Sférické souřadnice $(r,\ \vartheta,\ \varphi)$

Vztah ke kartézským souřadnicím:

$$x = r\sin\vartheta\,\cos\varphi \qquad r = \sqrt{x^2+y^2+z^2}$$
$$y = r\sin\vartheta\,\sin\varphi \qquad \vartheta = \arctan\frac{\sqrt{x^2+y^2}}{z}$$
$$z = r\cos\vartheta \qquad \varphi = \arctan\frac{y}{x}$$

Metrika a objemový element:

$$\boldsymbol{g} = \mathbf{d}r\,\mathbf{d}r + r^2\,\mathbf{d}\vartheta\,\mathbf{d}\vartheta + r^2\sin^2\vartheta\;\mathbf{d}\varphi\,\mathbf{d}\varphi \qquad dV = r^2\sin\vartheta\;dr\,d\vartheta\,d\varphi$$

Laméovy koeficienty:

$$h_r = 1 \quad,\quad h_\vartheta = r \quad,\quad h_\varphi = r\sin\vartheta$$

Triáda (normovaná báze):

$$\boldsymbol{e}_{\hat r} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} \qquad \boldsymbol{e}_{\hat\vartheta} = \frac{1}{r}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} \qquad \boldsymbol{e}_{\hat\varphi} = \frac{1}{r\sin\vartheta}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$
$$\boldsymbol{e}^{\hat r} = \mathbf{d}r \qquad \boldsymbol{e}^{\hat\vartheta} = r\,\mathbf{d}\vartheta \qquad \boldsymbol{e}^{\hat\varphi} = r\sin\vartheta\;\mathbf{d}\varphi$$

Vztah souřadnic tenzorů (různé druhy indexů):

$$a^r = a^{\hat r} = a_{\hat r} = a_r$$
$$r\,a^\vartheta = a^{\hat\vartheta} = a_{\hat\vartheta} = \frac{1}{r}\,a_\vartheta$$
$$r\sin\vartheta\;a^\varphi = a^{\hat\varphi} = a_{\hat\varphi} = \frac{1}{r\sin\vartheta}\,a_\varphi$$

Vektorové násobení a antisymetrický tenzor:

$$\boldsymbol{\epsilon} = r^2\sin\vartheta\;\mathbf{d}r\wedge\mathbf{d}\vartheta\wedge\mathbf{d}\varphi$$
$$\epsilon_{r\vartheta\varphi} = r^2\sin\vartheta \qquad \epsilon_{\hat r\hat\vartheta\hat\varphi} = 1 \qquad \epsilon^{r\vartheta\varphi} = \frac{1}{r^2\sin\vartheta}$$
$$\epsilon^r{}_{\vartheta\varphi} = r^2\sin\vartheta \qquad \epsilon^\vartheta{}_{\varphi r} = \sin\vartheta \qquad \epsilon^\varphi{}_{r\vartheta} = \frac{1}{\sin\vartheta}$$

$$\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{b}$$
$$c^r = r^2\sin\vartheta\,(a^\vartheta b^\varphi - a^\varphi b^\vartheta) \qquad c^\vartheta = \sin\vartheta\,(a^\varphi b^r - a^r b^\varphi) \qquad c^\varphi = \frac{1}{\sin\vartheta}\,(a^r b^\vartheta - a^\vartheta b^r)$$

Vztah sférických a kartézských bází vektorů a forem

$$\boldsymbol{e}_{\hat{x}} = \sin\vartheta\cos\varphi\,\boldsymbol{e}_{\hat{r}} + \cos\vartheta\cos\varphi\,\boldsymbol{e}_{\hat{\vartheta}} - \sin\varphi\,\boldsymbol{e}_{\hat{\varphi}} = \frac{x}{r}\,\boldsymbol{e}_{\hat{r}} + \frac{z}{r}\frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{\vartheta}} - \frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{\varphi}}$$

$$\boldsymbol{e}_{\hat{y}} = \sin\vartheta\sin\varphi\,\boldsymbol{e}_{\hat{r}} + \cos\vartheta\sin\varphi\,\boldsymbol{e}_{\hat{\vartheta}} + \cos\varphi\,\boldsymbol{e}_{\hat{\varphi}} = \frac{y}{r}\,\boldsymbol{e}_{\hat{r}} + \frac{z}{r}\frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{\vartheta}} + \frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{\varphi}}$$

$$\boldsymbol{e}_{\hat{z}} = \cos\vartheta\,\boldsymbol{e}_{\hat{r}} - \sin\vartheta\,\boldsymbol{e}_{\hat{\vartheta}} = \frac{z}{r}\,\boldsymbol{e}_{\hat{r}} - \frac{\sqrt{x^2+y^2}}{r}\,\boldsymbol{e}_{\hat{\vartheta}}$$

