\infty}^{\infty} dt dt' \chi_A(t)\chi_B(t')\mathrm{Re}(b_A e^{i\Omega_A t})[\hat{m}_B(t'),\rho_{0,B}]\mathcal{C}(t,t') .} \tag{5.15}$$

This gives the probability of signalling in the leading order of the perturbation theory. See [31, 32] for more details.

- Some perhaps surprising features:
  - If spacelike separated point, $[\ ,\ ] = 0$ (micro-causality); B knows nothing about A unless causally connected![1]
  - Note that for this communication we need coherences, that is, we need quantum antennas:

    $$|\psi_A\rangle = \alpha_A|e_A\rangle + \beta_A|g_A\rangle , \quad |\psi_B\rangle = \alpha_B|e_B\rangle + \beta_B|g_B\rangle . \tag{5.16}$$

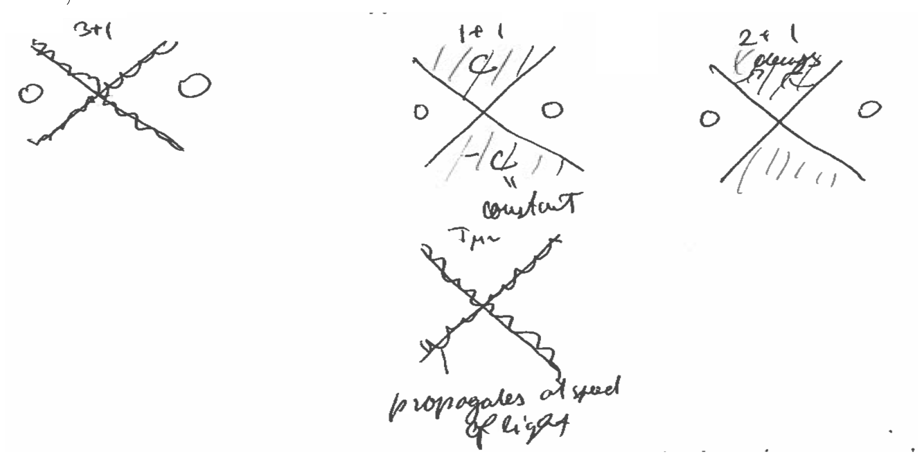
    Classical antenna does not have b-terms! So the effect is not there!
  - Probability of exchanging a photon, while field is in vacuum is of order 4 (exchange of energy)! We need to de-excite A while ϕ in $|0\rangle$ (order 2), and excite B while ϕ excited in $1\rangle$ (order 2), see formulas from the first lecture ($P \propto \lambda^2$)!

    Here it is order 2! This is not a real photon exchange. It is in a way similar to Casimir type interaction: A interacted with ϕ and altered it, now we pick up that altered field in B.
  - Since C is independent of the initial state – we do not mind a noisy field – the signalling does not care! This is intuitively very unusual.

[1] Had we used the detector in rotating wave approximation in quantum optics – would find that we can have communication for spacelike separated detectors [33]. The same goes for single mode approximation.

- Commutators. In $(3+1)$, commutator is only supported on a light cone. Namely, we have the following:
  Huygens principle: *The Green's function of the (massless) wave equation in (3+1) Minkowski space has support only on the light cone. Hence, any disturbances propagate strictly along null geodesics (at the speed of light).*[2]

  How general is this? Even in Minkowski, if number of spatial dimensions is even or one, it is not satisfied!:



  In $(1+1)$ energy propagates on a light cone, but communication happens inside it (it cannot be due to energy exchange, rather a Casimir force). So we can miss the energy signal, but still know about A.

  If beyond Minkowski, it generally does not work! Namely, if there is curvature (and no full conformal invariance) it is probably violated. Thence, in general spacetimes there will be a leakage of info towards the inside of the light-cone decoupled from energy propagation. We shall see an explicit example below.

- Summary: We have challenged the following 'reasonable' properties of wireless communication:

  - Communication mediated by 'real' energy-carrying quanta (emitter emits photon, receiver captures it). Info reaches when energy reaches.
  - Communication only possible at the speed of light (miss the beam – miss the message).

  In flat (3+1) this is the case, but not true for optical fibres, in curved spacetime, important in cosmology (after glow,...). Instead:

[2]Field commutator is the radiation Green's function (retarded-advanced):

$$\nabla^2 G(x,x') = -4\pi\delta(x-x')\,, \quad [\phi(x),\phi(x')] = \frac{i}{4\pi}G(x,x')\,. \tag{5.17}$$

However, Wightmann is a kernel ($\nabla^2 W = 0$).

- Info propagates arbitrarily slow (even for massless fields).
- Can recover message even if the beam is missed.
- Info flow is not supported by real quanta (photons) flow.
- There is info flow in the absence of energy flow.