$$\boldsymbol{e}_{\hat{r}} = \frac{1}{\sqrt{x^2+y^2+z^2}}\,(x\,\boldsymbol{e}_{\hat{x}} + y\,\boldsymbol{e}_{\hat{y}} + z\,\boldsymbol{e}_{\hat{z}}) = \sin\vartheta\cos\varphi\,\boldsymbol{e}_{\hat{x}} + \sin\vartheta\sin\varphi\,\boldsymbol{e}_{\hat{y}} + \cos\vartheta\,\boldsymbol{e}_{\hat{z}}$$

$$\boldsymbol{e}_{\hat{\vartheta}} = \frac{z}{\sqrt{x^2+y^2+z^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{x}} + \frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{y}} - \frac{\sqrt{x^2+y^2}}{z}\,\boldsymbol{e}_{\hat{z}}\right)$$

$$= \cos\vartheta\cos\varphi\,\boldsymbol{e}_{\hat{x}} + \cos\vartheta\sin\varphi\,\boldsymbol{e}_{\hat{y}} - \sin\vartheta\,\boldsymbol{e}_{\hat{z}}$$

$$\boldsymbol{e}_{\hat{\varphi}} = -\frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{x}} + \frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{y}} = -\sin\varphi\,\boldsymbol{e}_{\hat{x}} + \cos\varphi\,\boldsymbol{e}_{\hat{y}}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x} = \sin\vartheta\cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}r} + \frac{\cos\vartheta\cos\varphi}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} - \frac{\sin\varphi}{r\sin\vartheta}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = \frac{x}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}r} + \frac{z}{r^2}\frac{x}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} - \frac{y}{x^2+y^2}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}y} = \sin\vartheta\sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}r} + \frac{\cos\vartheta\sin\varphi}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} + \frac{\cos\varphi}{r\sin\vartheta}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = \frac{y}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}r} + \frac{z}{r^2}\frac{y}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} + \frac{x}{x^2+y^2}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z} = \cos\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}r} - \frac{\sin\vartheta}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} = \frac{z}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}r} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}r} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x} + y\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}y} + z\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z}\right) = \sin\vartheta\cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x} + \sin\vartheta\sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}y} + \cos\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} = \frac{xz}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x} + \frac{yz}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}y} - \sqrt{x^2+y^2}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z} = r\cos\vartheta\cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x} + r\cos\vartheta\sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}y} - r\sin\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = -y\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x} + x\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}y} = -r\sin\vartheta\sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}x} + r\sin\vartheta\cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}y}$$

$$\mathbf{d}x = \sin\vartheta\cos\varphi\,\mathbf{d}r + r\cos\vartheta\cos\varphi\,\mathbf{d}\vartheta - r\sin\vartheta\sin\varphi\,\mathbf{d}\varphi = \frac{x}{r}\,\mathbf{d}r + \frac{xz}{\sqrt{x^2+y^2}}\,\mathbf{d}\vartheta - y\,\mathbf{d}\varphi$$

$$\mathbf{d}y = \sin\vartheta\sin\varphi\,\mathbf{d}r + r\cos\vartheta\sin\varphi\,\mathbf{d}\vartheta + r\sin\vartheta\cos\varphi\,\mathbf{d}\varphi = \frac{y}{r}\,\mathbf{d}r + \frac{yz}{\sqrt{x^2+y^2}}\,\mathbf{d}\vartheta + x\,\mathbf{d}\varphi$$

$$\mathbf{d}z = \cos\vartheta\,\mathbf{d}r - r\sin\vartheta\,\mathbf{d}\vartheta = \frac{z}{r}\,\mathbf{d}r - \sqrt{x^2+y^2}\,\mathbf{d}\vartheta$$

$$\mathbf{d}r = \frac{1}{\sqrt{x^2+y^2+z^2}}\,(x\,\mathbf{d}x + y\,\mathbf{d}y + z\,\mathbf{d}z) = \sin\vartheta\cos\varphi\,\mathbf{d}x + \sin\vartheta\sin\varphi\,\mathbf{d}y + \cos\vartheta\,\mathbf{d}z$$

$$\mathbf{d}\vartheta = \frac{1}{x^2+y^2+z^2}\left(\frac{xz}{\sqrt{x^2+y^2}}\,\mathbf{d}x + \frac{yz}{\sqrt{x^2+y^2}}\,\mathbf{d}y - \sqrt{x^2+y^2}\,\mathbf{d}z\right)$$

$$= \frac{1}{r}\,(\cos\vartheta\cos\varphi\,\mathbf{d}x + \cos\vartheta\sin\varphi\,\mathbf{d}y - \sin\vartheta\,\mathbf{d}z)$$

$$\mathbf{d}\varphi = \frac{1}{x^2+y^2}(-y\,\mathbf{d}x + x\,\mathbf{d}y) = \frac{1}{r\sin\vartheta}(-\sin\varphi\,\mathbf{d}x + \cos\varphi\,\mathbf{d}y)$$

Derivace bázových vektorů a forem

$$\nabla \boldsymbol{e}_{\hat r} = \frac{1}{r}\,\boldsymbol{e}^{\hat\vartheta}\,\boldsymbol{e}_{\hat\vartheta} + \frac{1}{r}\,\boldsymbol{e}^{\hat\varphi}\,\boldsymbol{e}_{\hat\varphi} \qquad \nabla\cdot\boldsymbol{e}_{\hat r} = \frac{2}{r} \qquad \nabla\times\boldsymbol{e}_{\hat r} = 0$$

$$\nabla \boldsymbol{e}_{\hat\vartheta} = -\frac{1}{r}\,\boldsymbol{e}^{\hat\vartheta}\,\boldsymbol{e}_{\hat r} + \frac{1}{r}\cot\vartheta\,\boldsymbol{e}^{\hat\varphi}\,\boldsymbol{e}_{\hat\varphi} \qquad \nabla\cdot\boldsymbol{e}_{\hat\vartheta} = \frac{1}{r}\cot\vartheta \qquad \nabla\times\boldsymbol{e}_{\hat\vartheta} = \frac{1}{r}\,\boldsymbol{e}_{\hat\varphi}$$

$$\nabla \boldsymbol{e}_{\hat\varphi} = -\frac{1}{r}\,\boldsymbol{e}^{\hat\varphi}\,\boldsymbol{e}_{\hat r} - \frac{1}{r}\cot\vartheta\,\boldsymbol{e}^{\hat\varphi}\,\boldsymbol{e}_{\hat\vartheta} \qquad \nabla\cdot\boldsymbol{e}_{\hat\varphi} = 0 \qquad \nabla\times\boldsymbol{e}_{\hat\varphi} = -\frac{1}{r}\,\boldsymbol{e}_{\hat\vartheta} + \frac{1}{r}\cot\vartheta\,\boldsymbol{e}_{\hat r}$$

$$\nabla\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} = \frac{1}{r}\,\mathbf{d}\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} + \frac{1}{r}\,\mathbf{d}\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} \qquad \nabla\cdot\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} = \frac{2}{r} \qquad \nabla\times\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} = 0$$

$$\nabla\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} = -r\,\mathbf{d}\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} + \frac{1}{r}\,\mathbf{d}r\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} + \cot\vartheta\,\mathbf{d}\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} \qquad \nabla\cdot\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} = \cot\vartheta \qquad \nabla\times\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} = \frac{2}{r\sin\vartheta}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$

$$\nabla\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = \frac{1}{r}\,\mathbf{d}r\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} + \cot\vartheta\,\mathbf{d}\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} - r\sin^2\vartheta\,\mathbf{d}\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} - \sin\vartheta\cos\vartheta\,\mathbf{d}\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta}$$

$$\nabla\cdot\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = 0 \qquad \nabla\times\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = -\frac{2}{r}\sin\vartheta\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} + 2\cos\vartheta\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r}$$

$$\nabla\,\mathbf{d}r = r\;\mathbf{d}\vartheta\,\mathbf{d}\vartheta + r\sin^2\vartheta\;\mathbf{d}\varphi\,\mathbf{d}\varphi \qquad \nabla^2\,r = \frac{2}{r}$$

$$\nabla\,\mathbf{d}\vartheta = -\frac{1}{r}(\,\mathbf{d}r\,\mathbf{d}\vartheta + \,\mathbf{d}\vartheta\,\mathbf{d}r) + \sin\vartheta\cos\vartheta\,\mathbf{d}\varphi\,\mathbf{d}\varphi \qquad \nabla^2\,\vartheta = \frac{1}{r^2}\cot\vartheta$$

$$\nabla\,\mathbf{d}\varphi = -\frac{1}{r}(\,\mathbf{d}r\,\mathbf{d}\varphi + \,\mathbf{d}\varphi\,\mathbf{d}r) - \cot\vartheta(\,\mathbf{d}\vartheta\,\mathbf{d}\varphi + \,\mathbf{d}\varphi\,\mathbf{d}\vartheta) \qquad \nabla^2\,\varphi = 0$$