5.2 Example: Echoes from Early Universe

- Consider a spatially flat open FRW Universe:

$$ds^2 = a(\eta)^2(-d\eta^2 + dr^2 + r^2 d\Omega^2)\,. \tag{5.18}$$

Here, cosmological time t is related to the conformal time η via $dt = a d\eta$, or

$$a \propto \eta^{\alpha+1/2} \propto t^{\frac{2\alpha+1}{2\alpha+3}}\,, \tag{5.19}$$

where $\alpha = (3-3w)/(6w+2)$ and $p = w\rho$ is the equation of state the perfect fluid. Consider test scalar field in Bunch–Davies vacuum.[3] This can either be minimally coupled ($\xi = 0$) or conformal ($\xi = 1$):

$$\Big(\Box + \frac{1}{6}\xi R\Big)\phi = 0\,. \tag{5.20}$$

The field couples to 2 UdW detectors (antennas) in the following initial states:

$$|\psi_\nu\rangle = \alpha_\nu|e_\nu\rangle + \beta_\nu|g_\nu\rangle\,. \tag{5.21}$$

with the following communication protocol: i) Alice encodes "1" by coupling her detector A to the field and "0" by not coupling it. ii) Later Bob switches B and measures its energy. If B is excited, Bob interprets that as "1", and "0" otherwise.

- Thus we have the initial state:

$$\rho_0 = |\psi_0\rangle\langle\psi_0| \otimes \rho_\phi\,, \quad |\psi_0\rangle = (\alpha_A|e_A\rangle + \beta_A|g_A\rangle) \otimes (\alpha_B|e_B\rangle + \beta_B|g_B\rangle)\,. \tag{5.22}$$

Bob's evolved state is

$$\rho_{Bf} = \mathrm{Tr}_A \mathrm{Tr}_\phi(U\rho_0 U^+)\,, \tag{5.23}$$

and the probability of exciting Bob's detector is given by

$$P_B(t) = \underbrace{|\alpha_B|^2}_{\text{initial prob}} + \underbrace{O(\lambda_B) + O(\lambda_B^2) + O(\lambda_B)^4}_{\text{local noise}} + \underbrace{O(\lambda_A\lambda_B) + O(\lambda_A^2\lambda_B^2)}_{S\text{:Casimir-like+real photon exchange}}\,. \tag{5.24}$$

[3] This is an example of adiabatic vacuum (a 'closest' to conformal vacuum): starting with this vacuum as I evolve I have the smallest possible number of particles in the future. In BH physics we have Boulvare, Hartle–Hawking, or Unruh vacuum. Some diverge at the horizon, other at infinity, and then something in between.

Here, S is the signalling estimator (influence of the presence of A on B) $S = \lambda_A \lambda_B S_2 + O(\lambda^4)$, where as derived in (5.15) (note the appearance of additional scaling factors a^3 due to the proper volume element in FRW):

$$S_2 = 4\int a^3(t)d^3x dt \int a^3(t')d^3x'dt'\chi_A(t)\chi_B(t')\mathrm{Re}(\alpha_A^*\beta_A e^{i\Omega_A t})F(x-x_A,t) \times F(x'-x_B,t')\mathrm{Re}(\alpha_B^*\beta_B e^{i\Omega_B t'})C(t,x,t',x')\,. \tag{5.25}$$

- For conformal scalar we then find

$$C(t,x,t',x') = \frac{i}{4\pi}\Big[\frac{\delta(\Delta\eta + |x-x'|) - \delta(\Delta\eta - |x-x'|)}{a(t)a(t')|x-x'|}\Big]\,. \tag{5.26}$$

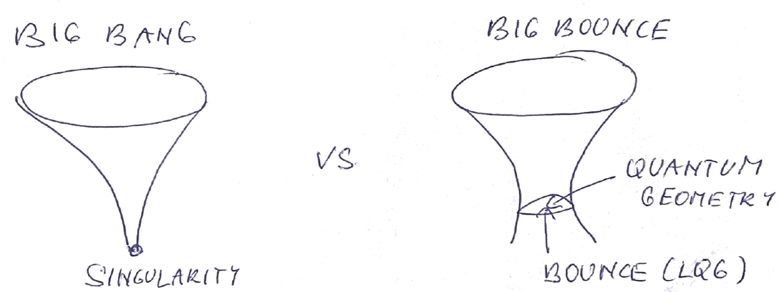
This is a 'boring' propagator, that i) has only support on the lightcone and ii) decays with spatial separation as $1/|x-x'|$. On the other hand, in the minimally coupled case, one obtains a complicated propagator (with Bessel functions), which in particular case of the matter dominated universe ($\alpha = 2$) simplifies to

$$C(t,x,t',x') = \frac{i}{4\pi}\Big(\text{conformal piece} + \frac{\theta(-\Delta\eta - |x-x'|) - \theta(\Delta\eta - |x-x'|)}{a(t)a'(t')\eta(t)\eta(t')}\Big)\,. \tag{5.27}$$

Obviously, the second term violates Huygens principle, as it has timelike leakage. Moreover, it does not decay with spatial separation.