Souřadnice kovariantní derivace

$$a^k{}_{;m} = a^k{}_{,m} + \Gamma^{\,k}_{mn}a^n \qquad , \qquad a_{n;m} = a_{n,m} - \Gamma^{\,k}_{mn}a_k$$

$$\Gamma^{\,r}_{\vartheta\vartheta} = -r \qquad \Gamma^{\,\vartheta}_{r\vartheta} = \Gamma^{\,\vartheta}_{\vartheta r} = \frac{1}{r} \qquad \Gamma^{\,\varphi}_{r\varphi} = \Gamma^{\,\varphi}_{\varphi r} = \frac{1}{r}$$

$$\Gamma^{\,r}_{\varphi\varphi} = -r\sin^2\vartheta \qquad \Gamma^{\,\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \qquad \Gamma^{\,\varphi}_{\vartheta\varphi} = \Gamma^{\,\varphi}_{\varphi\vartheta} = \cot\vartheta$$

ostatní Γ-koeficienty jsou nulové

Operátory ve sférických souřadnicích

gradient skaláru: $\boldsymbol{a} = \mathbf{d}f = \nabla f$

$$a_r = f_{,r} \qquad a_{\hat r} = f_{,r} \qquad a^r = f_{,r}$$

$$a_\vartheta = f_{,\vartheta} \qquad a_{\hat\vartheta} = \frac{1}{r}\, f_{,\vartheta} \qquad a^\vartheta = \frac{1}{r^2}\, f_{,\vartheta}$$

$$a_\varphi = f_{,\varphi} \qquad a_{\hat\varphi} = \frac{1}{r\sin\vartheta}\, f_{,\varphi} \qquad a^\varphi = \frac{1}{r^2\sin^2\vartheta}\, f_{,\varphi}$$

divergence: $f = \nabla\cdot\boldsymbol{a}$

$$\begin{aligned} f &= \frac{1}{r^2}(r^2\, a^r)_{,r} + \frac{1}{\sin\vartheta}(\sin\vartheta\, a^\vartheta)_{,\vartheta} + a^\varphi{}_{,\varphi} \\ &= \frac{1}{r^2}(r^2\, a^{\hat r})_{,r} + \frac{1}{r\sin\vartheta}\,(\sin\vartheta\, a^{\hat\vartheta})_{,\vartheta} + \frac{1}{r\sin\vartheta}\, a^{\hat\varphi}{}_{,\varphi} \\ &= \frac{1}{r^2}(r^2\, a_r)_{,r} + \frac{1}{r^2\sin\vartheta}\,(\sin\vartheta\, a_\vartheta)_{,\vartheta} + \frac{1}{r^2\sin^2\vartheta}\, a_{\varphi,\varphi} \end{aligned}$$

rotace: $\boldsymbol{b} = \nabla\times\boldsymbol{a}$

$$b_r = \frac{1}{r^2\sin\vartheta}\,(a_{\varphi,\vartheta} - a_{\vartheta,\varphi}) \qquad b_{\hat r} = \frac{1}{r\sin\vartheta}\,(\sin\vartheta\, a_{\hat\varphi})_{,\vartheta} - a_{\hat\vartheta,\varphi} \qquad b^r = \frac{1}{\sin\vartheta}\,((\sin^2\vartheta\, a^\varphi)_{,\vartheta} - a^\vartheta{}_{,\varphi})$$

$$b_\vartheta = \frac{1}{\sin\vartheta}\,(a_{r,\varphi} - a_{\varphi,r}) \qquad b_{\hat\vartheta} = \frac{1}{r}\left(\frac{1}{\sin\vartheta}\, a_{\hat r,\varphi} - (r\, a_{\hat\varphi})_{,r}\right) \qquad b^\vartheta = \frac{1}{r^2\sin\vartheta}\, a^r{}_{,\varphi} - \frac{1}{r^2}(r^2\, a^\varphi)_{,r})$$

$$b_\varphi = \sin\vartheta\,(a_{\vartheta,r} - a_{r,\vartheta}) \qquad b_{\hat\varphi} = \frac{1}{r}\,((r\, a_{\hat\vartheta})_{,r} - a_{\hat r,\vartheta}) \qquad b^\varphi = \frac{1}{r^2\sin\vartheta}\,\big((r^2\, a^\vartheta)_{,r} - a^r{}_{,\vartheta}\big)$$

laplace:

$$\begin{aligned} \nabla^2 f &= \frac{1}{r^2}\,(r^2\, f_{,r})_{,r} + \frac{1}{r^2\sin\vartheta}\,(\sin\vartheta\, f_{,\vartheta})_{,\vartheta} + \frac{1}{r^2\sin^2\vartheta}\, f_{,\varphi\varphi} \\ &= f_{,rr} + \frac{1}{r^2} f_{,\vartheta\vartheta} + \frac{1}{r^2\sin^2\vartheta} f_{,\varphi\varphi} + \frac{2}{r} f_{,r} + \frac{\cot\vartheta}{r^2} f_{,\vartheta} \end{aligned}$$