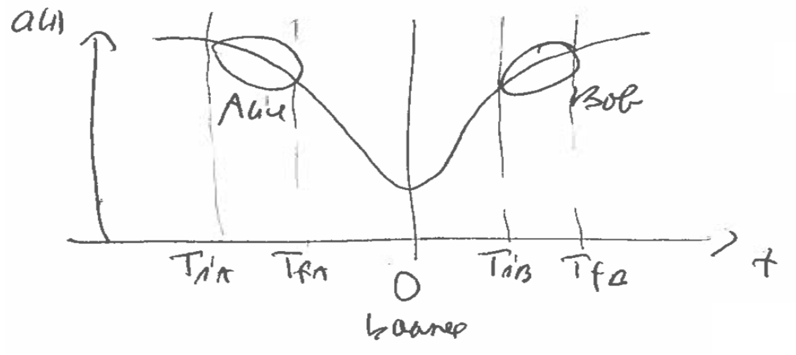
In the case of de Sitter Universe, we have exponential expansion, and the timelike signal does not decay in time! Can we detect signals from the Early Universe (not carried by light)?

- How much information survives cosmological cataclysm? Quantum Bounce vs. Big Bang?

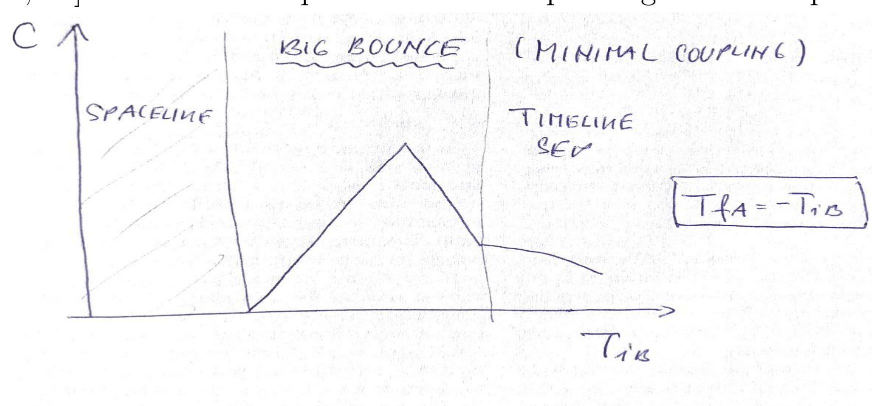


Can we get an RQI echo of an ancient civilization? (Atoms or any complex systems will not survive a quantum bounce.) They will encode the info in quantum field – due to unitarity such info cannot be destroyed (it can only be scrambled).

Namely, consider the following scenario:



Provided the spacetime still exists during the bounce (although with some quantum corrections), if we violate Huygens principle, we can still get signal S, see [34, 35] for details and plots of the corresponding channel capacities:



Here, the channel capacity, c describes how much information can be sent, which is given by

$$c \propto \lambda_A^2 \lambda_B^2 \Big(\frac{S_2}{4|\alpha_B||\beta_B|} \Big)^2 + O(\lambda^6) \tag{5.28}$$

(for the noisy asymmetric binary channel).

Note that already 1 detector switched on from Early Times till now can tell the difference between Big Bang and Big Bounce scenarios via the Gibbons–Hawking effect, e.g. [36, 37, 38]. In this case, however, we need a detector that can survive the Early times – not an atom, but perhaps a quantum field.

Chapter 6: Quantum energy teleportation

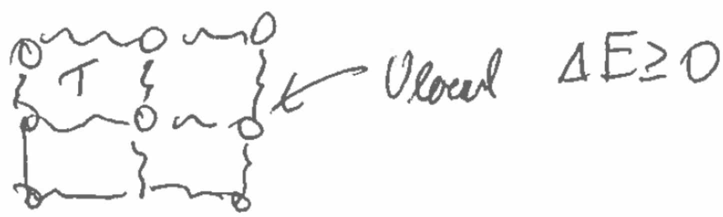
6.1 A few words about thermodynamics

- Thermal states are passive: *all unitary operations have a positive energy cost.* Namely, as shown by Pusz and Woronowicz in 1978, for arbitrary unitary operation U we have [39]:

$$\rho' = U\rho(\beta)U^{+} \quad \Rightarrow \quad \Delta E = \mathrm{Tr}(H\rho') - \mathrm{Tr}(H\rho(\beta)) \geq 0\,. \tag{6.1}$$

  This implies stability of thermal equilibrium state under unitary disturbances. (Classically, I cannot run a heat engine with just 1 thermal bath – cannot extract work from a thermal state.)

  In particular, this is true for the local unitary operations (local passivity):

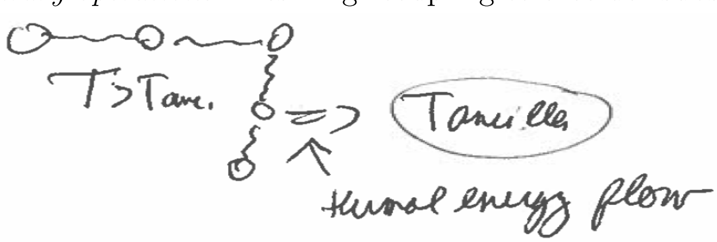


- Note: knowing ρ does not give me T, unless I also know the Hamiltonian:

$$\rho(\beta) = \frac{1}{Z}e^{-\beta H} \tag{6.2}$$

  remains invariant under $\beta \to \lambda\beta$ and $H \to H/\lambda$. Thence, thermometers need to register the dynamics! (Thermometers cannot take a snapshot, they need to take the video and watch it – they register dynamics.)

- Process of thermalizations. Can we extract energy from a thermal state if we allow *arbitrary operations*? Yes! E.g. coupling to a colder subsystem (thermometer):



We thus can extract energy from the system, as the state evolves: $\rho(\beta) \to \Gamma(\rho(\beta))$ and

$$\Delta E = \mathrm{Tr}(\Gamma(\rho(\beta)H) - \mathrm{Tr}(H\rho(\beta)) < 0\,. \tag{6.3}$$

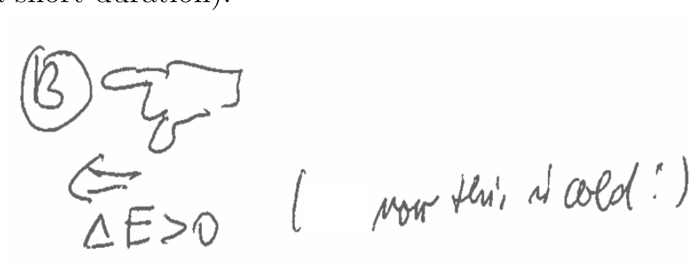
Thermal energy flow is non-local! The process takes a while and involves (the dynamics of) the whole system. (In particular, thermometer sees the dynamics of the whole system.)[1]

- Strong local passivity. Can we extract energy purely locally by using a colder Masahiro Hotta's hand?

Just touching quickly and locally (but not necessarily unitarily) – will the hand get burned? Not in general! In fact the hand gets cooler – strong local passivity!

Theorem: If the ground state contains max-rank entanglement (involves the whole Hilbert space, but not necessarily maximal entanglement), there exists a temperature T^* (dependent on the system), such that for $T < T^*$ (where T is the temperature of the system) it is not possible to extract work from the system through any local operation [40, 41] (even if the hand was actually colder than the system)!

In particular: can we extract energy from the ground state of vacuum with $T = 0$? (It is maximally entangled.) Touching it, we get colder! (Strong point contact in a short duration):



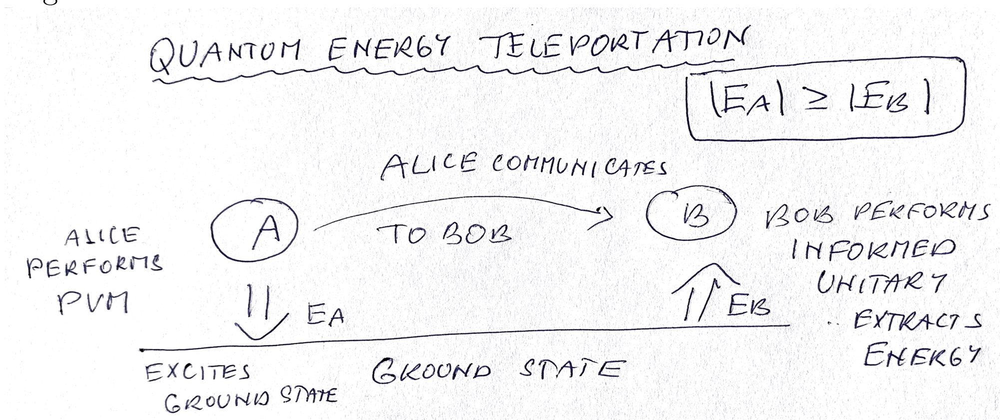
[1] In quantum information science, local operations, that is operations that act independently on each subsystem, do not affect the entanglement. Here is an example of a global operator:

$$\hat{W}\psi(x) = \int dx' W(x,x')\psi(x')\,. \tag{6.4}$$

Such operations can affect entanglement.

6.2 Breaking strong local passivity: quantum energy teleportation

- Can we cheat this? We can use ground state entanglement as a resource for local energy extraction! Namely, if we assist *local operations with classical communication* (LOCC), it is possible to extract energy with local operations. Even from the ground state:



"Alice does the projective measurement and gives Bob information what to do to extract energy."

Intuition: Because of the ground state entanglement, the measurement in A provides information about fluctuations in B. "Unlocking zero-point fluctuations at a cost." Extract energy with local operations assisted by the info in A. Energy does not travel from A to B! Info does!

- Minimal QET model. Following [40], let us consider two qubits A and B, and the following Hamiltonian:

$$H = H_A + H_B + V\,, \tag{6.5}$$

where

$$\begin{aligned} H_A &= h\sigma_z^A + f(h,k)\mathbb{I}\,, \quad H_B = h\sigma_z^B + f(h,k)\mathbb{I}\,, \\ V &= 2\Big[k\sigma_x^A\sigma_x^B + \frac{k^2}{h^2}f(h,k)\mathbb{I}\Big]\,. \end{aligned} \tag{6.6}$$

Here, $f(h,k)\mathbb{I}$ just rescales the groundstate energy of the system, and V is designed not to commute with the free Hamiltonians. If we pick

$$f(h,k) = \frac{h^2}{\sqrt{h^2+k^2}}\,, \tag{6.7}$$

then

$$\langle g|H_A|g\rangle = \langle g|H_B|g\rangle = \langle g|V|g\rangle = 0\,, \tag{6.8}$$

and we know that H is non-negative. Here $|g\rangle$ is the ground state of the whole system:

$$|g\rangle = \frac{1}{\sqrt{2}}\Big(\sqrt{1-\frac{f}{h}}|1\rangle_A|1\rangle_B - \sqrt{1+\frac{f}{h}}|0\rangle_A|0\rangle_B\Big)\,, \tag{6.9}$$

where

$$\sigma_z^\nu|0\rangle_\nu = -|0\rangle_\nu\,, \quad \sigma_z^\nu|1\rangle_\nu = |1\rangle_\nu \tag{6.10}$$

for $\nu = A, B$. Note that when $k = 0$ (no interaction), we have a ground state of σ_z's and there is no entanglement. When $k \to \infty$ ground state is maximally entangled. The state satisfies the assumptions of the theorem. Thence, there exists T^*.

- Protocol.
  i) We start from the ground state $|g\rangle$.
  ii) Alice carries out a PVM of σ_x^A and repeats it many times. This will have an average energy cost $E_{p_A} > 0$. Thus increase the energy of the system.
  iii) The result of the measurement (1 bit: $\alpha = \pm 1$) is announced to Bob through a classical channel. This can be fast – timescale much smaller than $1/k$ (timescale of H).
  iv) With the information of α, Bob carries out an informed local unitary $U(\alpha)$. After many repetitions of this protocol, the average energy cost of Bob's unitary will be negative (B looks like in a ground state).

- Step 1. Average energy cost of PVM on A. Alice measures σ_x^A and obtains $\alpha = \pm 1$. In a single shot PVM, the post measurement state is

$$|\psi_{PM}(\alpha)\rangle = \frac{1}{\sqrt{p(\alpha)}}P_A(\alpha)|g\rangle\,, \quad p(\alpha) = \langle g|P_A(\alpha)|g\rangle\,, \tag{6.11}$$

where

$$P_A(\alpha) = \frac{1}{2}(\mathbb{I} + \alpha\sigma_x^A)\,. \tag{6.12}$$

If we repeat this projection on an ensemble of identical setups, the post-measurement state of the ensemble is

$$\rho_1 = \sum_{\alpha=\pm1} p(\alpha)|\psi_{PM}(\alpha)\rangle\langle\psi_{PM}(\alpha)| = \sum_{\alpha=\pm1} P_A(\alpha)|g\rangle\langle g|P_A(\alpha)\,. \tag{6.13}$$

The average energy cost of step 1 is

$$E_{P_A} = \mathrm{Tr}(\rho_1 H) - \underbrace{\mathrm{Tr}(\rho_0 H)}_{0} = \sum_{\alpha=\pm1}\langle g|P_A(\alpha)\underbrace{H}_{H_A+H_B+V}P_A(\alpha)|g\rangle\,. \tag{6.14}$$

Lemma: We have

$$[P_A(\alpha), H_B] = 0 = [P_A(\alpha), V]\,, \tag{6.15}$$

$$\langle g|\sigma_x^A|g\rangle = \langle g|\Big(|1\rangle_A\langle 0| + |0\rangle_A\langle 1|\Big)|g\rangle = 0\,,$$

$$\langle g|\sigma_z^B\sigma_x^A|g\rangle = \langle g|\Big(|1\rangle_B\langle 1| - |0\rangle_B\langle 0|\Big)\otimes\Big(|1\rangle_A\langle 0| + |0\rangle_A\langle 1|\Big)|g\rangle = 0\,. \tag{6.16}$$

Using this lemma, we find:

$$\begin{aligned}\sum_{\alpha=\pm1}\langle g|P_A(\alpha)H_BP_A(\alpha)|g\rangle &= \sum_{\alpha=\pm1}\langle g|H_BP_A(\alpha)|g\rangle = \frac{1}{2}\sum_{\alpha=\pm1}\Big(\underbrace{\langle g|H_B|g\rangle}_{0} + \alpha\langle g|H_B\sigma_x^A|g\rangle\Big)\\ &= \frac{1}{2}\sum_{\alpha=\pm1}\alpha\,\underbrace{\langle g|H_B\sigma_x^A|g\rangle}_{\propto\langle g|\sigma_z^B\sigma_x^A|g\rangle} = 0\,. \end{aligned} \tag{6.17}$$

Similarly

$$\begin{aligned}\sum_{\alpha=\pm1}\langle g|P_A(\alpha)VP_A(\alpha)|g\rangle &= \sum_{\alpha=\pm1}\langle g|VP_A(\alpha)|g\rangle\\ &= \frac{1}{2}\sum_{\alpha=\pm1}\Big(\underbrace{\langle g|V|g\rangle}_{0} + \alpha\,\underbrace{\langle g|V\sigma_x^A|g\rangle}_{\propto\langle g|\sigma_x^B|g\rangle}\Big) = 0\,. \end{aligned} \tag{6.18}$$

So we have

$$E_{P_A} = \mathrm{Tr}(\rho_1 H) - \mathrm{Tr}(|g\rangle\langle g|H) = \sum_{\alpha=\pm1}\langle g|P_A(\alpha)H_AP_A(\alpha)|g\rangle = f(h,k) > 0\,. \tag{6.19}$$

So on average, changing the ground state costs energy!

- Step 2. Classical communication of α and (informed) local unitary on B:

$$U_B(\alpha) = \cos\theta\mathbb{I} - i\alpha\sin\theta\sigma_B^y\,, \tag{6.20}$$

where

$$\cos 2\theta = \frac{h^2+2k^2}{\sqrt{(h^2+2k^2)^2+h^2k^2}}\,, \quad \sin 2\theta = \frac{hk}{\sqrt{(h^2+2k^2)^2+h^2k^2}}\,. \tag{6.21}$$

In a single shot application of the protocol: start from $|g\rangle$, apply PVM $P_A(\alpha)$, and then $U_B(\alpha)$:

$$|\psi_2\rangle = \frac{1}{\sqrt{p(\alpha)}}U_B(\alpha)P_A(\alpha)|g\rangle\,. \tag{6.22}$$

The ensemble is thus

$$\rho_2 = \sum_{\alpha=\pm1}U_B(\alpha)P_A(\alpha)|g\rangle\langle g|P_A(\alpha)U_B^{+}(\alpha)\,. \tag{6.23}$$

The energy cost of applying the unitary is then

$$E_{U_B} = \mathrm{Tr}(\rho_2 H) - \underbrace{\mathrm{Tr}(\rho_1 H)}_{E_{P_A}} \tag{6.24}$$

To calculate

$$\mathrm{Tr}(\rho_2 H) = \sum_{\alpha=\pm 1} \langle g|P_A(\alpha)U_B^{+}(\alpha)HU_B(\alpha)P_A(\alpha)|g\rangle\,, \tag{6.25}$$

is some work. To aid this calculation, we can use that apart from (6.15), we also have $[H_A, U_B] = 0$. In the end this yields

$$E_{U_B} = -\frac{1}{h^2+k^2}\Big[hk\sin 2\theta - (h^2+k^2)(1-\cos 2\theta)\Big]\,. \tag{6.26}$$

If $0 < \theta \ll 1$, this yields

$$E_{U_B} \approx -\frac{2hk\theta}{h^2+k^2} < 0\,. \tag{6.27}$$

We are unlocking the zero point fluctuations energy. B acts on what looks like a ground state for him (communication is much faster than the energy transfer). We are breaking the strong passivity of the ground state by local operations! No matter how negative, we always have

$$|E_{U_B}| \leq |E_{P_A}|\,. \tag{6.28}$$

Energy is not coming from A, rather it is 'energy teleportation'.

- Natural energy flow from A to B: how fast is this?

$$\langle H_B(t)\rangle_{\rho_1} = \sum_{\alpha=\pm 1} \langle g|P_A(\alpha)e^{iHt}H_B e^{-iHt}P_A(\alpha)|g\rangle = \frac{1}{2}f\big[1-\cos(4kt)\big]\,, \tag{6.29}$$

which has characteristic speed

$$\frac{1}{k}\,. \tag{6.30}$$

We may also need the interaction part energy, but we find

$$\langle V(t)\rangle_{\rho_1} = \sum_{\alpha=\pm 1} \langle g|P_A(\alpha)e^{iHt}Ve^{-iHt}P_A(\alpha)|g\rangle = 0\,. \tag{6.31}$$

Thus, communication has to happen faster than $1/k$. This means that *QET can be arbitrarily faster than the natural energy flow!*

- Can the protocol work if the bit from Alice is lost? If Bob does not know α, but still does a local unitary W_B, which does not depend on α, in a single shot we get:

$$|\psi_2\rangle = \frac{1}{\sqrt{p(\alpha)}}W_B P_A(\alpha)|g\rangle\,, \tag{6.32}$$

and repeat many times:

$$\rho_2 = \sum_{\alpha=\pm 1} W_B P_A(\alpha)|g\rangle\langle g|P_A(\alpha)W_B^+ = W_B \underbrace{\Big(\sum_{\alpha=\pm 1} P_A(\alpha)|g\rangle\langle g|P_A(\alpha)\Big)}_{\rho_1} W_B^+ \,. \tag{6.33}$$

Thus

$$\begin{aligned} E_{W_B} &= \mathrm{Tr}(\rho_2 H) - \underbrace{\mathrm{Tr}(\rho_1 H)}_{E_{P_A}} = \sum_{\alpha=\pm 1} \langle g|P_A(\alpha)W_B^+(H_B+V)W_B P_A(\alpha)|g\rangle \\ &= \langle g| \sum_{\alpha=\pm 1} \underbrace{P_A^2(\alpha)}_{P_A(\alpha)} W_B^+(H_B+V)W_B|g\rangle = \langle g|W_B^+(H_B+V)W_B|g\rangle \,. \end{aligned} \tag{6.34}$$

However, since $[H_A, W_B] = 0$, we have

$$\langle g|W_B^+ H_A W_B|g\rangle = \langle g|H_A|g\rangle = 0 \,. \tag{6.35}$$

Thus,

$$E_{W_B} = \langle g|W_B^+(H_B + V + H_A)W_B|g\rangle = \langle g|W_B^+ H W_B|g\rangle \geq 0 \,, \tag{6.36}$$

as H is a non-negative operator. Therefore, we see that exchanging the information is crucial for QET!

6.3 Warping the fabric of spacetime

- We can prepare states of spacetime, e.g. [42]. If the *Weak Energy Condition (WEC)*:

$$\rho \equiv T_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \,, \quad \text{for every timelike vector } \xi^\mu \,, \tag{6.37}$$

is violated, we can have exotic solutions, such as wormholes, warp drives, anti-gravity/screening,...

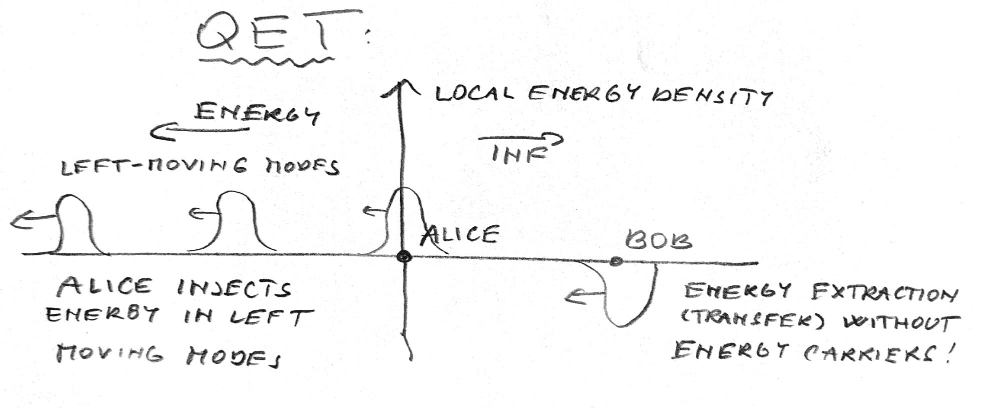
However, it is well known that quantum fields can violate AWEC, e.g. [43, 44, 45]:

$$\Delta\hat{\rho} = \frac{\tau_0}{\pi}\int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu}\xi^\mu\xi^\nu\rangle}{\tau^2+\tau_0^2} \geq -\frac{3}{32\pi^2\tau_0^4} \,. \tag{6.38}$$

Note the Lorentzian weight $\tau_0/[\pi(\tau^2+\tau_0^2)]$, determining the effective probing of the energy density (τ_0 corresponds to the duration of reading).[2]

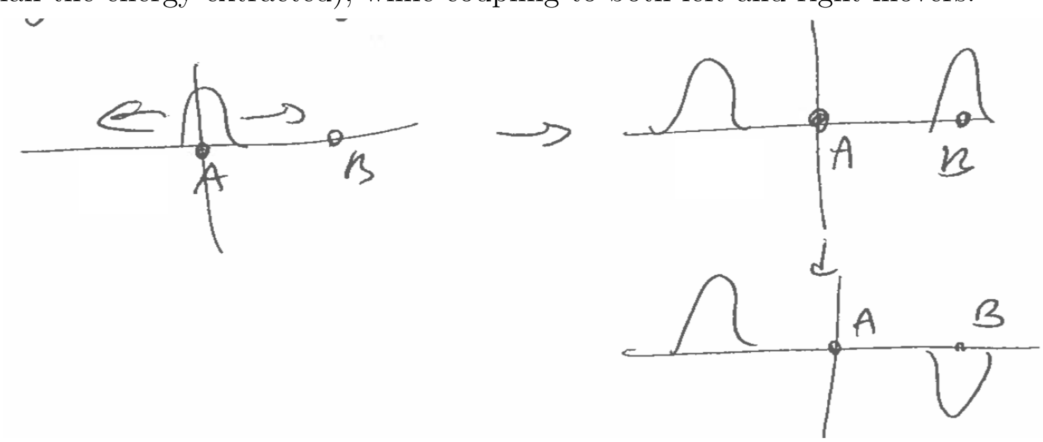
[2] See also [46] for Kip Thorne's discussion of a possibility of violating energy conditions with Casimir effect (accelerating mirrors). But that is tough.

- QET: we can unlock zero-point energy consuming entanglement (correlations). Instead of two qubits let's do it with quantum fields, replacing projective measurements with atoms (detectors) – Hotta 2008 [47]:

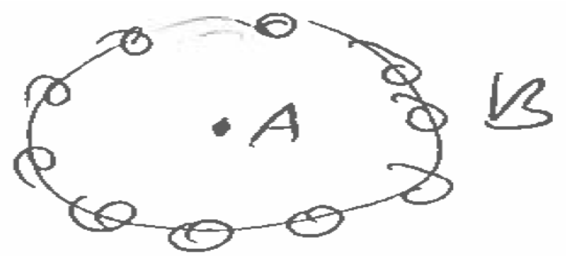


Here, Alice measures – injects energy in left-moving modes, and sends a signal to the right to Bob who extracts energy (also from left-moving modes). Thus, in this setup energy always flows to the left, while information flows to the right. The energy extracted is smaller than the one injected.

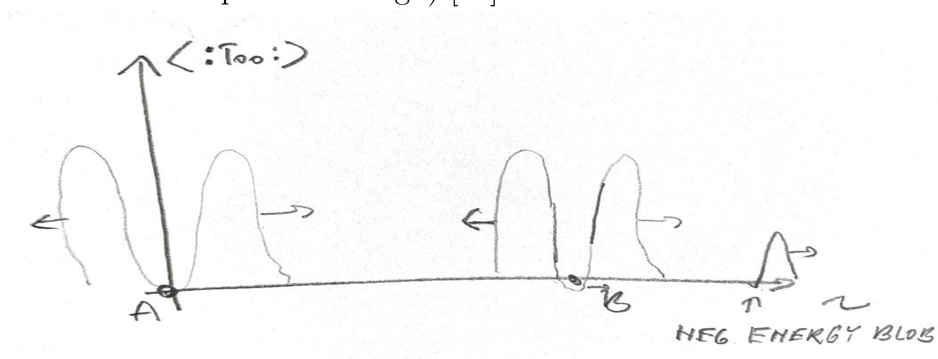
- Engineering energy densities in 3+1, We will focus on the state of the field (rather than the energy extracted), while coupling to both left and right movers:



Since everything is relativistic now, the information to B arrives at the same time as the excited energy packet from Alice – need to overcome the positive energy due to Alice! Moreover, in 3+1 we need a distribution of detectors (cannot clone states) – trick with many Bobs and Alices:



Protocol: i) Alice measures the field by coupling an atom to it ii) Alice measures her non-relativistic atom iii) Alice broadcasts her results to agency of Bobs iv) Bob's agents use that info to prepare atoms and couple to the field. In this way Bob can create a negative energy state with as much negative energy as we want (provided we compress it enough) [42]:



Violates quantum inequalities optimally and saturates quantum interest conjecture.[3]

- Summary.
  - QET can be used to operationally generate negative energy distributions. Does so 'consuming' space-like vacuum entanglement.
  - The negative energy packets are accompanied by positive energy packets (quantum interest conjecture). This suggests interesting scenarios under gravitational backreaction.
  - QET protocol scaling saturates the Quantum Interest Conjecture. QET protocol scaling optimally violates AWEC.
- *Remember: not everything that is allowed by laws of physics happens – I am allowed to get a Nobel Prize, GR allows for it, but it is unlikely to happen! And that is all folks!*

[3]This conjecture states that a positive energy pulse must overcompensate the negative energy pulse by an amount which is a monotonically increasing function of the pulse separation.

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